Journal of Nonlinear and Convex Analysis
Volume 16, Number 3, 2015, 551–562

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SOME FIXED POINT THEOREMS FOR A PAIR TYPE OF BOGIN-POPESCU MAPPINGS IN COMPLETE METRIC SPACES*

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ABSTRACT. By using a concept of generalized commuting mappings, we study a new class pair of mappings. Some fixed point theorems and corresponding example are considered and discussed on such introduced class. The presented results in this work are generalizations and improvements of many important results, in the sense that we are providing more choices of tool implements to check whether a fixed point of considered mapping exists.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space. If $T : X \to X$ is a mapping, then an element $x \in X$ is called a *fixed point* of T if $x = Tx$. In 1922, Banach [2] considered a class of mappings, so-called *contraction mappings*, that is a mapping $T : X \to X$ such that we can find a real number $k \in [0, 1)$ such that

$$d(Tx, Ty) \le kd(x, y), \tag{1.1}$$

for all $x, y \in X$. He proved that if the metric space (X, d) is a complete and a considered mapping T satisfied the condition (1.1) then T must has a unique fixed point. A such result is well known and called the Banach contraction mapping principle. Evidently, the Banach contraction mapping principle are very useful and there exists a huge research on this line, where the contractive condition (1.1) is replaced by some more relaxed conditions. Ones may see [5, 11, 10] for a good monograph on this topic.

In 1976, Bogin[4] proved the following generalization of the Banach contraction mapping principle.

Theorem 1.1 ([4]). *Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \le ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \tag{1.2}$$

where $a \ge 0, b > 0, c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Recently, may be inspired by Theorem 1.2, Popescu[15] showed the following result.

2010 *Mathematics Subject Classification.* 37C25, 47H10.

Key words and phrases. Fixed point, common fixed point, coincidence point, commuting mappings, weakly compatible mappings.

*This work is partially supported by the National Research Council of Thailand (Project No. 2557A13702021).

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Theorem 1.2 ([15])**.** *Let (X,d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfies the following condition: for each $x, y \in X$ such that*

$$\frac{1}{2}\, d(x,Tx) \leq d(x,y)$$

implies

$$d(Tx,Ty) \leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)] \tag{1.3}$$

where $a \geq 0, b > 0, c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Remark 1.3. Obviously, Theorem 1.2 is a generalization of Theorem 1.1. In fact, in [15], the author also gave an example showing that the class of mappings that satisfies (1.3) is properly larger than that of condition (1.2).

On the other hand, the Banach contraction mapping principle was extended by Jungck[7], in 1976, as following.

Theorem 1.4 ([7])**.** *Let (X,d) be a complete metric space. Let S be a continuous self-map on X and T be any self-map on X that $STx = TSx$, for all $x \in X$. If S and T satisfy $T(X) \subset S(X)$ and there exists a constant $k \in [0,1)$ such that*

$$d(Tx,Ty) \leq kd(Sx,Sy),$$

for all $x, y \in X$. Then S and T have the unique common fixed point, that is, there exists the unique $x \in X$ such that $Tx = x = Sx$.

Notice that the two self mappings S and T such that $STx = TSx$ for all $x \in X$, is called *commuting mappings*. Observe that, in a special situation as S is the identity mapping then Theorem 1.4 is nothing but the Banach contraction principle. Recently, there are many fixed point theorems which are concerning to common fixed points of two mappings (or finite family of mappings), and there exist several concepts which are more general than the commuting class of mappings, see [1, 3, 6, 8, 9, 12, 14, 13] for examples.

Motivated by above literatures, we will introduce a new class of mappings. Some fixed point theorems on such introduced class will be given, by using a concept of generalized commuting mappings. Moreover, an interesting example will be provided and discussed. To do this, the following basic concepts are needed.

Definition 1.5. Let (X,d) be a metric space and $f, T : X \to X$. An element $x \in X$ is called a *coincidence point* of f and T if $fx = Tx$. We will denote the set of all coincidence points of f and T by $C(f,T)$.

Definition 1.6. Let (X,d) be a metric space and $f, T : X \to X$. Then f and T are said to be *weakly compatible* if for each $x \in C(f,T)$ we have $Tfx = fTx$.

Definition 1.7. Let (X,d) be a nonempty metric space and E be a nonempty subset of X. Let $T : E \to E$ and $f : E \to X$ be two mappings such that $T(E) \subset f(E)$. For any fixed $x_0 \in E$, a sequence $\{x_n\} = \{x_0, x_1, x_2, \dots\} \subset X$ such that $Tx_n = fx_{n+1}$ is called an *f-orbit of T at x_0*.

