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HYPER-EXTENSIONS IN METRIC FIXED POINT THEORY

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. We apply a modern axiomatic system of nonstandard analysis in metric fixed point theory. In particular, we formulate a nonstandard iteration scheme for nonexpansive mappings and present a nonstandard approach to fixed-point problems in direct sums of Banach spaces.

1. Introduction

Nonstandard analysis was originated in the 1960s by A. Robinson. By considering a hyper-extension of real numbers he was able to provide logically rigorous foundations for infinitely small and infinitely large numbers.

Nonstandard methods came to Banach space theory from the work of W. A. J. Luxemburg who introduced the notion of a nonstandard hull. Another approach, based on the concept of a Banach space ultraproduct, was proposed by J. Bretagnolle, D. Dacunha-Castelle and J.-L. Krivine. In 1980, B. Maurey [21] applied the Banach space ultraproduct construction to solve several difficult problems in metric fixed point theory. His methods have been extended by numerous authors to obtain a lot of strong results in that theory (see [1, 13, 27]). It seems that nonstandard analysis has some conceptual advantages over the ultraproduct method because it provides us with techniques which are not very easy to express in the ultraproduct setting. But the early approaches to nonstandard analysis appeared too technical to many mathematicians and required a good background in logic.

At present there exist several interesting frameworks for nonstandard analysis. In this paper we shall use a modern axiomatic approach based on Alpha-Theory introduced by V. Benci and M. Di Nasso in [3]. In this approach, all axioms of ZFC (except foundation) are assumed and for every set A, there exists a set ${}^*A$ called the hyper-extension of A. The resulting theory overcomes the distinction between "standard" and "nonstandard" objects and is closer to mathematical practice. Our aim is to signal new possibilities in metric fixed point theory by applying modern infinitesimal techniques.

Section 2 contains a brief presentation of basic notions including a nonstandard hull of a Banach space, an intra-convergence of an ${}^*\mathbb{N}$-sequence and a counterpart of Mazur's lemma. In Section 3 we formulate a nonstandard iteration scheme for nonexpansive mappings in uniformly convex spaces. Although it is not clear to what extent nonstandard analysis can be done constructively, there are some interesting attempts to give infinitesimal analysis computational content (see [6, 25, 26]). In Section 4 we present a nonstandard approach to fixed-point problems in direct

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sums of Banach spaces. We show how to use the notion of intra-convex sets and a counterpart of Mazur's lemma to improve the results in [24]. The reader may compare this approach, closer to the original idea, with its classical translation in [31]. A brief presentation of Alpha-Theory is given in the Appendix.

2. Nonstandard preliminaries

In this paper we work in the system of nonstandard analysis based on Alpha-Theory introduced by Benci and Di Nasso in [3]. In this approach, for every set A, there exists a set ${}^*A$ called the hyper-extension (or the star-transform) of A, see Appendix A.

The most important for our purposes is the following theorem called the transfer principle.

Theorem 2.1. *For every bounded formula $\sigma(x_1, \ldots, x_k)$ and for any sets $a_1, \ldots, a_k$,*

$$\sigma(a_1, \ldots, a_k) \Longleftrightarrow \sigma({}^*a_1, \ldots, {}^*a_k).$$

We will use this theorem several times. See, e.g., [4, 14] for more details how to apply the transfer principle correctly.

Let X be a real Banach space and let ${}^*X$ be its hyper-extension endowed with a function

$${}^*\|\cdot\| : {}^*X \to {}^*\mathbb{R}$$

called an internal norm (or $*$-norm) of ${}^*X$. There is a common practice to omit "stars" when no confusion can arise and we abbreviate ${}^*\|\cdot\|$ to $\|\cdot\|$. Recall that an element $x \in {}^*X$ is bounded if $\|x\|$ is bounded in ${}^*\mathbb{R}$. It is infinitesimal if $\|x\|$ is infinitesimal in ${}^*\mathbb{R}$, see Appendix A. Let $\operatorname{gal}({}^*X)$ denote the set of bounded elements and $\operatorname{mon}(0)$ the set of infinitesimal elements of ${}^*X$. Notice that $\operatorname{gal}({}^*X)$ and $\operatorname{mon}(0)$ are vector spaces over $\mathbb{R}$ and we may define $\widetilde{X}$ as the quotient vector space

$$\operatorname{gal}({}^*X)/\operatorname{mon}(0).$$

Let $\pi : \operatorname{gal}({}^*X) \to \widetilde{X}$ denote the quotient linear mapping and define a norm on $\widetilde{X}$ by $\|y\| = \operatorname{st}(\|x\|)$ for all $x \in \operatorname{gal}({}^*X)$, $y = \pi(x)$, where $\operatorname{st}(\|x\|)$ is the standard part of $\|x\|$ in $\mathbb{R}$. The vector space $\widetilde{X}$ with the above norm becomes a Banach space and is called the nonstandard hull of X, see, e.g., [10, 15, 22]. It is clear that X is isometric to a subspace of $\widetilde{X}$ via the mapping $z \to \pi({}^*z)$. Virtually, π is an extension of the standard part mapping $\operatorname{st} : \operatorname{mon}(X) \to X$ and we denote $\pi(x)$ by $\operatorname{sh}(x)$ or ${}^\circ x$. Thus we have $\operatorname{sh}(x) = \operatorname{mon}(\operatorname{st}(x))$ for every $x \in \operatorname{mon}(X)$. We refer to

$$\operatorname{sh} : \operatorname{gal}({}^*X) \to \widetilde{X}$$

as the shadow mapping. Set ${}^\circ A = \{{}^\circ x : x \in A\}$ for any set $A \subset \operatorname{gal}({}^*X)$ and $\widetilde{B} = {}^\circ({}^*B \cap \operatorname{gal}({}^*X))$ for any $B \subset X$.

