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EXISTENCE RESULTS FOR A CLASS OF DIRICHLET PARAMETRIC BOUNDARY VALUE PROBLEMS WITH NON SMOOTH POTENTIAL

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Abstract. We study the existence and multiplicity of nonnegative solutions for some parametric variational-hemivariational inequalities, driven by a possibly nonhomogeneous operator and involving a nonlinear term that may have critical growth. Our problem incorporates, as a special case the classical p-Laplacian with small perturbations.

1. Introduction

The aim of the present paper is to establish some existence and multiplicity results for a class of hemivariational inequalities depending on a parameter $\lambda > 0$, involving a possibly nonhomogeneous operator and slow perturbations of it:

$$(P_{\lambda,\mu}) \begin{cases} \text{Find } u \in W_0^{1,p}(\Omega),\ u \leq \psi \text{ in } \Omega, \text{ satisfying} \\ \langle Au, v-u\rangle - \mu \int_\Omega |u|^{p-2}u(v-u)dx + \lambda \int_\Omega J^0(x,u(x);(v-u)(x))\,dx \geq 0 \\ \text{for all } v \in W_0^{1,p}(\Omega),\ v \leq \psi \text{ in } \Omega, \end{cases}$$

where Ω is a nonempty open bounded subset of $\mathbb{R}^N$, λ is a real parameter, ψ is a nonegative function of $W_0^{1,p}(\Omega)$, and the operator A is $\langle Au, v\rangle = \int_\Omega (a(x,Du),Dv)_N\,dx$, $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map such that for every $z \in \overline{\Omega}$ $a(z,\cdot)$ is strictly monotone in $\mathbb{R}^N$, while $(z,y) \to a(z,y)$ is C^1 on $\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$. The precise hypotheses on $a(z,y)$ are given in $H(a)$ (see Section 2). These hypotheses incorporate as a special case the p-Laplacian defined by $\Delta_p(u) = div(|Du|^{p-2}Du)$ for all $u \in W^{1,p}(\Omega)$, $p > 1$. When $p = 2$ we can consider also a problem driven by a homogeneous elliptic operator B for which the regularity assumptions (see $H(b)$ in Section 2) are weaker than $H(a)$. We show that the problem obtained by suitable perturbations of B (see $(P_{\lambda,\nu})$ in the sequel) has nontrivial solution too.

The potential J has the same structure in all the problems considered. J^0 is the generalized directional derivative with respect to the second variable of a nonsmooth function $J(x,\xi) = -q(x)F(\xi)$ that may have critical growth (see Section 3 for all the hypotheses on the functions q and F) and for which, due to the structure of the problem, we avoid any assumption at $\pm\infty$.

Here we consider only the constraint $v \leq \psi$; this situation generalizes $v \leq 0$, but in literature one can find different obstacles. There are many existence results for

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parametric problems in the whole space, and for non parametric problems with constraint (it corresponds to the case $\lambda = 1$; for the obstacle $v \leq \psi$ see, for instance [1, 7, 13]), but to the best of our knowledge, the same is no longer true for the mixed problem (see for instance [5, 6, 4, 11]). In [5], the author proves that for some interval of the parameter μ, the solution set of the problem, $S(\mu)$, is compact in K (here the constraint is a closed convex subset K). The operator is the $p-$Laplacian, but the nonlinear term is the sum of two functions and the parameter μ is involved only in one of them. In [11], the operator A is close to ours but the nonlinear term is totally different. In [6] the problem, driven by a $p-$homogeneous operator, is: find $u \in W_0^{1,p}(\Omega),\ u \geq \psi$ satisfying

$$\int_\Omega A(x, \nabla u)\nabla(v-u)dx \geq \lambda \int_\Omega (u^+)^{p^*-1}(v-u)dx \text{ for all } v \geq \psi \text{ in } \Omega,$$

where $\psi \leq 0$, so the situation is totally different from ours. Finally, in [4] the problem has a structure more similar to $(P_{\lambda,0})$, but the operator A is the $p-$Laplacian, the function $q(x)$ is positive a.e. and the constraint K is a general closed convex and bounded subset of the space. Here we take into account a changing sign function $q(x)$, so also the results in [4] are not comparable with those exposed in this paper. Nevertheless our assumptions incorporate many elliptical obstacle problems with slow perturbations (like some of those treated in [1, 13]), where the nonlinear term is given by $-q(x)f(\xi)$ (see Section 5 for some examples). The Mountain Pass Theorem, and in general variational methods based on nonsmooth critical point theory, are very useful in the study of problems like $(P_{\lambda,\mu})$, $(P_{\lambda,\nu})$ and in effect they are a tool in most of the papers cited above. In the present paper, using a recent critical point result established by Bonanno-Winkert in [3] (see Theorem 2.5 in Section 2), we show that our problems have at least two nontrivial nonnegative solutions. In order to apply Theorem 2.5 to our contest, we need the functional associated to the potential J to be sequentially weakly upper semicontinuous. This property is true for a potential having subcritical growth but it fails in the critical one; to overcome this difficulty we opportunely truncate the functional (see Lemma 3.1 for a detailed proof).