2. Main results

In this section, we start by introducing a new class of mappings, which can be viewed as a generalization of those mappings that considered in Theorem 1.2.

Definition 2.1. Let (X, d) be a metric space and E be a nonempty subset of X. A mapping $T : E \to E$ is called *f-Bogin-Popsecu type mapping*, if there exist a mapping $f : E \to X$ and real numbers $a \geq 0, b > 0, c > 0$ which $a + 2b + 2c = 1$, and for each $x, y \in X$ such that

$$\frac{1}{2}\, d(fx, Tx) \leq d(fx, fy)$$

implies

$$d(Tx, Ty) \leq M^f_{(a,b,c)}(x, y), \tag{2.1}$$

where $M^f_{(a,b,c)}(x, y) = ad(fx, fy)+b[d(fx, Tx)+d(fy, Ty)]+c[d(fx, Ty)+d(fy, Tx)]$.

Remark 2.2. If $f = I$, the identity map on E, then we see that the condition (2.1) is reduced to condition (1.3).

Next, we present some coincident point theorems for a subclass of Bogin-Popsecu type mappings.

Theorem 2.3. *Let (X, d) be a nonempty complete metric space and let E be a nonempty closed subset of X. Let $T : E \to E$ be a f-Bogin-Popescu type mapping such that $T(E) \subset f(E)$ and $f(E)$ be a closed subset of X. Then*

(a) *$C(f, T)$ is a nonempty set.*
(b) *$f(C(f, T))$ is a singleton set.*
(c) *for any $x_0 \in E$, there exists an f-orbit of T at x_0 and $u \in C(f, T)$ such that $fx_n \to fu$. Moreover,*

$$d(fx_n, fu) \leq \frac{h^n}{h(1-h)} d(fx_0, fx_1), \tag{2.2}$$

for all $n \geq 1$, where $h = \sqrt{a + 2b + c \max\left\{\frac{1+a}{1-b}, \frac{1+a}{1-c}\right\}}$.

Proof. Firstly, let us prove (b). Suppose that z and w are coincident points of f and T. We see that

$$\frac{1}{2}d(fz, Tz) = 0 \leq d(fz, fw).$$

Thus, by (2.1), it follows that

$$\begin{aligned} d(fz, fw) &= d(Tz, Tw) \\ &\leq ad(fz, fw) + b[d(fz, Tz) + d(fw, Tw)] + c[d(fz, Tw) + d(fw, Tz)] \\ &= ad(fz, fw) + b[d(fz, fz) + d(fw, fw)] + c[d(fz, fw) + d(fw, fz)] \\ &= (a + 2c)d(fz, fw). \end{aligned}$$

This implies that $fz = fw$, and (b) is proved.

Now, we prove (a) and (c). Let x_0 be an arbitrary element in E. Since $T(E) \subset f(E)$, it follows that there is $x_1 \in E$ such that $Tx_0 = fx_1$. Subsequently, since

$Tx_1 \in T(E) \subset f(E)$, there exists $x_2 \in E$ such that $Tx_1 = fx_2$. Similarly, since $Tx_2 \in T(E) \subset f(E)$, there exists $x_3 \in E$ such that $Tx_2 = fx_3$. By continuing this process, we obtain a sequence $\{fx_n\}_{n=1}^{\infty}$ and $\{Tx_n\}_{n=0}^{\infty}$ such that

$$Tx_n = fx_{n+1}, \quad n \geq 0.$$

From now on, for the sake of simplicity, we will denote the constructed sequence $\{Tx_n\}$ by $\{y_n\}$, that is

$$y_n := Tx_n = fx_{n+1}, \quad n \geq 0.$$

Next, we divide proof into four steps.