Let $\mathbb{R}_+$ denote the set of positive reals. By an ${}^*\mathbb{N}$-sequence $(x_n)_{n\in {}^*\mathbb{N}}$ in Y we mean a function $x : {}^*\mathbb{N} \to Y$.

Definition 2.2. An ${}^*\mathbb{N}$-sequence $(x_n)_{n\in {}^*\mathbb{N}}$ in ${}^*X$ is said to intra-converge (or $*$-converge) to $a \in {}^*X$ if

$$\forall \varepsilon \in {}^*\mathbb{R}_+ \; \exists k \in {}^*\mathbb{N} \; \forall n \in {}^*\mathbb{N} \; (n \geq k \Rightarrow \|x_n - a\| < \varepsilon).$$

In a similar way, we can define intra-convergence for the weak topology. Let $\mathcal{T}$ denote the weak topology on a Banach space X.

Definition 2.3. An ${}^*\mathbb{N}$-sequence $(x_n)_{n\in {}^*\mathbb{N}}$ in ${}^*X$ is said to weakly intra-converge (or $*$-weakly converge) to $a \in {}^*X$ if

$$\forall U \in {}^*\mathcal{T}\ \exists k \in {}^*\mathbb{N}\ \forall n \in {}^*\mathbb{N}\ (a \in U \wedge n \geq k \Rightarrow x_n \in U).$$

Notice that if (x_n) is a sequence in X converging (resp., weakly converging) to x_0, then it follows from transfer that its hyper-extension $(x_n)_{n\in {}^*\mathbb{N}}$ intra-converges (resp., weakly intra-converges) to ${}^*x_0$ in ${}^*X$.

Definition 2.4. We say that a set $A \subset {}^*X$ is intra-convex (or $*$-convex) if

$$\forall \alpha, \beta \in {}^*[0,1]\ \forall x, y \in A\ (\alpha + \beta = 1 \Rightarrow \alpha x + \beta y \in A).$$

For $A \subset {}^*X$, define

$$\mathrm{conv}_{\mathrm{int}}(A) = \bigcup_{n\in {}^*\mathbb{N}} \left\{ \sum_{i=0}^{n} \lambda_i x_i : \lambda_i \in {}^*[0,1], x_i \in A, 0 \leq i \leq n, \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

The following lemma is a simple application of the transfer principle and Mazur's lemma.

Lemma 2.5. *Assume that an internal ${}^*\mathbb{N}$-sequence $(x_n)_{n\in {}^*\mathbb{N}}$ in ${}^*X$ intra-converges weakly to a. Then ${}^\circ a \in {}^\circ\, \mathrm{conv}_{int}(\{x_n : n \in {}^*\mathbb{N}\})$.*

Proof. Let $(x_n)_{n\in {}^*\mathbb{N}}$ be an internal ${}^*\mathbb{N}$-sequence in ${}^*X$ intra-converging weakly to $a \in {}^*X$. It follows from the transfer of Mazur's lemma that for every $\varepsilon \in {}^*\mathbb{R}_+$ there exists $k \in {}^*\mathbb{N}$ and $\lambda_0, \ldots, \lambda_k \in {}^*[0,1]$ with $\sum_{i=0}^{k} \lambda_i = 1$ such that $\left\| \sum_{i=0}^{k} \lambda_i x_i - a \right\| \leq \varepsilon$. Fix a positive $\varepsilon \simeq 0$. Then there exists $y \in \mathrm{conv}_{\mathrm{int}}(\{x_n : n \in {}^*\mathbb{N}\})$ such that $\|y - a\| \simeq 0$. Hence

$${}^\circ a \in {}^\circ\, \mathrm{conv}_{\mathrm{int}}(\{x_n : n \in {}^*\mathbb{N}\}).$$

□

A routine application of the transfer principle shows that if A is internal and $*$-relatively compact (i.e., for every internal ${}^*\mathbb{N}$-sequence $(x_n)_{n\in {}^*\mathbb{N}}$ of elements in A, there exists an internal intra-convergent ${}^*\mathbb{N}$-subsequence $(x_{n_k})_{k\in {}^*\mathbb{N}}$), then $\mathrm{conv}_{\mathrm{int}}(A)$ is $*$-relatively compact, too. We will use this fact together with Lemma 2.5 in Section 4.

3. Nonstandard Picard Iteration

Let (M, ρ) be a metric space. An internal mapping $T : {}^*M \to {}^*M$ is said to be an intra-contraction if there exists $k \in {}^*(0,1)$ such that

$${}^*\rho(Tx, Ty) \leq k\, {}^*\rho(x, y)$$

for all $x, y \in {}^*M$.

Let $T : {}^*M \to {}^*M$ be an intra-contraction and fix $x_0 \in {}^*M$. Set $x_{n+1} = Tx_n$ for each $n \in {}^*\mathbb{N}$. Since T is internal, we obtain the ${}^*\mathbb{N}$-sequence $(T^n x_0)_{n\in {}^*\mathbb{N}}$ by internal induction.

