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ITERATIVE METHODS FOR THE SPLIT FEASIBILITY PROBLEM IN BANACH SPACES

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Abstract. In this paper, we consider the split feasibility problem in Banach spaces. Then using the idea of Mann's iteration, we first prove a weak convergence theorem for finding a solution of the split feasibility problem in Banach spaces. Furthermore, using the idea of Halpern's iteration, we obtain a strong convergence theorem for finding a solution of the problem in Banach spaces. It seems that these results are first in Banach spaces.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\kappa > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \kappa \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [5] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [4] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [4] is to find a point $z \in H_1$ such that

$$z \in \left(\cap_{i=1}^m A_i^{-1}0 \right) \cap \left(\cap_{j=1}^n T_j^{-1}(B_j^{-1}0) \right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility peoblem and generalized split feasibility peoblems including the split common null point problem in Hilbert spaces; see , for instance, [4, 6, 10, 16].

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On the other hand, in 1953, Mann [9] introduced the following iteration process. Let C be a nonempty, closed and convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. In 1967, Halpern [7] also gave an iteration process as follows: Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations of iterative processes for finding fixed points of nonexpansive mappings.

In this paper, motivated by these problems and results, we consider the split feasibility problem in Banach spaces. Then using the idea of Mann's iteration, we first prove a weak convergence theorem for finding a solution of the split feasibility problem in Banach spaces. Furthermore, using the idea of Halpern's iteration, we obtain a strong convergence theorem for finding a solution of the problem in Banach spaces. It seems that these results are first in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot\, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [14] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [12]. The following result was proved by Takahashi and Toyoda [15].

Lemma 2.1 ([15])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $n \in \mathbb{N}$ and $u \in C$, then $\{P_C x_n\}$ converges strongly to some $z \in C$, where P_C is the metric projection on H onto C.*

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [12] and [13]. We know the following result.

Lemma 2.2 ([12])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.3 ([12])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \ge 0, \quad \forall y \in C$.

We also know the following lemmas:

Lemma 2.4 ([2], [17])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative* *real numbers with* $\sum_{n=1}^{\infty}\beta_n < \infty$, *and let* $\{\gamma_n\}$ *be a sequence of real numbers with* $\limsup_{n\to\infty}\gamma_n \le 0$. *Suppose that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. *Then* $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([8]). *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Define the sequence* $\{\tau(n)\}_{n\ge n_0}$ *of integers as follows:*

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ *such that* $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. *Then, the following hold:*

(i) $\tau(n_0) \le \tau(n_0+1) \le \cdots$ *and* $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \le \Gamma_{\tau(n)+1}$, $\forall n \ge n_0$.

3. Weak convergence theorem

In this section, we prove a weak convergence theorem of Mann's type iteration for the split feasibility problem in Banach spaces.

Theorem 3.1. *Let* H *be a Hilbert space and let* F *be a strictly convex, reflexive and smooth Banach space. Let* J_F *be the duality mapping on* F. *Let* C *and* D *be nonempty, closed and convex subsets of* H *and* F, *respectively. Let* P_C *and* P_D *be the metric projections of* H *onto* C *and* F *onto* D, *respectively. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $C \cap A^{-1}D \neq \emptyset$. *For any* $x_1 = x \in H$, *define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C\big(I - rA^*J_F(A - P_DA)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ *and* $r \in (0,\infty)$ *satisfy the following:*

$$0 < a \le \beta_n \le b < 1 \quad \text{and} \quad 0 < r\|A\|^2 < 2$$

for some $a, b \in \mathbb{R}$. *Then* $x_n \rightharpoonup z_0 \in C \cap A^{-1}D$, *where* $z_0 = \lim_{n\to\infty} P_{C\cap A^{-1}D}x_n$.