Step 1 We show $\{d(y_n, y_{n+1})\}_{n=0}^{\infty}$ is a nonincreasingness sequence of real numbers. Let n be a fixed natural number. By above constructive method, we see that

$$\frac{1}{2}d(fx_{2n}, Tx_{2n}) \leq d(fx_{2n}, fx_{2n+1}).$$

Thus, by (2.1), it follows that

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\leq ad(y_{2n-1}, y_{2n}) + b[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\quad + c[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})] \\ &\leq ad(y_{2n-1}, y_{2n}) + b[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\quad + c[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &= (a + b + c)d(y_{2n-1}, y_{2n}) + (b + c)d(y_{2n}, y_{2n+1}). \end{aligned}$$

This implies that

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\leq \frac{a + b + c}{1 - b - c} d(y_{2n-1}, y_{2n}) \\ &= d(y_{2n-1}, y_{2n}). \end{aligned}$$

That is,

$$d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) \tag{2.3}$$

Similarly, since

$$1/2d(fx_{2n+1}, Tx_{2n+1}) \leq d(fx_{2n+1}, Tx_{2n+1}) = d(fx_{2n+1}, fx_{2n+2}),$$

we know, by (2.1), that

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &\leq ad(y_{2n}, y_{2n+1}) + b[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] \\ &\quad + c[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})] \\ &\leq ad(y_{2n}, y_{2n+1}) + b[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] \\ &\quad + c[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] \\ &= (a + b + c)d(y_{2n}, y_{2n+1}) + (b + c)d(y_{2n+1}, y_{2n+2}). \end{aligned}$$

Thus,

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{a + b + c}{1 - b - c} d(y_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}).$$

This means,

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}) \tag{2.4}$$

Therefore, by (2.3) and (2.4), we conclude that

$$d(y_{n+1}, y_n) \leq d(y_n, y_{n-1}), \quad \text{for all} \quad n \geq 0.$$

This proves Step 1.

Step 2 We show that there exists a nonnegative real number $m < 2$ such that

$$d(y_{n+1}, y_{n+3}) \leq md(y_n, y_{n+1}), \quad \text{for all} \quad n \geq 0. \tag{2.5}$$

To do this, let us consider the following two possible cases.

Case (i) : Suppose that $d(Tx_n, Tx_{n+2}) \geq d(Tx_n, Tx_{n+1})$.

It follows that

$$1/2d(Tx_n, Tx_{n+1}) \leq d(Tx_n, Tx_{n+2}) = d(fx_{n+1}, fx_{n+3}).$$

Subsequently, by (2.1) and the nonincreasingness of $\{d(y_n, y_{n+1})\}$, we have

$$\begin{aligned} d(y_{n+1}, y_{n+3}) &\leq ad(y_n, y_{n+2}) + b[d(y_n, y_{n+1}) + d(y_{n+2}, y_{n+3})] \\ &\quad + c[d(y_n, y_{n+3}) + d(y_{n+2}, y_{n+1})] \\ &\leq a[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})] \\ &\quad + b[d(y_n, y_{n+1}) + d(y_{n+2}, y_{n+3})] \\ &\quad + c[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+3}) + d(y_{n+2}, y_{n+1})] \\ &= (a+b+c)d(y_n, y_{n+1}) + (a+c)d(y_{n+1}, y_{n+2}) \\ &\quad + bd(y_{n+2}, y_{n+3}) + cd(y_{n+1}, y_{n+3}). \\ &\leq (2a+2b+2c)d(y_n, y_{n+1}) + cd(y_{n+1}, y_{n+3}). \end{aligned}$$

This implies,

$$d(y_{n+1}, y_{n+3}) \leq \frac{a+2b+2c+a}{1-c} d(y_n, y_{n+1}) = \frac{1+a}{1-c} d(y_n, y_{n+1}).$$

Setting $m_1 := (1+a)/(1-c)$, we see that $m_1 < 2$ and

$$d(y_{n+1}, y_{n+3}) \leq m_1 d(y_n, y_{n+1}).$$

Case (ii): Suppose that $d(Tx_n, Tx_{n+2}) < d(Tx_n, Tx_{n+1})$.

It follows that

$$1/2d(Tx_n, Tx_{n+1}) \leq d(Tx_n, Tx_{n+1}) = d(fx_{n+1}, fx_{n+2}).$$

Thus, by (2.1), we have

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &= d(Tx_{n+1}, Tx_{n+2}) \\ &\leq ad(fx_{n+1}, fx_{n+2}) + b[d(fx_{n+1}, Tx_{n+1}) + d(fx_{n+2}, Tx_{n+2})] \\ &\quad + c[d(fx_{n+1}, Tx_{n+2}) + d(fx_{n+2}, Tx_{n+1})] \\ &= ad(Tx_n, Tx_{n+1}) + b[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2})] \\ &\quad + c[d(Tx_n, Tx_{n+2}) + d(Tx_{n+1}, Tx_{n+1})] \\ &\leq ad(Tx_n, Tx_{n+1}) + b[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2})] \\ &\quad + cd(Tx_n, Tx_{n+1}) \\ &= ad(y_n, y_{n+1}) + b[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})] + cd(y_n, y_{n+1}). \end{aligned}$$