In the next section, for the convenience of the reader we recall the main mathematical tools that we use here and we state the precise hypotheses on the functions a and b, while Section 3 is devoted to our main results for $(P_{\lambda,\mu})$. In Section 4 we deal with

$$(P_{\lambda,\nu}) \begin{cases} \text{Find } u \in H_0^1(\Omega),\ u \leq \psi \text{ in } \Omega, \text{ satisfying} \\ \langle Bu, v-u\rangle - \nu \int_\Omega u(v-u)dx + \lambda \int_\Omega J^0(x, u(x); (v-u)(x))\, dx \geq 0 \\ \text{for all } v \in H_0^1(\Omega),\ v \leq \psi \text{ in } \Omega, \end{cases}$$

We give the details of the proofs only for $(P_{\lambda,\mu})$, because for $(P_{\lambda,\nu})$ they are the same. We point out that our last result (see Theorem 4.4) extends to the critical case Theorem 3.1 of [1] and Theorem 1 of [13]. Moreover, the results in those papers guarantee the existence of a nontrivial nonnegative solution, whereas we get at least two.

Finally, for the sake of completeness, in Section 5 we present some problems to which our existence results apply. In all of them the function $q(x)$ has no constant sign.

2. Preliminaries

Let Ω be a nonempty open bounded subset of the real Euclidean $N-$space $(\mathbb{R}^N, |\cdot|)$, with a sufficiently smooth boundary $\partial\Omega$. We denote by $(\cdot,\cdot)_N$ and $|\cdot|_{N\times N}$ the standard scalar product in $\mathbb{R}^N$ and the norm on $\mathbb{R}^{N\times N}$ respectively. Let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p},$$

and let $\langle\cdot,\,\cdot\rangle$ be the duality pairing between $W_0^{1,p}(\Omega)$ and its dual space $W^{-1,p'}(\Omega)$. We put

$$p^* = \begin{cases} \frac{Np}{N-p} & if\ N > p\,, \\ +\infty & if\ N \le p\,, \end{cases}$$

and denote by c_t the constant of the embedding of $W_0^{1,p}(\Omega)$ into $L^t(\Omega)$, with $t \in [1,p^*]$ if $p \neq N$, $t \in [1,+\infty[$ if $p = N$ and point out that it is compact for any $t \in [1,p^*[$. The following notion will be useful in the sequel.

Definition 2.1. Let X be a Banach space and X^* its topological dual. A map $A : X \to X^*$ is of type $(S)_+$, if for every sequence $\{x_n\}_{n\ge1} \subseteq X$ such that $x_n \rightharpoonup x$ in X and $\limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle \le 0$, one has $x_n \to x$ in X.

In this paper we deal with the operator $A :\ W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, defined as

$$\langle Au, v\rangle = \int_\Omega (a(x,Du), Dv)_N\, dx. \tag{2.1}$$

The hypotheses on the map $a(z,y)$ are the following:
$\underline{H(a)}$: $a(z,y) = h(z,|y|)y$ for all $(z,y) \in \overline{\Omega}\times\mathbb{R}^N$ with $h(z,t) > 0$ for all $(z,t) \in \overline{\Omega}\times(0,+\infty)$ and

- (i) $a \in C^{0,\alpha}(\overline{\Omega}\times\mathbb{R}^N,\mathbb{R}^N)\cap C^1(\overline{\Omega}\times\mathbb{R}^N\setminus\{0\},\mathbb{R}^N)$ with $0<\alpha<1$;
- (ii) for all $(z,y) \in \overline{\Omega}\times\mathbb{R}^N\setminus\{0\}$, we have $|D_y a(z,y)|_{N\times N} \le \tilde{c}|y|^{p-2}$ for some $\tilde{c} > 0$, $1 < p < \infty$;
- (iii) for all $(z,y) \in \overline{\Omega}\times(\mathbb{R}^N\setminus\{0\})$ and all $\xi \in \mathbb{R}^N$, we have
$$(D_y a(z,y)\xi,\xi)_N \ge \hat{c}|y|^{p-2}|\xi|^2 \text{ for some } \hat{c} > 0\,;$$

Let $G(z,y)$ be the real-valued function defined by $D_yG(z,y) = a(z,y)$ and $G(z,0) = 0$ for all $(z,y) \in \overline{\Omega}\times\mathbb{R}^N$. The following Lemma summarizes basic properties of G, a and A. For more details see [2], Lemma 2.1 and Corollary 2.1 and [9], Proposition 3.1.