The following theorem is an internal version of the Banach's Contraction Principle. We leave its proof to the reader.

Theorem 3.1. *Let (M, ρ) be a complete metric space and $T : {}^*M \to {}^*M$ an intra-contraction. Then T has a unique fixed point in ${}^*M$ and for each $x_0 \in {}^*M$ the ${}^*\mathbb{N}$-sequence $(T^n x_0)_{n\in {}^*\mathbb{N}}$ intra-converges to this fixed point.*

Now let C be a nonempty bounded closed and convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. It is well known that unlike in the case of contractions, the Picard iteration $(T^n x_0)_{n\in\mathbb{N}}$, $x_0 \in C$, may fail to converge. In the last few decades, iterative methods for finding fixed points of nonexpansive mappings have been studied extensively. It is worth pointing out two types of such methods. The Mann iteration is defined by the recursive scheme

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\ n \in \mathbb{N},$$

where $x_0 \in C$ and $\alpha_n \in [0, 1]$. The Halpern iteration is defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T x_n,\ n \in \mathbb{N},$$

where $x_0, u \in C$ and $\alpha_n \in [0, 1],\ n \in \mathbb{N}$. Unlike Mann's iteration, a sequence generated by Halpern's scheme is strongly convergent provided the underlying Banach space is smooth enough and (α_n) satisfies some mild conditions. However, in general, the problem of the convergence of Halpern's iteration is still open even in the case of uniformly convex spaces. For a deeper discussion of this topic we refer the reader to [33] and the references given there.

New possibilities arises if we consider infinitesimal perturbations of nonexpansive mappings. Let $T : C \to C$ be a nonexpansive mapping. By the transfer principle, we obtain an (intra-nonexpansive) mapping ${}^*T : {}^*C \to {}^*C$ and we can define a nonexpansive mapping $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ in the nonstandard hull of a Banach space X by putting $\widetilde{T}({}^\circ x) = {}^\circ({}^*Tx)$ for $x \in {}^*C$. We may regard C as a subset of $\widetilde{C}$ via the mapping $x \to {}^\circ({}^*x)$ and $\widetilde{T}$ as an extension of T.

Let $u \in {}^*C$. Fix a positive infinitesimal ε and define

$$Sx = (1 - \varepsilon){}^*Tx + \varepsilon u,\ x \in {}^*C.$$

It is not difficult to check that $S : {}^*C \to {}^*C$ is an intra-contraction and we can consider for a fixed $x_0 \in {}^*C$ a nonstandard Picard iteration

$$x_{n+1} = S^n x_0,\ n \in {}^*\mathbb{N}. \tag{3.1}$$

It follows from Theorem 3.1 that the ${}^*\mathbb{N}$-sequence $(S^n x_0)_{n\in {}^*\mathbb{N}}$ intra-converges to a point $z_0 \in {}^*C$. Notice that ${}^\circ z_0 \in \widetilde{C}$ is a fixed point of $\widetilde{T}$ since

$$\|{}^*Tz_0 - z_0\| \leq \varepsilon \simeq 0.$$

Denote by $P_C : \widetilde{X} \to C$ a metric projection onto C:

$$P_C x = \left\{ y \in C : \|x - y\| = \inf_{z\in C} \|x - z\| \right\}.$$

It is well known that in uniformly convex spaces $P_C x$ is a singleton for every $x \in \widetilde{X}$. Furthermore

$$\left\| \widetilde{T} P_C {}^\circ z_0 - {}^\circ z_0 \right\| = \left\| \widetilde{T} P_C {}^\circ z_0 - \widetilde{T} {}^\circ z_0 \right\| \leq \| P_C {}^\circ z_0 - {}^\circ z_0 \| = \inf_{z \in C} \| {}^\circ z_0 - z \| .$$

But $\widetilde{T} P_C {}^\circ z_0 \in C$ and hence $\widetilde{T} P_C {}^\circ z_0 = P_C {}^\circ z_0$, i.e., $P_C {}^\circ z_0$ is a fixed point of T. In this way, we obtain the following theorem.

Theorem 3.2. *Let C be a nonempty bounded closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ a nonexpansive mapping. Then the nonstandard Picard iteration given by (3.1) intra-converges to a point $z_0 \in {}^*C$. Furthermore, $P_C {}^\circ z_0$ is a fixed point of T.*

A natural question arises whether the projection P_C is at all necessary, i.e., whether ${}^\circ z_0 \in C$ if $u, x_0 \in C$. An affirmative answer to this question should result in the study of Halpern's iteration.

4. Fixed points of direct sums

Recall that a Banach space X is said to have the fixed point property (FPP) if every nonexpansive self-mapping defined on a nonempty bounded closed and convex set $C \subset X$ has a fixed point. A Banach space X is said to have the weak fixed point property (WFPP) if the additional assumption is added that C is weakly compact.

The problem of whether FPP or WFPP is preserved under direct sum of Banach spaces is an old one. In 1968, L. P. Belluce, W. A. Kirk and E. F. Steiner [2] proved that the direct sum of two Banach spaces with normal structure, endowed with the "maximum" norm, also has normal structure. Since then, the preservation of normal structure and conditions which guarantee normal structure have been studied extensively and the problem is now quite well understood (see [9] for a survey). But the situation is much more difficult if at least one of these spaces lacks weak normal structure. We note here the results of S. Dhompongsa, A. Kaewcharoen and A. Kaewkhao [8], and M. Kato and T. Tamura (see [19, 20]).