Proof. Let $z \in C \cap A^{-1}D$. Then we have that $z = P_Cz$ and $Az - P_DAz = 0$. Put $y_n = P_C\big(x_n - rA^*J_F(Ax_n - P_DAx_n)\big)$ for all $n \in \mathbb{N}$. Since P_C is nonexpansive, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \big\|P_C\big(x_n - rA^*J_F(Ax_n - P_DAx_n)\big) - P_Cz\big\|^2 \\
&\le \|x_n - rA^*J_F(Ax_n - P_DAx_n) - z\|^2 \\
&= \|x_n - z - rA^*J_F(Ax_n - P_DAx_n)\|^2 \\
&= \|x_n - z\|^2 - 2\langle x_n - z, rA^*J_F(Ax_n - P_DAx_n)\rangle \\
&\quad + \|rA^*J_F(Ax_n - P_DAx_n)\|^2 \\
&\le \|x_n - z\|^2 - 2r\langle Ax_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\
&\quad + r^2\|A\|^2\|J_F(Ax_n - P_DAx_n)\|^2 \qquad (3.1)\\
&= \|x_n - z\|^2 - 2r\langle Ax_n - P_DAx_n + P_DAx_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\
&\quad + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2
\end{aligned}$$

$$= \|x_n - z\|^2 - 2r\|Ax_n - P_D Ax_n\|^2$$
$$- 2r\langle P_D Ax_n - Az, J_F(Ax_n - P_D Ax_n)\rangle + r^2\|A\|^2\|Ax_n - P_D Ax_n\|^2$$
$$\leq \|x_n - z\|^2 - 2r\|Ax_n - P_D Ax_n\|^2 + r^2\|A\|^2\|Ax_n - P_D Ax_n\|^2$$
$$= \|x_n - z\|^2 + r(r\|A\|^2 - 2)\|Ax_n - P_D Ax_n\|^2.$$

From $0 < r\|A\|^2 < 2$ we have that $\|y_n - z\| \leq \|x_n - z\|$ for all $n \in \mathbb{N}$ and hence

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n x_n + (1-\beta_n)y_n - z\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|y_n - z\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\ &\leq \|x_n - z\|.\end{aligned}$$

Then $\lim_{n\to\infty} \|x_n - z\|$ exists. Thus $\{x_n\}$, $\{Ax_n\}$ and $\{y_n\}$ are bounded. Using the equality (2.2), we have that for $n \in \mathbb{N}$ and $z \in C \cap A^{-1}D$

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\beta_n x_n + (1-\beta_n)y_n - z\|^2 \\ &= \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1-\beta_n)\|x_n - z\|^2 \\ &\quad + (1-\beta_n)r(r\|A\|^2 - 2)\|Ax_n - P_D Ax_n\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\ &= \|x_n - z\|^2 + (1-\beta_n)r(r\|A\|^2 - 2)\|Ax_n - P_D Ax_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - y_n\|^2.\end{aligned}$$

Therefore, we have that $\beta_n(1-\beta_n)\|x_n - y_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2$ and

$$(1-\beta_n)r(2 - r\|A\|^2)\|Ax_n - P_D Ax_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Thus we have from $0 < a \leq \beta_n \leq b < 1$ that

$$\lim_{n\to\infty} \|x_n - y_n\|^2 = 0 \text{ and } \lim_{n\to\infty} \|Ax_n - P_D Ax_n\|^2 = 0. \tag{3.2}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. From (3.2) we have that $\{y_{n_i}\}$ converges weakly to w. Since $y_n \in C$, we have $w \in C$. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using this and $\lim_{n\to\infty} \|Ax_n - P_D Ax_n\| = 0$, we have that $P_D Ax_{n_i} \rightharpoonup Aw$. Since P_D is the metric projection of F onto D, we have from [3] and [14] that $\langle P_D Ax_n - P_D Aw, J_F(Ax_n - P_D Ax_n)\rangle \geq 0$ and

$$\langle P_D Aw - P_D Ax_n, J_F(Aw - P_D Aw)\rangle \geq 0$$

and hence

$$\langle P_D Ax_n - P_D Aw, J_F(Ax_n - P_D Ax_n) - J_F(Aw - P_D Aw)\rangle \geq 0.$$

Since $P_D Ax_{n_i} \rightharpoonup Aw$ and $\|J_F(Ax_n - P_D Ax_n)\| \to 0$, we have that

$$-\|Aw - P_D Aw\|^2 = \langle Aw - P_D Aw, -J_F(Aw - P_D Aw)\rangle \geq 0$$

and hence $Aw = P_D Aw$. This implies that $w \in C \cap A^{-1}D$. We next show that if $x_{n_i} \rightharpoonup x^*$ and $x_{n_j} \rightharpoonup y^*$, then $x^* = y^*$. We know $x^*, y^* \in C \cap A^{-1}D$ and hence