Lemma 2.2. *If Hypotheses $H(a)$ hold, then*

- (a) *for all $z \in \overline{\Omega}$, $y \to a(z,y)$ is maximal monotone and strictly monotone;*
- (b) *for all $(z,y) \in \overline{\Omega}\times\mathbb{R}^N$ we have*
$$|a(z,y)| \le \frac{\tilde{c}}{p-1}|y|^{p-1},\ (a(z,y),y)_N \ge \frac{\hat{c}}{p-1}|y|^p \quad and$$
$$\frac{\hat{c}}{p(p-1)}|y|^p \le G(z,y) \le \frac{\tilde{c}}{p(p-1)}|y|^p\,;$$

(c) *$A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by (2.1) is maximal monotone and of type $(S)_+$.*

Remark 2.3. (a) holds under the only conditions (i) and (iii). In effect we have

$$(a(z,\xi) - a(z,\eta), \xi - \eta)_N \geq \hat{c}_0(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \text{ for some } \hat{c}_0 > 0\,.$$

When $p = 2$ we consider also a problem arising from a uniformly elliptic operator operator

$$B = -\sum_{i,j=1}^{N} D_i \left(b_{ij}(x) D_j\right)$$

satisfying the following standard conditions:
$\underline{H(b)}$:

(b_1) the functions $b_{ij} : \Omega \to \mathbb{R}$ belong to $L^\infty(\Omega)$ for any $i, j = 1...N$;
(b_2) $b_{ij}(x) = b_{ji}(x)$ a.e. in Ω;
(b_3) there are two positive constants, k_1, k_2, such that

$$k_1|\xi|^2 \leq \sum_{i,j=1}^{N} b_{ij}(x)\xi_i\xi_j \leq k_2|\xi|^2 \;\forall \xi \in \mathbb{R}^N, \text{ for a.e. } x \in \Omega\,.$$

It is well known that the norm induced by B on $H_0^1(\Omega)$ is equivalent to the usual one, so we can put

$$\|v\| = \langle Bv,\; v\rangle^{\frac{1}{2}}\;,\;\forall\, v\, \in H_0^1(\Omega)\;.$$

From now on, to shorten the notation, we will omit the dependence from the x variable of the functions involved. For any $u \in W_0^{1,p}(\Omega)$ we will often adopt the standard decomposition

$$u(x) = u^+(x) - u^-(x)\,,\; x \in \Omega,$$

where $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$. If $\psi \in W_0^{1,p}(\Omega)$, then the set

$$K := \{v \in W_0^{1,p}(\Omega) \;:\; v \leq \psi \;\text{ in } \Omega\}$$

turns out nonempty, convex and closed, hence weakly closed, in $W_0^{1,p}(\Omega)$. In this paper we are interested to problems involving a nonnegative obstacle ψ; for such ψ and for any $u \in K$ then both u^+ and $-u^-$ belong to K too, but the same doesn't hold generally true for u^-.
The nonlinear term of $(P_{\lambda,\mu})$ and $(P_{\lambda,\nu})$ is nonsmooth. The precise hypotheses on it are given in Section 3. Since we deal with the generalized directional derivative rather than with the usual one, it is useful to recall that, given a real Banach space X and a locally Lipschitz function $g : X \to \mathbb{R}$, then its generalized directional derivative at the point $u \in X$ along the direction $v \in X$ is

$$g^0(u; v) := \limsup_{z \to u,\, t \to 0^+} \frac{g(z + tv) - g(z)}{t};$$

the generalized gradient of g at u is

$$\partial g(u) = \{u^* \in X^* :\; \langle u^*, z\rangle \leq g^0(u; z) \;\forall\; z \in\; X\}\,.$$

Let $I = g + h$, where g is as above and $h : X \to]-\infty, +\infty]$ is convex proper (i.e. $h \not\equiv +\infty$) and lower semicontinuous.