Recently, a few general fixed point theorems in direct sums were proved in [31, 32] (see also [24, 30]). Although their proofs were formulated in standard terms, the original ideas came from nonstandard analysis. In this section we present the original proof of the main result in [31] which is, in our opinion, more insightful than its classical translation.

Let us first recall terminology concerning direct sums. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be monotone if

$$\|(x_1, y_1)\| \leq \|(x_2, y_2)\| \quad \text{whenever} \quad 0 \leq x_1 \leq x_2, 0 \leq y_1 \leq y_2.$$

A norm $\|\cdot\|$ is said to be strictly monotone if

$$\|(x_1, y_1)\| < \|(x_2, y_2)\| \quad \text{whenever} \quad 0 \leq x_1 \leq x_2, 0 \leq y_1 < y_2 \quad \text{or} \quad 0 \leq x_1 < x_2, 0 \leq y_1 \leq y_2.$$

It is easy to see that ℓ_p^2-norms, $1 \leq p < \infty$, are strictly monotone. We will assume that the norm is normalized, i.e.,

$$\|(1, 0)\| = \cdots = \|(0, 1)\| = 1.$$

F. F. Bonsall and J. Duncan [5] showed that the set of all monotone and normalized norms on $\mathbb{R}^2$ is in one-to-one correspondence with the set Ψ of all continuous convex functions on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $0 \leq t \leq 1$, where the correspondence is given by

$$\psi(t) = \|(1-t,t)\|\,,\ 0 \leq t \leq 1. \tag{4.1}$$

Conversely, for any $\psi \in \Psi$ define

$$\|(x_1, x_2)\|_\psi = (|x_1| + |x_2|)\psi(|x_2| \,/\, |x_1| + |x_2|)$$

for $(x_1, x_2) \neq (0,0)$ and $\|(0,0)\|_\psi = 0$. Then $\|\cdot\|_\psi$ is an absolute and normalized norm which satisfies (4.1). It was proved in [29, Corollary 3] that a norm $\|\cdot\|_\psi$ in $\mathbb{R}^2$ is normalized and strictly monotone iff

$$\psi(t) > \psi_\infty(t)$$

for all $0 < t < 1$. Let X, Y be Banach spaces and $\psi \in \Psi$. We shall write $X \oplus_\psi Y$ for the ψ-direct sum of X, Y with the norm $\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$, where $(x,y) \in X \times Y$.

A Banach space X is said to have the generalized Gossez-Lami Dozo property (GGLD, in short) if

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - x_m\| > 1$$

whenever (x_n) converges weakly to 0 and $\lim_{n\to\infty} \|x_n\| = 1$. It is known that the GGLD property is weaker than weak uniform normal structure (see, e.g., [28]).

The following lemma was proved in [24, Lemma 4] (see also [11, 28]).

Lemma 4.1. *Let $X \oplus_\psi Y$ be a ψ-direct sum of Banach spaces X, Y with a strictly monotone norm. Assume that Y has the GGLD property, the vectors $w_n = (x_n, y_n) \in X \oplus_\psi Y$ tend weakly to 0 and*

$$\lim_{n,m\to\infty, n\neq m} \|w_n - w_m\|_\psi = \lim_{n\to\infty} \|w_n\|_\psi.$$

Then $\lim_{n\to\infty} \|y_n\| = 0$.

We are now in a position to give a nonstandard proof of the following theorem.

Theorem 4.2 ([31])**.** *Let X be a Banach space with WFPP and suppose Y has the GGLD property. Then $X \oplus_\psi Y$ with a strictly monotone norm has WFPP.*

Proof. The proof will be divided into 5 steps.

Step 1. We follow the classical arguments in metric fixed point theory. Assume that $X \oplus_\psi Y$ does not have WFPP. Then, there exist a weakly compact convex subset C of $X \oplus_\psi Y$ and a nonexpansive mapping $T : C \to C$ without a fixed point. By the Kuratowski-Zorn lemma, there exists a convex and weakly compact set $K \subset C$ which is minimal invariant under T and which is not a singleton. Let $(w_n) = ((x'_n, y'_n))$ be an approximate fixed point sequence for T in K, i.e., $\lim_{n\to\infty} \|Tw_n - w_n\|_\psi = 0$. Without loss of generality we can assume that $\operatorname{diam} K = 1$, (w_n) converges weakly to $(0,0) \in K$ and the double limit $\lim_{n,m\to\infty, n\neq m} \|w_n - w_m\|_\psi$ exists. It follows from the Goebel-Karlovitz lemma (see [12, 18]) that

$$\lim_{n,m\to\infty, n\neq m} \|w_n - w_m\|_\psi = 1 = \lim_{n\to\infty} \|w_n - w\|_\psi \tag{4.2}$$

for every $w \in K$. Hence $\lim_{n\to\infty} \|y'_n\| = 0$ by Lemma 4.1.