$\lim_{n\to\infty}\|x_n - x^*\|$ and $\lim_{n\to\infty}\|x_n - y^*\|$ exist. Suppose $x^* \neq y^*$. Since H satisfies Opial's condition, we have that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x^*\| &= \lim_{i\to\infty}\|x_{n_i} - x^*\| < \lim_{i\to\infty}\|x_{n_i} - y^*\| \\ &= \lim_{n\to\infty}\|x_n - y^*\| = \lim_{j\to\infty}\|x_{n_j} - y^*\| \\ &< \lim_{j\to\infty}\|x_{n_j} - x^*\| = \lim_{n\to\infty}\|x_n - x^*\|.\end{aligned}$$

This is a contradiction. Then we have $x^* = y^*$. Therefore, $x_n \rightharpoonup x^* \in C \cap A^{-1}D$. Moreover, since for any $z \in C \cap A^{-1}D$

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad \forall n \in \mathbb{N},$$

we have from Lemma 2.1 that $P_{C\cap A^{-1}D}x_n \to z_0$ for some $z_0 \in C \cap A^{-1}D$. The property of metric projection implies that

$$\langle x^* - P_{C\cap A^{-1}D}x_n, x_n - P_{C\cap A^{-1}D}x_n\rangle \leq 0.$$

Therefore, we have

$$\|x^* - z_0\|^2 = \langle x^* - z_0, x^* - z_0\rangle \leq 0.$$

This means that $x^* = z_0$, i.e., $x_n \rightharpoonup z_0$. □

4. Strong convergence theorem

In this section, we prove a strong convergence theorem of Halpern's type iteration for the split feasibility problem in Banach spaces.

Theorem 4.1. *Let H be a Hilbert space and let F be a strictly convex, reflexive and smooth Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)P_C(I - rA^*J_F(I - P_D)A)x_n)$$

for all $n \in \mathbb{N}$, where $r \in (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < r\|A\|^2 < 2, \quad \lim_{n\to\infty}\alpha_n = 0,$$

$$\sum_{n=1}^{\infty}\alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

where $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, for some $z_0 = P_{C\cap A^{-1}D}u$.

Proof. Put $z_n = P_C(I - rA^*J_F(I - P_D)A)x_n$ for all $n \in \mathbb{N}$. Let $z \in C \cap A^{-1}D$. We have that $z = P_C z$ and $Az - P_D Az = 0$. As in the proof of Theorem 3.1, we have that

$$\begin{aligned}\|z_n - z\|^2 &= \|P_C(I - rA^*J_F(I - P_D)A)x_n - P_C z\|^2 \\ &\leq \|x_n - rA^*J_F(I - P_D)Ax_n - z\|^2\end{aligned}$$

$$\begin{aligned}(4.1)\qquad &\le \|x_n - z\|^2 - 2r\|Ax_n - P_D Ax_n\|^2 - 2r\langle P_D Ax_n - Az, J_F(I - P_D)Ax_n\rangle \\ &\quad + (r)^2\|A\|^2 \|(I - P_D)Ax_n\|^2 \\ &\le \|x_n - z\|^2 - 2r\|Ax_n - P_D Ax_n\|^2 + (r)^2\|A\|^2 \|(I - P_D)Ax_n\|^2 \\ &= \|x_n - z\|^2 + r(r\|A\|^2 - 2)\|(I - P_D)Ax_n\|^2 .\end{aligned}$$

From $0 < r\|A\|^2 < 2$ we have that $\|z_n - z\| \le \|x_n - z\|$ for all $n \in \mathbb{N}$. Put $y_n = \alpha_n u_n + (1 - \alpha_n)P_C(x_n - rA^* J_F(I - P_D)Ax_n)$. We have that

$$\begin{aligned}\|y_n - z\| &= \|\alpha_n(u_n - z) + (1 - \alpha_n)(z_n - z)\| \\ &\le \alpha_n\|u_n - z\| + (1 - \alpha_n)\|z_n - z\| \\ &\le \alpha_n \|u_n - z\| + (1 - \alpha_n)\|x_n - z\| .\end{aligned}$$