This implies,

$$d(y_{n+1}, y_{n+2}) \le (a+b+c)d(y_n, y_{n+1}) + bd(y_{n+1}, y_{n+2}).$$

Hence,

$$d(y_{n+1}, y_{n+2}) \le \frac{a+b+c}{1-b} d(y_n, y_{n+1}). \tag{2.6}$$

Meanwhile, by the nonincreasingness of $\{d(y_n, y_{n+1})\}$, we observe that

$$d(y_{n+1}, y_{n+3}) \le d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) \le 2d(y_{n+1}, y_{n+2}).$$

Using this one together with (2.6), we have

$$d(y_{n+1}, y_{n+3}) \le \frac{2a+2b+2c}{1-b} d(y_n, y_{n+1}) = \frac{1+a}{1-b} d(y_n, y_{n+1}).$$

Setting $m_2 = (1+a)/(1-b)$, we have $m_2 < 2$ and

$$d(y_{n+1}, y_{n+3}) \le m_2 d(y_n, y_{n+1}).$$

Puting $m = \max\{m_1, m_2\}$, in view of Cases (i) and (ii), we see that $0 < m < 2$ and

$$d(y_{n+1}, y_{n+3}) \le md(y_n, y_{n+1}), \quad \text{for all} \quad n \ge 0.$$

This proves Step 2.

Step 3 We show that there exists a nonnegative number $k < 1$ such that

$$d(y_n, y_{n+1}) \le k^{\frac{n-1}{2}} d(y_0, y_1) \tag{2.7}$$

for all $n \ge 0$.

Since $\frac{1}{2}d(Tx_{n+1}, Tx_{n+2}) \le d(Tx_{n+1}, Tx_{n+2}) = d(fx_{n+2}, fx_{n+3})$, it follows by (2.1) and the nonincreasingness of $\{d(y_n, y_{n+1})\}$, that

$$\begin{aligned} d(y_{n+2}, y_{n+3}) &= d(Tx_{n+2}, Tx_{n+3}) \\ &\le ad(fx_{n+2}, fx_{n+3}) + b[d(fx_{n+2}, Tx_{n+2}) + d(fx_{n+3}, Tx_{n+3})] \\ &\quad + c[d(fx_{n+2}, Tx_{n+3}) + d(fx_{n+3}, Tx_{n+2})] \\ &= ad(y_{n+1}, y_{n+2}) + b[d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3})] \\ &+ c[d(y_{n+1}, y_{n+3}) + d(y_{n+2}, y_{n+2})] \\ &\le (a+2b)d(y_{n+1}, y_{n+2}) + cd(y_{n+1}, y_{n+3}). \end{aligned}$$

Subsequently, in light of (2.5), we obtain

$$\begin{aligned} d(y_{n+2}, y_{n+3}) &\le (a+2b)d(y_n, y_{n+1}) + mcd(y_n, y_{n+1}) \\ &= (a+2b+mc)d(y_n, y_{n+1}). \end{aligned} \tag{2.8}$$

Setting $k = a + 2b + mc$. Note that $k < 1$. Now, we will prove (2.7) by considering the following two cases:

If n is an even natural number and t is a natural number such that $n = 2t$. So, by (2.8), we have

$$\begin{aligned} d(y_n, y_{n+1}) = d(y_{2t}, y_{2t+1}) &\le kd(y_{2t-2}, y_{2t-1}) \\ &\le k^2 d(y_{2t-4}, y_{2t-3}) \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\leq k^{\frac{n}{2}} d(y_0, y_1) \\
&\leq k^{\frac{n-1}{2}} d(y_0, y_1). \qquad (2.9)
\end{aligned}
$$

If n is an odd natural number and s is a natural number such that $n = 2s + 1$. So, by (2.8) and the nonincreasingness of $\{d(y_n, y_{n+1})\}$, we have

$$
\begin{aligned}
d(y_n, y_{n+1}) = d(y_{(2s-1)+2}, y_{(2s-1)+3}) &\leq kd(y_{2s-1}, y_{2s}) \\
&= kd(y_{(2s-3)+2}, y_{(2s-3)+3}) \\
&\leq k^2 d(y_{2s-3}, y_{2s-2}) \\
&= k^2 d(y_{(2s-5)+2}, y_{(2s-5)+3}) \\
&\vdots \\
&\leq k^{\frac{n-1}{2}} d(y_1, y_2) \\
&\leq k^{\frac{n-1}{2}} d(y_0, y_1). \qquad (2.10)
\end{aligned}
$$

Hence, from (2.9) and (2.10), we obtain (2.7).