Step 2. Let $(w_n)_{n\in {}^*\mathbb{N}}$ be a hyper-extension of the sequence $(w_n)_{n\in\mathbb{N}}$. Since $(\mathbb{R}^2, \|\cdot\|_\psi)$ is a finite dimensional space, the norm $\|\cdot\|_\psi$ is strictly monotone iff it is uniformly monotone. It follows, using transfer, that for every $\varepsilon \in {}^*\mathbb{R}_+$, there exists $\delta(\varepsilon) \in {}^*\mathbb{R}_+$ such that if $(\bar{a}, \bar{b}), (\bar{a}, \bar{c})$ belong to ${}^*B_{(\mathbb{R}^2,\|\cdot\|_\psi)}$ and $\left\|(\bar{a}, \bar{b})\right\|_\psi < \|(\bar{a}, \bar{c})\|_\psi + \delta(\varepsilon)$, then $\left\|\bar{b}\right\| < \|\bar{c}\| + \varepsilon$. Fix an unbounded $\omega \in {}^*\mathbb{N}$ and put $\eta = \frac{1}{\omega} \simeq 0$. Let

$$\varepsilon_i = \min\{\eta\delta(\eta^i)/3, \eta^{i+1}\}, \quad i \in {}^*\mathbb{N}.$$

By transfer, $\|{}^*Tw_n - w_n\|_\psi$ and $\|y'_n\|$ intra-converge to 0 and hence we can fix $v_0 = w_{n_0} = (x_0, y_0)$ such that $\|{}^*Tv_0 - v_0\|_\psi < \varepsilon_0$ and $\|y_0\| < \varepsilon_0$. For hypernatural numbers $1 \le j \le \omega$, write $D_j^0 = \{v_0\}$. We shall define an internal ${}^*\mathbb{N}$-subsequence $(v_n)_{n\in {}^*\mathbb{N}}$ of $(w_n)_{n\in {}^*\mathbb{N}}$ and an internal family $\left\{D_j^i\right\}_{1\le j\le\omega, i\in {}^*\mathbb{N}}$ of $*$-relatively compact subsets of ${}^*K$ by internal induction. Choose $v_1 = w_{n_1} = (x_1, y_1)$, $n_0 < n_1 \in {}^*\mathbb{N}$ in such a way that $\|{}^*Tv_1 - v_1\|_\psi < \varepsilon_1$, $\|y_1\| < \varepsilon_1$ and $\|v_1 - v_0\|_\psi > 1 - \varepsilon_1$ (notice that, by transfer of (4.2), $\|w_n - v_0\|$ intra-converges to 1). Let us put

$$D_1^1 = \mathrm{conv}_{\mathrm{int}}\{v_0, v_1\}$$

and

$$D_{j+1}^1 = \mathrm{conv}_{\mathrm{int}}(D_j^1 \cup {}^*T(D_j^1))$$

for $1 \le j < \omega$. By internal induction, $\{D_1^1, \ldots, D_\omega^1\}$ is a well-defined internal family of $*$-relatively compact subsets of ${}^*K$ with $D_1^1 \subset \cdots \subset D_\omega^1$.

Now suppose that we have chosen an internal k-tuple $n_1 < \cdots < n_k$ ($k \in {}^*\mathbb{N}\smallsetminus\{0\}, n_1 > n_0$), $v_i = w_{n_i} = (x_i, y_i), 0 \le i \le k$, and internal k-tuple $(\{D_1^i, \ldots, D_\omega^i\})_{1\le i\le k}$ of subsets of ${}^*K$ such that for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \omega - 1\}$:

(i) $\|{}^*Tv_i - v_i\|_\psi < \varepsilon_i$,
(ii) $\|y_i\| < \varepsilon_i$,
(iii) $\|v_i - v\|_\psi > 1 - \varepsilon_i$ for all $v \in D_\omega^{i-1}$,
(iv) $D_1^i = \mathrm{conv}_{\mathrm{int}}(D_1^{i-1} \cup \{v_i\})$,
(v) $D_{j+1}^i = \mathrm{conv}_{\mathrm{int}}(D_j^i \cup {}^*T(D_j^i))$.

Then, there exist (internally chosen) $n_{k+1} > n_k$, $v_{k+1} = w_{n_{k+1}} = (x_{k+1}, y_{k+1})$ such that $\|{}^*Tv_{k+1} - v_{k+1}\| < \varepsilon_{k+1}$, $\|y_{k+1}\| < \varepsilon_{k+1}$ and $\|v_{k+1} - v\| > 1 - \varepsilon_{k+1}$ for all $v \in D_\omega^k$ (the last inequality follows from the $*$-relative compactness of D_ω^k). Let us put

$$D_1^{k+1} = \mathrm{conv}_{\mathrm{int}}(D_1^k \cup \{v_{k+1}\})$$

and

$$D_{j+1}^{k+1} = \mathrm{conv}_{\mathrm{int}}(D_j^{k+1} \cup {}^*T(D_j^{k+1}))$$

for $1 \le j < \omega$. Then, by internal induction on j, $\left\{D_1^{k+1}, \ldots, D_\omega^{k+1}\right\}$ is a well-defined internal family of $*$-relatively compact subsets of ${}^*K$. Hence, by internal induction on i, we obtain an internal sequence $(v_n)_{n\in {}^*\mathbb{N}}$ and an internal family of sets $\left\{D_j^i\right\}_{1\le j\le\omega, i\in {}^*\mathbb{N}}$ such that (i)-(v) are satisfied for every $j \in \{1, \ldots, \omega - 1\}$ and $i \in {}^*\mathbb{N}\smallsetminus\{0\}$.