Using this, we get that

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1 - \beta_n)(y_n - z)\| \\ &\le \beta_n \|x_n - z\| + (1 - \beta_n)\|y_n - z\| \\ &\le \beta_n \|x_n - z\| + (1 - \beta_n)(\alpha_n \|u_n - z\| + (1 - \alpha_n)\|x_n - z\|) \\ &= (1 - \alpha_n(1 - \beta_n))\|x_n - z\| + \alpha_n(1 - \beta_n)\|u_n - z\|.\end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \le M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_k - z\| \le K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}\|x_{k+1} - z\| &\le (1 - \alpha_k(1 - \beta_k))\|x_k - z\| + \alpha_k(1 - \beta_k)\|u_k - z\| \\ &\le (1 - \alpha_k(1 - \beta_k))K + \alpha_k(1 - \beta_k)K = K.\end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. Take $z_0 = P_{C\cap A^{-1}D}u$. Since $z_n = P_C(I - rA^* J_F(I - P_D)A)x_n$, from the definition of $\{x_n\}$ we have that

$$x_{n+1} - x_n = \beta_n x_n + (1 - \beta_n)\{\alpha_n u_n + (1 - \alpha_n)z_n\} - x_n$$

and hence

$$\begin{aligned}x_{n+1} - x_n - &(1 - \beta_n)\alpha_n u_n \\ &= \beta_n x_n + (1 - \beta_n)(1 - \alpha_n)z_n - x_n \\ &= (1 - \beta_n)\{(1 - \alpha_n)z_n - x_n\} \\ &= (1 - \beta_n)\{z_n - x_n - \alpha_n z_n\}.\end{aligned}$$

Thus we have that

$$\begin{aligned}&\langle x_{n+1} - x_n - (1 - \beta_n)\alpha_n u_n, x_n - z_0\rangle \\ (4.2)\qquad &= (1 - \beta_n)\langle z_n - x_n, x_n - z_0\rangle - (1 - \beta_n)\langle \alpha_n z_n, x_n - z_0\rangle \\ &= -(1 - \beta_n)\langle x_n - z_n, x_n - z_0\rangle - (1 - \beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.\end{aligned}$$

From (2.3) and (4.1), we have that

$$\begin{aligned}&2\langle x_n - z_n, x_n - z_0\rangle \\ (4.3)\qquad &= \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|z_n - z_0\|^2\end{aligned}$$

$$\geq \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|x_n - z_0\|^2$$
$$= \|z_n - x_n\|^2.$$

From (4.2) and (4.3), we have that

$$\begin{aligned}
2\langle x_{n+1}-x_n, x_n - z_0\rangle & \\
&= 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\quad - 2(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \\
&\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.
\end{aligned} \tag{4.4}$$

Furthermore, using (2.3) and (4.4), we have that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 - \|x_n - x_{n+1}\|^2 - \|x_n - z_0\|^2 & \\
&\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.
\end{aligned}$$

Setting $\mathit{\Gamma}_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned}
\mathit{\Gamma}_{n+1} - \mathit{\Gamma}_n - \|x_n - x_{n+1}\|^2 & \\
&\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.
\end{aligned} \tag{4.5}$$

Noting that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n x_n + (1-\beta_n)\{\alpha_n u_n + (1-\alpha_n)z_n\} - x_n\| \\
&= \|(1-\beta_n)\alpha_n(u_n - z_n) + (1-\beta_n)(z_n - x_n)\| \\
&\leq (1-\beta_n)\big(\|z_n - x_n\| + \alpha_n\|u_n - z_n\|\big),
\end{aligned} \tag{4.6}$$

we have that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &\leq (1-\beta_n)^2\big(\|z_n - x_n\| + \alpha_n\|u_n - z_n\|\big)^2 \\
&= (1-\beta_n)^2\|z_n - x_n\|^2 \\
&\quad + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big).
\end{aligned} \tag{4.7}$$

Thus we have from (4.5) and (4.7) that

$$\begin{aligned}
\mathit{\Gamma}_{n+1} - \mathit{\Gamma}_n &\leq \|x_n - x_{n+1}\|^2 + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \\
&\leq (1-\beta_n)^2\|z_n - x_n\|^2 \\
&\quad + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big) \\
&\quad + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - (1-\beta_n)\|z_n - x_n\|^2 \\
&\quad - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle
\end{aligned}$$

and hence

$$\begin{aligned}
\mathit{\Gamma}_{n+1} - \mathit{\Gamma}_n + \beta_n(1-\beta_n)\|z_n - x_n\|^2 & \\
&\leq (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big)
\end{aligned} \tag{4.8}$$

$$+ 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.$$

We will divide the proof into two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < a \le \beta_n \le b < 1$, we have from (4.8) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{4.9}$$