Step 4 We show that $C(f,T)$ is a nonempty set, and moreover, the sequence $\{y_n\}_{n=0}^{\infty}$ converges to z, for some $z \in f(C(f,T))$.

Let n be a fixed natural number. For each natural number m such that $m > n$, we have

$$
\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\
&\leq \frac{(k^n)^{1/2} d(y_0, y_1)}{k^{1/2}} + \frac{(k^{n+1})^{1/2} d(y_0, y_1)}{k^{1/2}} + ... + \frac{(k^{m-1})^{1/2} d(y_0, y_1)}{k^{1/2}} \\
&= \frac{1}{k^{1/2}} d(y_0, y_1)[(k^n)^{1/2} + (k^{n+1})^{1/2} + (k^{n+2})^{1/2} + ... + (k^{m-1})^{1/2}] \\
&= \frac{1}{k^{1/2}} d(y_0, y_1)(k^n)^{1/2}[1 + k^{1/2} + k^{2/2} + ... + (k^{m-1-n})^{1/2}] \\
&\leq \frac{1}{k^{1/2}} d(y_0, y_1)(k^n)^{1/2} \left[\frac{1}{1 - k^{1/2}}\right]. \qquad (2.11)
\end{aligned}
$$

Since $k \in (0,1)$, we conclude that $\{y_n\}$ is a Cauchy sequence in $f(E)$. Thus, by the completeness of $f(E)$, we know that there is $u \in E$ such that $\{y_n\}$ converges to $f(u)$ as $n \to \infty$.

Finally, we now show that $u \in C(T,f)$. In order to complete the proof of this one, we first show that

$$
d(fu, y_n) \geq 1/2 d(y_n, y_{n+1}) \text{ or } d(fu, y_{n+1}) \geq 1/2 d(y_{n+1}, y_{n+2}), \qquad (2.12)
$$

for all $n \geq 0$. Suppose, on the contrary, that there exists a natural number n such that

$$
d(fu, y_n) < 1/2 d(y_n, y_{n+1}) \text{ and } d(fu, y_{n+1}) < 1/2 d(y_{n+1}, y_{n+2}).
$$

It would follow that

$$
d(y_n, y_{n+1}) \leq d(fu, y_n) + d(fu, y_{n+1})
$$

$$
\begin{aligned}
&< \frac{1}{2}\left[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})\right] \\
&\leq d(y_n, y_{n+1}),
\end{aligned}
$$

which is a contradiction. Thus, (2.12) is true. Subsequently, by (2.12), we can find a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$d(fu, y_{n_j}) \geq 1/2 d(y_{n_j}, y_{n_j+1})$$

for all $j \geq 0$. Then, by (2.1), we have

$$
\begin{aligned}
d(Tu, y_{n_j+1}) &\leq ad(fu, y_{n_j}) + b[d(fu, Tu) + d(y_{n_j}, y_{n_j+1})] \\
&\quad + c[d(fu, y_{n_j+1}) + d(y_{n_j}, Tu)].
\end{aligned}
$$

Letting $j \to \infty$, we obtain $d(Tu, fu) \leq (b+c)d(Tu, fu)$. Since $b + c \in (0, 1)$, this implies that $d(Tu, fu) = 0$. Hence, $u \in C(T, f)$ and the proof is complete. □

As special case of Theorem 2.3, we have a result which presented by Popescu in [15].

Theorem 2.4 ([15]). *Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping such that the condition (1.3) is satisfied, then T has a unique fixed point.*

Proof. Observe that if T satisfies the condition (1.3) then it is I-Bogin-Popescu type mapping, where I is the identity mapping on X. Subsequently, by Theorem 2.3, we know that $C(I, T)$ is a singleton set. By using this fact, the required result is easily obtained. □

Next, we give a theorem of a unique fixed point.

Theorem 2.5. *Assume that all conditions of Theorem 2.3 are satisfied. If, in addition, f and T are weakly compatible then f and T have a unique common fixed point.*

Proof. Notice that, since $f(C(f, T))$ is a singleton set, if the set of common fixed point of f and T is a nonempty set then it must be a singleton set. This means that f and T have a unique common fixed point.