Step 3. We claim that for every $1 \le j \le \omega$, $i \in {}^*\mathbb{N}\setminus\{0\}$ and $u \in D_j^{i+1}$ there exists $v \in D_j^i$ such that

$$\|v-u\|_\psi + \|u-v_{i+1}\|_\psi \le \|v-v_{i+1}\|_\psi + 3(j-1)\varepsilon_{i+1}. \tag{4.3}$$

Fix $i \in {}^*\mathbb{N}\setminus\{0\}$. We shall proceed by internal induction with respect to j. For $j=1$ and $u \in D_1^{i+1} = \mathrm{conv}_{\mathrm{int}}\,(D_1^i \cup \{v_{i+1}\})$ there exists $v \in D_1^i$ such that

$$\|v-u\|_\psi + \|u-v_{i+1}\|_\psi = \|v-v_{i+1}\|_\psi\,.$$

Fix $1 \le j < \eta$ and suppose that for every $u \in D_j^{i+1}$ there exists $v \in D_j^i$ such that (4.3) is satisfied. Let $u \in D_{j+1}^{i+1} = \mathrm{conv}_{\mathrm{int}}(D_j^{i+1} \cup {}^*T(D_j^{i+1}))$. The inductive step is obvious if $u \in D_j^{i+1}$ so take $u \in {}^*T(D_j^{i+1})$. Then $u = {}^*T\bar{u}$ for some $\bar{u} \in D_j^{i+1}$ and, by assumption, there exists $\bar{v} \in D_j^i$ such that

$$\|\bar{v}-\bar{u}\|_\psi + \|\bar{u}-v_{i+1}\|_\psi \le \|\bar{v}-v_{i+1}\|_\psi + 3(j-1)\varepsilon_{i+1}.$$

Let $v = {}^*T\bar{v} \in D_{j+1}^i$. Then

$$\begin{aligned}\|v-u\|_\psi + \|u-v_{i+1}\|_\psi &\le \|\bar{v}-\bar{u}\|_\psi + \|\bar{u}-v_{i+1}\|_\psi + \|{}^*Tv_{i+1}-v_{i+1}\|_\psi \\ &\le \|\bar{v}-v_{i+1}\|_\psi + (3j-2)\varepsilon_{i+1} \\ &< \|v-v_{i+1}\|_\psi + (3j-1)\varepsilon_{i+1},\end{aligned} \tag{4.4}$$

since $\|{}^*Tv_{i+1}-v_{i+1}\|_\psi < \varepsilon_{i+1}$ and $\|v-v_{i+1}\|_\psi > 1-\varepsilon_{i+1} \ge \|\bar{v}-v_{i+1}\|_\psi - \varepsilon_{i+1}$.

Now let $u = \sum_{s=1}^t \lambda_s u_s$ for some $u_s \in D_j^{i+1} \cup {}^*T(D_j^{i+1}), \lambda_s \in {}^*[0,1], 1 \le s \le t \in {}^*\mathbb{N}$, $\sum_{s=1}^t \lambda_s = 1$. Then, by (4.3) and (**??**), there exist $\bar{v}_1,\dots,\bar{v}_t \in D_{j+1}^i$ such that

$$\|\bar{v}_s-u_s\|_\psi + \|u_s-v_{i+1}\|_\psi \le \|\bar{v}_s-v_{i+1}\|_\psi + (3j-1)\varepsilon_{i+1}, 1 \le s \le t.$$

Hence

$$\begin{aligned}\Big\|\sum\nolimits_{s=1}^t \lambda_s\bar{v}_s - u\Big\|_\psi + \|u-v_{i+1}\|_\psi &\le \sum\nolimits_{s=1}^t \lambda_s \|\bar{v}_s-v_{i+1}\|_\psi + (3j-1)\varepsilon_{i+1} \\ &\le 1+(3j-1)\varepsilon_{i+1} \\ &< \Big\|\sum\nolimits_{s=1}^t \lambda_s\bar{v}_s - v_{i+1}\Big\|_\psi + 3j\varepsilon_{i+1},\end{aligned}$$

since, by (iii), $\mathrm{dist}(D_\omega^i, v_{i+1}) > 1-\varepsilon_{i+1}$, and the claim is proved.

Step 4. Let $1 \le j \le \omega$, $i \in {}^*\mathbb{N}$ and $u=(a,b) \in D_j^i$. We claim that $\|b\| \simeq 0$. Fix $i \ge 2$. By Step 3, take $v=(x,y) \in D_j^{i-1}$ such that

$$\|v-u\|_\psi + \|u-v_i\|_\psi \le \|v-v_i\|_\psi + 3(j-1)\varepsilon_i < \|v-v_i\|_\psi + 3\omega\varepsilon_i.$$

Hence

$$\|(\|x-x_i\|, \|y-b\|+\|b-y_i\|)\|_\psi < \|(\|x-x_i\|, \|y-y_i\|)\|_\psi + 3\omega\varepsilon_i$$

which yields

$$\|y-b\|+\|b-y_i\| < \|y-y_i\| + \eta^i$$

since $3\omega\varepsilon_i \le \delta(\eta^i)$. Consequently,

$$\|b\| < \|y\| + \|y_i\| + \frac{1}{2}\eta^i.$$

Repeating this procedure $(i-1)$ times we obtain by internal induction an element $(\bar{x}, \bar{y}) \in D^1_\omega$ such that

$$\|b\| < \|\bar{y}\| + \|y_2\| + \frac{1}{2}\eta^2 + \cdots + \|y_i\| + \frac{1}{2}\eta^i.$$

Furthermore, it is not difficult to show that $\|\bar{y}\| < \omega\varepsilon_1$ (see [32, Lemma 3.1]). Hence $\|b\| < \omega\varepsilon_1 + (\varepsilon_2 + \cdots + \varepsilon_i) + \frac{1}{2}(\eta^2 + \cdots + \eta^i) < \eta + 2\eta^3 + \eta^2 \simeq 0$.