From (4.6) we have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{4.10}$$

We also have that

$$\begin{aligned}\|y_n - z_n\| &= \|\alpha_n u_n + (1-\alpha_n)z_n - z_n\| \\ &= \alpha_n\|u_n - z_n\| \to 0.\end{aligned} \tag{4.11}$$

Furthermore, from $\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|$, we have that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{4.12}$$

We show that $\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0$, where $z_0 = P_{C\cap A^{-1}D}u$. Put $l = \limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w \in H$. From $\|x_n - y_n\| \to 0$, $\{x_{n_i}\}$ converges weakly to $w \in H$. Since $\|z_n - x_n\| \to 0$, we also have that $\{z_{n_i}\}$ converges weakly to $w \in H$. From $\{z_n\} \subset C$, we have that $w \in C$. On the other hand, from (4.1) we have that

$$\begin{aligned} r(2 - r\|A\|^2)\|(I - P_D)Ax_n\|^2 &\le \|x_n - z\|^2 - \|z_n - z\|^2 \\ &= (\|x_n - z\| - \|z_n - z\|)(\|x_n - z\| + \|z_n - z\|) \\ &\le \|x_n - z_n\|(\|x_n - z\| + \|z_n - z\|).\end{aligned} \tag{4.13}$$

Then we get from $\|x_n - z_n\| \to 0$ that

$$\lim_{n\to\infty}\|Ax_n - P_DAx_n\| = 0. \tag{4.14}$$

Since $\{x_{n_i}\}$ converges weakly to $w \in H$ and A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using this and $\lim_{n\to\infty}\|Ax_n - P_DAx_n\| = 0$, we have that $P_DAx_{n_i} \rightharpoonup Aw$. Since P_D is the metric projection of F onto D, as in the proof of Theorem 3.1, we have that

$$\langle P_DAx_n - P_DAw, J_F(Ax_n - P_DAx_n) - J_F(Aw - P_DAw)\rangle \ge 0.$$

Since $P_DAx_{n_i} \rightharpoonup Aw$ and $\|J_F(Ax_n - P_DAx_n)\| \to 0$, we have that

$$-\|Aw - P_DAw\|^2 = \langle Aw - P_DAw, -J_F(Aw - P_DAw)\rangle \ge 0$$

and hence $Aw = P_DAw$. This implies that $w \in C \cap A^{-1}D$. Since $\{y_{n_i}\}$ converges weakly to $w \in C \cap A^{-1}D$, we have that

$$l = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle = \langle u - z_0, w - z_0\rangle \le 0.$$

Since $y_n - z_0 = \alpha_n(u_n - z_0) + (1-\alpha_n)(P_C(x_n - rA^*J_F(I-P_D)Ax_n) - z_0)$, we have from (2.1) that

$$\begin{aligned}\|y_n - z_0\|^2 \le{}& (1-\alpha_n)^2\|P_C(x_n - rA^*J_F(I-P_D)Ax_n) - z_0\|^2 \\ &+ 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle.\end{aligned}$$

From (4.1), we have

$$\|y_n - z_0\|^2 \le (1-\alpha_n)^2\,\|x_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle.$$

This implies that

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &\le \beta_n\,\|x_n - z_0\|^2 + (1-\beta_n)\,\|y_n - z_0\|^2 \\ &\le \beta_n\,\|x_n - z_0\|^2 \\ &\quad + (1-\beta_n)\Big((1-\alpha_n)^2\,\|x_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle\Big) \\ &= \big(\beta_n + (1-\beta_n)(1-\alpha_n)^2\big)\,\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\ &\le (\beta_n + (1-\beta_n)(1-\alpha_n))\,\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\ &= (1-(1-\beta_n)\alpha_n)\,\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\ &= (1-(1-\beta_n)\alpha_n)\,\|x_n - z_0\|^2 \\ &\quad + 2(1-\beta_n)\alpha_n(\langle u_n - u, y_n - z_0\rangle + \langle u - z_0, y_n - z_0\rangle).\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$, by Lemma 2.4 we obtain that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.5 that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$. Thus we have from (4.8) that for all $n \in \mathbb{N}$,

$$\begin{aligned}\beta_{\tau(n)}(1-\beta_{\tau(n)})&\|z_{\tau(n)} - x_{\tau(n)}\|^2 \\ &\le (1-\beta_{\tau(n)})^2 2\alpha_{\tau(n)}\|z_{\tau(n)} - x_{\tau(n)}\|\|u_{\tau(n)} - z_{\tau(n)}\| \\ &\quad + (1-\beta_{\tau(n)})^2\alpha_{\tau(n)}^2\|u_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)}, x_{\tau(n)} - z_0\rangle \\ &\quad - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle z_{\tau(n)}, x_{\tau(n)} - z_0\rangle.\end{aligned} \tag{4.15}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < a \le \beta_n \le b < 1$, we have from (4.15) that

$$\lim_{n\to\infty}\|z_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{4.16}$$