Now, we show that f and T have a common fixed point. As proving the Theorem 2.3, we know that there are $u \in C(T, f)$ and $z \in f(E)$ such that $Tu = fu = z$. Thus, since f and T are weakly compatible, it follows that

$$Tz = Tfu = fTu = fz. \tag{2.13}$$

Considering this element z, by following the lines of proof proving (2.12), we know that

$$d(fz, y_n) \geq 1/2 d(y_n, y_{n+1}) \quad \text{or} \quad d(fz, y_{n+1}) \geq 1/2 d(y_{n+1}, y_{n+2}),$$

for all $n \geq 0$. Subsequently, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $d(fz, y_{n_k}) \geq 1/2 d(y_{n_k}, y_{n_k+1})$ for every integer $k \geq 0$. Then, by (2.1), we have

$$
\begin{aligned}
d(Tz, y_{n_k+1}) &\leq ad(fz, y_{n_k}) + b[d(fz, Tz) + d(y_{n_k}, y_{n_k+1})] \\
&\quad + c[d(fz, y_{n_k+1}) + d(y_{n_k}, Tz)] \\
&= ad(Tz, y_{n_k}) + b[d(Tz, Tz) + d(y_{n_k}, y_{n_k+1})]
\end{aligned}
$$

$$+ c[d(Tz, y_{n_k+1}) + d(y_{n_k}, Tz)].$$

Taking $k \to \infty$, we obtain $d(Tz, z) \leq (a + 2c)d(Tz, z)$. This implies that $Tz = z$. Further, by (2.13), we have $fz = Tz = z$. This means that z is a common fixed point of T and f, and the proof is complete. □

Now, we provide an example which shows that Theorem 2.5 is a genuine generalization of Theorem 2.4.

Example 2.6. *Let $X = (-\infty, \infty)$ with the usual metric and let $T : X \to X$ be a mapping defined by*

$$Tx = \begin{cases} \frac{-x}{4} & \text{if } x \in [0, \infty) =: A, \\ \frac{x}{2} & \text{if } x \in (-\infty, \frac{-1}{4}) \cup (\frac{-1}{4}, 0) =: B, \\ \frac{-1}{2} & \text{if } x = \frac{-1}{4} =: C. \end{cases}$$

Then,

(1) *T does not satisfy condition (1.3).*

(2) *T is a f-Bogin-Popescu type mapping, where $f : X \to X$ is defined by*

$$fx = \begin{cases} \frac{-x}{2} & \text{if } x \in [0, \infty), \\ x & \text{if } x \in (\infty, \frac{-1}{4}) \cup (\frac{-1}{4}, 0), \\ -1 & \text{if } x = \frac{-1}{4}, \end{cases}$$

and $a = 1/2, b = c = 1/8$.

(3) *T and f have a unique common fixed point.*

Proof. (1) Let us consider when $x = \frac{-1}{4}$, $y = \frac{-1}{2}$. We see that

$$\frac{1}{2}d(x, Tx) = \frac{1}{2}\left|\frac{-1}{4} + \frac{1}{2}\right| \leq \left|\frac{-1}{4} + \frac{1}{2}\right| = d(x, y).$$

If T satisfies the condition (1.3), it would follow that there is $(a, b, c) \in [0, \infty) \times (0, \infty) \times (0, \infty)$ such that

$$\begin{aligned}\left|\frac{-1}{2} + \frac{1}{4}\right| &= d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \\ &= a\left|\frac{-1}{4} + \frac{1}{2}\right| + b\left[\left|\frac{-1}{4} + \frac{1}{2}\right| + \left|\frac{-1}{2} + \frac{1}{4}\right|\right] + c\left[\left|\frac{-1}{4} + \frac{1}{4}\right| + \left|\frac{-1}{2} + \frac{1}{2}\right|\right] \\ &= \frac{1}{4}a + \frac{1}{2}b.\end{aligned}$$

This implies that $1 \leq a + 2b$. On the other hand, since $a + 2b = 1 - 2c$ and $c > 0$, then $a + 2b < 1$. These lead to a contradiction, and so (1) is showed.

(2) We have to consider the following essential five cases.