Step 5. Let $D_j = \bigcup_{i \in {}^*\mathbb{N}} D^i_j$ for $1 \leq j \leq \omega$. Then we can easily prove that $D_1 \subset D_2 \subset \cdots \subset D_\omega$ and ${}^*T(D_j) \subset D_{j+1}$ for $1 \leq j < \omega$. Moreover, a sequence $(v_n)_{n \in {}^*\mathbb{N}}$ intra-converges to $(0,0)$ and hence, by Lemma 2.5, ${}^\circ(0,0) \in {}^\circ D_1$. Let

$$D = \mathrm{cl}\Big(\bigcup_{j \in \mathbb{N}\setminus\{0\}} {}^\circ D_j\Big).$$

Notice that D is closed and convex subset of $\widetilde{K}$ which is invariant under $\widetilde{T}$. Moreover, ${}^\circ(0,0) \in D$ and consequently the set $M = D \cap \{{}^\circ({}^*x) : x \in K\}$ is nonempty, closed, convex and $\widetilde{T}$-invariant. It follows from Step 4 that $M \subset \{{}^\circ({}^*x) : x \in X\} \times \{0\}$ and therefore M is isometric to a subset of X. Since X has WFPP, $\widetilde{T}$ has a fixed point in M, which contradicts our assumption. □

Appendix A. Alpha-Theory

At present there exist several frameworks for nonstandard analysis. In this paper we use an axiomatic approach introduced in [3] (see also the related system *ZFC [7]). This approach is based on the existence of a new mathematical object α which can be seen as a new "ideal" number added to $\mathbb{N}$. Our exposition follows [17, Sect. 8.3d] (we do not assume the existence of atoms).

The Alpha-Theory is a theory in the language $\mathcal{L}' = \{\in, J\}$ of set theory extended by a new binary relation symbol J. The axioms include all of ZFC minus Regularity, together with the following five axioms:

J1. J is a function defined on the class of all sequences of arbitrary sets, i.e.,

$$\forall\varphi(\mathrm{Seq}(\varphi) \Rightarrow \exists! x J(\varphi, x)) \wedge \forall\varphi\forall x(J(\varphi, x) \Rightarrow \mathrm{Seq}(\varphi)),$$

where $\mathrm{Seq}(\varphi)$ means that φ is a sequence, i.e., a function with the domain $\mathbb{N}$.

Let $J(\varphi)$ be the unique x which satisfies $J(\varphi, x)$.

J2. If f is a function defined on a set A and $\varphi, \psi : \mathbb{N} \to A$, then $J(\varphi) = J(\psi)$ implies $J(f \circ \varphi) = J(f \circ \psi)$.

J3. $J(c_m) = m$ for any natural m, where $c_m(n) = m$ for all $n \in \mathbb{N}$, $J(\mathrm{id}) \notin \mathbb{N}$, where $\mathrm{id}(n) = n$ for all $n \in \mathbb{N}$.

J4. If $\vartheta(n) = \{\varphi(n), \psi(n)\}$ for all $n \in \mathbb{N}$, then $J(\vartheta) = \{J(\varphi), J(\psi)\}$.

J5. For any φ, $J(\varphi) = \{J(\psi) : \psi(n) \in \varphi(n)\}$ for all $n \in \mathbb{N}$.

Let us define ${}^*x = J(c_x)$ for any set x (where $c_x(n) = x$ for all $n \in \mathbb{N}$). Put

$$\alpha = J(\mathrm{id}).$$

By the axiom J3, $\alpha \notin \mathbb{N}$ and, by J5, $\alpha \in {}^*\mathbb{N}$. It turns out (see [3, Prop. 2.3]) that if $f : A \to B$, then ${}^*f$ is a function from ${}^*A$ to ${}^*B$ and ${}^*f(J(\varphi)) = J(f \circ \varphi)$ for any $\varphi : \mathbb{N} \to A$. Taking $\varphi = \mathrm{id}$ and $f = \varphi$ we obtain ${}^*\varphi(\alpha) = J(\varphi)$ for any sequence φ. Thus, J-extensions are simply values of the $*$-extended functions at a "non-standard natural number" α.

A set x is said to be internal if there exists a sequence φ such that $x = J(\varphi)$. Equivalently, x is internal if there exists y such that $x \in {}^*y$. A set x is external if it is not internal.