As in the proof of Case 1 we have that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \tag{4.17}$$

and

$$\lim_{n\to\infty}\|y_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{4.18}$$

Since $\|y_{\tau(n)} - x_{\tau(n)}\| \le \|y_{\tau(n)} - z_{\tau(n)}\| + \|z_{\tau(n)} - x_{\tau(n)}\|$, we have that

$$\lim_{n\to\infty} \|y_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{4.19}$$

For $z_0 = P_{C\cap A^{-1}D}u$, let us show that $\limsup_{n\to\infty}\langle z_0 - u, y_{\tau(n)} - z_0\rangle \ge 0$. Put

$$l = \limsup_{n\to\infty}\langle z_0 - u, y_{\tau(n)} - z_0\rangle.$$

Without loss of generality, there exists a subsequence $\{y_{\tau(n_i)}\}$ of $\{y_{\tau(n)}\}$ such that $l = \lim_{i\to\infty}\langle z_0 - u, y_{\tau(n_i)} - z_0\rangle$ and $\{y_{\tau(n_i)}\}$ converges weakly to some point $w \in H$. From $\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$, $\{x_{\tau(n_i)}\}$ converges weakly to $w \in H$. Furthermore, since $\|z_{\tau(n)} - x_{\tau(n)}\| \to 0$, we also have that $\{z_{\tau(n_i)}\}$ converges weakly to $w \in H$. From $\{z_{\tau(n_i)}\} \subset C$, we have that $w \in C$. As in the proof of Case 1 we have that $w \in C \cap A^{-1}D$. Then we have

$$l = \lim_{i\to\infty}\langle z_0 - u, y_{\tau(n_i)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \ge 0.$$

As in the proof of Case 1, we also have that

$$\left\|y_{\tau(n)} - z_0\right\|^2 \le (1-\alpha_{\tau(n)})^2 \left\|x_{\tau(n)} - z_0\right\|^2 + 2\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle$$

and then

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 &\le \beta_{\tau(n)}\left\|x_{\tau(n)} - z_0\right\|^2 + (1-\beta_{\tau(n)})\left\|y_{\tau(n)} - z_0\right\|^2 \\ &\le \big(1 - (1-\beta_{\tau(n)})\alpha_{\tau(n)}\big)\left\|x_{\tau(n)} - z_0\right\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$, we have that

$$(1-\beta_{\tau(n)})\alpha_{\tau(n)}\left\|x_{\tau(n)} - z_0\right\|^2 \le 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle.$$

Since $(1-\beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}\left\|x_{\tau(n)} - z_0\right\|^2 &\le 2\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle \\ &= 2\langle u_{\tau(n)} - u, y_{\tau(n)} - z_0\rangle + 2\langle u - z_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty}\left\|x_{\tau(n)} - z_0\right\|^2 \le 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$. From (4.17), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.5 again, we obtain that

$$\|x_n - z_0\| \le \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

References

[1] S. M. Alsulami and W. Takahashi, *The split common null point problem for maximal monotone mappings in Hilbert spaces and applications*, J. Nonlinear Convex Anal. **15** (2014), 793–808.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuous properties*, J. Nonlinear Convex Anal. **10** (2009), 131–147.
[4] C. Byrne, Y. Censor, A. Gibali and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759–775.
[5] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.
[6] Y. Censor and A. Segal, *The split common fixed-point problem for directed operators*, J. Convex Anal. **16** (2009), 587–600.
[7] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957-961.
[8] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.
[9] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[10] A. Moudafi, *The split common fixed point problem for demicontractive mappings*, Inverse Problems **26** (2010), 055007, 6 pp.
[11] Z. Opial, *Weak covergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.
[12] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
[13] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.
[14] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.
[15] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.
[16] W. Takahashi, H.-K. Xu and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal., to appear.
[17] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.

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