Case 1: Let $x, y \in A$. We have

$$\begin{aligned}M^f_{(\frac{1}{2},\frac{1}{8},\frac{1}{8})}(x, y) &= \frac{1}{2}\left|\frac{-x}{2} + \frac{y}{2}\right| + \frac{1}{8}\left[\left|\frac{-x}{2} + \frac{x}{4}\right| + \left|\frac{-y}{2} + \frac{y}{4}\right| + \left|\frac{-x}{2} + \frac{y}{4}\right| + \left|\frac{-y}{2} + \frac{x}{4}\right|\right] \\ &= \frac{1}{4}|x - y| + \frac{1}{8}\left[\frac{x}{4} + \frac{y}{4} + \left|\frac{y}{4} - \frac{x}{2}\right| + \left|\frac{x}{4} - \frac{y}{2}\right|\right],\end{aligned}$$

and

$$d(Tx,Ty) = \left|\frac{-x}{4} + \frac{y}{4}\right| = \frac{1}{4}\left|x - y\right|.$$

Case 2: Let $x \in A, y \in B$. We have

$$\begin{aligned} M^f_{(\frac{1}{2},\frac{1}{8},\frac{1}{8})}(x,y) &= \frac{1}{2}\left|\frac{-x}{2} - y\right| + \frac{1}{8}\left[\left|\frac{-x}{2} + \frac{x}{4}\right| + \left|y - \frac{y}{2}\right| + \left|\frac{-x}{2} - \frac{y}{2}\right| + \left|y + \frac{x}{2}\right|\right] \\ &= \frac{1}{2}\left|\frac{x}{2} + y\right| + \frac{1}{8}\left[\frac{x}{4} - \frac{y}{2} + \left|\frac{x}{2} + \frac{y}{2}\right| + \left|y + \frac{x}{2}\right|\right], \end{aligned}$$

and

$$d(Tx,Ty) = \left|\frac{-x}{4} - \frac{y}{2}\right| = \frac{1}{2}\left|\frac{x}{2} + y\right|.$$

Case 3: Let $x \in A, y = \frac{-1}{4}$. We have

$$\begin{aligned} M^f_{(\frac{1}{2},\frac{1}{8},\frac{1}{8})}(x,y) &= \frac{1}{2}\left|\frac{-x}{2} + 1\right| + \frac{1}{8}\left[\left|\frac{-x}{2} + \frac{x}{4}\right| + \left|-1 - \frac{1}{2}\right| + \left|\frac{-x}{2} + \frac{1}{2}\right| + \left|-1 + \frac{x}{4}\right|\right] \\ &= \frac{1}{2}\left|1 - \frac{x}{2}\right| + \frac{1}{8}\left[\frac{x}{4} + \frac{1}{2} + \left|\frac{1}{2} - \frac{x}{2}\right| + \left|\frac{x}{4} - 1\right|\right], \end{aligned}$$

and

$$d(Tx,Ty) = \left|\frac{-x}{4} + \frac{1}{2}\right| = \frac{1}{2}\left|1 - \frac{x}{2}\right|.$$

Case 4: Let $x \in B, y \in A$. We have

$$\begin{aligned} M^f_{(\frac{1}{2},\frac{1}{8},\frac{1}{8})}(x,y) &= \frac{1}{2}\left|x + \frac{y}{2}\right| + \frac{1}{8}\left[\left|x - \frac{x}{2}\right| + \left|\frac{-y}{2} + \frac{y}{4}\right| + \left|x + \frac{y}{4}\right| + \left|\frac{-y}{2} - \frac{x}{2}\right|\right] \\ &= \frac{1}{2}\left|x + \frac{y}{2}\right| + \frac{1}{8}\left[\frac{-x}{2} + \frac{y}{4} + \left|x + \frac{y}{4}\right| + \left|\frac{y}{2} + \frac{x}{2}\right|\right], \end{aligned}$$

and

$$d(Tx,Ty) = \left|\frac{x}{2} + \frac{y}{4}\right| = \frac{1}{2}\left|x + \frac{y}{2}\right|.$$

Case 5: Let $x \in B, y = \frac{-1}{4}$. We have

$$\begin{aligned} M^f_{(\frac{1}{2},\frac{1}{8},\frac{1}{8})}(x,y) &= \frac{1}{2}|x + 1| + \frac{1}{8}\left[\left|x - \frac{x}{2}\right| + \left|-1 + \frac{1}{2}\right| + \left|-x + \frac{1}{2}\right| + \left|-1 - \frac{x}{2}\right|\right] \\ &= \frac{1}{2}|x + 1| + \frac{1}{8}\left[-\frac{x}{2} - \frac{1}{2} + \left|x + \frac{1}{2}\right| + \left|\frac{y}{2} + 1\right|\right], \end{aligned}$$

and

$$d(Tx,Ty) = \left|\frac{x}{2} + \frac{1}{2}\right| = \frac{1}{2}|x + 1|.$$

Thus, from all above cases, we see that T is a f-Bogin-Popescu type mapping. This proves (2).