One of the fundamental tools in nonstandard analysis is the transfer principle which is an application of a famous theorem of Łoś. Recall that a formula σ is bounded if it is constructed from atomic formulae using connectives and bounded quantifiers $\forall x \in y$ (i.e., $\forall x\ x \in y \Rightarrow \dots$), $\exists x \in y$ (i.e., $\exists x\ x \in y \wedge \dots$). The following theorem (see [3, Th. 6.2], [17, Cor. 8.3.13]) shows that the transfer principle is satisfied in Alpha-Theory.

Theorem A.1. *For every bounded formula $\sigma(x_1, \dots, x_k)$ in the first-order language $\mathcal{L} = \{\in\}$ and for any sets $a_1, \dots, a_k$,*

$$\sigma(a_1, \dots, a_k) \Longleftrightarrow \sigma({}^*a_1, \dots, {}^*a_k).$$

The following useful theorem (known as the Internal Definition Principle) is a rather straightforward consequence of the transfer principle.

Theorem A.2. *If $\sigma(x, x_1, \dots, x_k)$ is a bounded formula in the first-order language $\mathcal{L} = \{\in\}$ and $b, b_1, \dots, b_k$ are internal sets, then $\{x \in b : \sigma(x, b_1, \dots, b_k)\}$ is an internal set.*

Another notion which is frequently used in nonstandard analysis is the so-called countable saturation.

Theorem A.3 (see [3, Th. 4.4])**.** *Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of internal sets with the finite intersection property. Then the intersection $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.*

Let $(\mathbb{R}, +, \cdot, \leq)$ be the complete ordered field of real numbers. Then, by transfer, we obtain an ordered field $({}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq)$. There is a common practice to omit "stars" when no confusion can arise. Notice that ${}^*\mathbb{R} = \{\varphi(\alpha) : \varphi : \mathbb{N} \to \mathbb{R}\}$ and hence $\{{}^*x : x \in \mathbb{R}\} \subset {}^*\mathbb{R}$. Although, in general, $x \neq {}^*x$ we do not usually distinguish between x and ${}^*x$ and regard the set of reals as a subset of ${}^*\mathbb{R}$. Elements of ${}^*\mathbb{R}$ are called hyperreals.

Definition A.4. A hyperreal number x is said to be

(i) bounded if $x = O(1)$, i.e., $|x| \leq c$ for some $c \in \mathbb{R}$,
(ii) infinitesimal if $x = o(1)$, i.e., $|x| \leq \varepsilon$ for every positive $\varepsilon \in \mathbb{R}$,
(iii) unbounded if $1/x = o(1)$.

Notice that $\alpha > n$ for every $n \in \mathbb{N}$ (see [3, Prop. 2.5]) and hence $1/\alpha$ is an example of a (nonzero) infinitesimal. We say that x and y are infinitely close, denoted by $x \simeq y$, if $x - y$ is infinitesimal. This defines an equivalence relation on ${}^*\mathbb{R}$ and the monad (or the halo) of x is the equivalence class

$$\mathrm{mon}(x) = \{y \in {}^*\mathbb{R} : x \simeq y\}.$$

We say that x and y are of bounded distance apart, denoted by $x \sim y$, if $x - y$ is bounded. The galaxy of x is the equivalence class

$$\mathrm{gal}(x) = \{y \in {}^*\mathbb{R} : x \sim y\}$$

(see [14, 23] and references therein). If a hyperreal x is bounded, i.e., $x \in \mathrm{gal}(0)$, the unique $a \in \mathbb{R}$ such that $x \simeq a$ is called the standard part of x and is denoted by $\mathrm{st}(x)$. These notions can be generalized in the following way. Let X be a real Banach space and let $^*X$ be its hyper-extension endowed with a function $^*\|\cdot\| : {}^*X \to {}^*\mathbb{R}$ called an internal norm (or $*$-norm) of $^*X$. By transfer, $^*\|\cdot\|$ is homogeneous over $^*\mathbb{R}$ and satisfies the triangle inequality. As before, we do not distinguish between x and $^*x$, and abbreviate $^*\|\cdot\|$ to $\|\cdot\|$. The monad of $x \in {}^*X$ is the equivalence class $\mathrm{mon}(^*X, x) = \{y \in {}^*X : \|x - y\| \simeq 0\}$ ($\mathrm{mon}(x)$ for brevity) and the galaxy of x is the equivalence class $\mathrm{gal}(^*X, x) = \{y \in {}^*X : \|x - y\| \sim 0\}$. The set $\mathrm{gal}(^*X, 0)$ is called the principal galaxy and denoted by $\mathrm{gal}(^*X)$. Let $\mathrm{mon}(X) = \bigcup_{x \in X} \mathrm{mon}(x)$. Notice that in general $\mathrm{mon}(X)$ is a proper subset of $\mathrm{gal}(^*X)$. If $x \in \mathrm{mon}(X)$, the unique $a \in X$ such that $\|x - a\| \simeq 0$ is called the standard part of x and is denoted, as in a real case, by $\mathrm{st}(x)$. We refer to $\mathrm{st} : \mathrm{mon}(X) \to X$ as the standard part mapping.

It was proved in [3, Th. 6.4], that ZFC is faithfully interpretable in the Alpha-Theory, i.e., a sentence σ in the language $\mathcal{L} = \{\in\}$ is a theorem of ZFC if and only if its relativization σ^{WF} to the class of well-founded sets is a theorem of the Alpha-Theory. In other words, the Alpha Theory proves those and only those statements ($\in$-statements, to be precise) about well-founded sets which ZFC proves about all sets.

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