(3) We will show that 0 is the unique common fixed point of f and T.

If $x \in [0, \infty)$. Then $Tx = \frac{-x}{4}$ and $fx = \frac{-x}{2}$.

We see that

$$\begin{aligned} Tx = x = fx &\Leftrightarrow -\frac{x}{4} = x = -\frac{x}{2} \\ &\Leftrightarrow x = 0. \end{aligned}$$

This means 0 is the unique common fixed point of f and T on $[0, 1]$.

If $x \in [-\infty, \frac{-1}{4}) \cup (\frac{-1}{4}, 0)$. Then $Tx = \frac{x}{2}$ and $fx = x$.

Consider

$$\begin{aligned} Tx = x = fx &\Leftrightarrow \frac{x}{2} = x \\ &\Leftrightarrow x = 0. \end{aligned}$$

However, since $0 \notin [-\infty, \frac{-1}{4}) \cup (\frac{-1}{4}, 0)$, we conclude that f and T have no common fixed point on $[-\infty, \frac{-1}{4}) \cup (\frac{-1}{4}, 0)$.

If $x = \frac{-1}{4}$. It is obvious that x is not a common fixed point of f and T.

Hence 0 is the unique common fixed point of T and f on X. □

3. Conclusion

In this paper, we introduce a new class of mappings and prove some fixed point theorems in a complete metric spaces setting. Subsequently, as shown by Remark 2.2 and Example 2.6, our results is a genuine generalization of some important existing results that have been presented in [4, 15]. Indeed, we would like to point out that the usefulness of our presented results are providing more choices of tool implements to check whether a fixed point of considered mapping exists. At this point, we also desire that the results presented here will be useful for further research works, because this paper may be extended to many classes of mappings such as weakly commuting mappings; R-weakly commuting, C_q-commuting, and many concepts.

References

[1] M. Abbas, Y. J. Cho and T. Nazir, *Common fixed points of Ćirić-type contractive mappings in two ordered generalized metric spaces*, Fixed Point Theory and Applications 2012, 2012:139.

[2] S. Banach, *Surles opérations dans les ensembles abstraits et leurs applications aux êquations intégrales*, Fund. Math. **3** (1922), 133–181.

[3] V. Berinde, *A common fixed point theorem for compatible quasi contractive self mappings in metric spaces*, Appl. Math. and Comp. **213** (2009), 348–354.

[4] J. Bogin, *A generalization of a fixed point theorem of Goebel Kirk and Shimi*, Canad. Math. Bull. **19** (1976), 7–12.

[5] L. B. Ćirić and B. Fisher, *Fixed Point Theory: contraction mapping principle*, Faculty of Mechanical Engineering, (2003).

[6] L. B. Ćirić, *Common fixed point theorems for a family of non-self mappings in convex metric spaces*, Nonlinear Anal. **71** (2009), 1662–1669.

[7] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly **83** (1976), 261–263.

[8] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. and Math. Sci. **9** (1986), 771–779.

[9] G. Jungck, *Common fixed points for commuting and compatible maps on compacta*, Proc. Amer. Math. Soc. **103** (1988), 977–983.

[10] W. A. Kirk and M. A. Khamsi, *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley, New York, 2001.

[11] W. A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.

[12] V. Lakshmikanthama and L. B. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), 4341–4349.

[13] H. K. Pathak, Y. J. Cho and S. M. Kang and B. S. Lee, *Fixed point theorems for compatible mappings of type (P) and application to dynamic programming*, Le Matematiche (Fasc. I) **50** (1995), 15–33.

[14] H. K. Pathak, Y. J. Cho and S. M. Kang, *Remarks on R-weakly commuting mappings*, Bull. Korean Math. Soc. **34** (1997), 247–257.

[15] O. Popescu, *Two generalization of some fixed point theorem*, Math. Anal. **62** (2011), 3912–3919.

Manuscript received September 24, 2013
revised March 10, 2014

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