Definition 2.4. We say that $u \in X$ is a critical point of I if it satisfies

$$g^0(u, v-u) + h(v) - h(u) \geq 0 \quad \forall v \in X\,.$$

We say that I satisfies the Palais-Smale condition at a level c (briefly $(PS)_c$) if

every sequence $\{u_n\} \subseteq X$ satisfying $I(u_n) \to c$ and

$$g^0(u_n, v-u_n) + h(v) - h(u_n) \geq -\varepsilon_n \|v - u_n\|,\ \forall n \in \mathbb{N} \text{ and } \forall v \in X,$$

where $\varepsilon_n \to 0^+$, has a convergent subsequence.

In our approach to problem $(P_{\lambda,\mu})$ we use a critical point theorem due to Bonanno and Winkert (see Theorem 2.5 below). Now, let us give the assumptions concerning the abstract result we need. X is a reflexive Banach space, the functional $\Phi : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous and coercive, $\Upsilon : X \to \mathbb{R}$ is sequentially weakly upper semicontinuous, $j : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D(j)$ stands for the effective domain of j. Then we put $I_\lambda = \Phi - \lambda\Upsilon + \lambda j = \Phi - \lambda\Psi$ and suppose that $D(j) \cap\ \Phi^{-1}(]-\infty; r[) \neq \emptyset$ for all $r > \inf_X \Phi$. Now we define

$$\varphi_1(r) = \inf_{\Phi(v)\,<\,r} \frac{\sup_{\Phi(u)\,<\,r} \Psi(u) - \Psi(v)}{r - \Phi(v)} \text{ for all } r > \inf_X \Phi$$

and

$$\varphi_2(r) = \sup_{\Phi(v)>r} \frac{\Psi(v) - \sup_{\Phi(u)\leq r} \Psi(u)}{\Phi(v) - r} \text{ for all } r < \sup_X \Phi$$

The three critical point abstract result we need (see Theorem 2.5 below) can be found in ([3]). This type of theorems can be inserted in a new field originated by the seminal paper [12].

Theorem 2.5. *Let X be a reflexive Banach space and let $I_\lambda = \Phi - \lambda\Upsilon + \lambda j : X \to \mathbb{R}$, $\lambda > 0$. Assume there is $r \in]\inf_X \Phi, sup_X \Phi[$ such that $\varphi_1(r) < \varphi_2(r)$. Further suppose that the functional I_λ is bounded from below and satisfies the (PS)-condition for each $\lambda \in \Lambda := \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$. Then I_λ has three distinct critical points.*

3. Main results

We deal with a changing sign nonlinearity of the kind $q(x)f(\xi)$. We begin by stating the properties of $q : \Omega \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$; for q we assume:

$\underline{H(q)}$: (q_1) $q \in L^\infty(\Omega)$,
(q_2) $q^+ \not\equiv 0$ in Ω,
(q_3) $meas\{x \in \Omega : q(x) = 0\} = 0$,

while f satisfies:

$\underline{H(f)}$: (f_1) f is measurable;
(f_2) $\exists\, a_1,\ a_2 \geq 0$ $(a_1, a_2) \neq (0,0)$: $|f(\xi)| \leq a_1 + a_2|\xi|^s$, $\forall\, \xi \in \mathbb{R}$, where $s \in]0,\, p^* - 1]$,

As usual, we put

$$F(\xi)=\int_0^\xi f(t)dt,\ J(x,\xi)=-q(x)F(\xi)\ \forall x\in\Omega,\ \forall\xi\in\mathbb{R}\,;$$

finally, we define

$$\Upsilon(u)=-\int_\Omega J(x,u(x))dx\ \forall u\in\ W_0^{1,p}(\Omega)\,.$$

J is locally Lipschitz with respect to the second variable (we simply denote with J^0 its directional derivative with respect to ξ) and so is Υ. We define Υ on $W_0^{1,p}(\Omega)$, but it is well posed on $L^t(\Omega)$ too, for $t\in[1,p^*]$ if $p\neq N$, $t\in[1,+\infty[$ if $p=N$, and it turns out locally Lipschitz in this space too. Furthermore, well known properties on the gradients of integral functionals (see [8] for the details) ensure that any $w\in\partial\Upsilon(u)$ belongs to $\big(L^t(\Omega)\big)'$ too, as well as $w(x)\in J^0(x,u(x))$ a.e. in Ω and

$$\langle w,v\rangle=\int_\Omega w(x)v(x)dx\ \forall v\in W_0^{1,p}(\Omega)\,.$$

It is worthwhile to point out that the equality above holds for any $v\in L^t(\Omega)$.
In this Section we discuss the existence of solutions to $(P_{\lambda,\mu})$, arising from a slow perturbation of A, with $\mu<\frac{\hat c\mu_1}{p-1}$, μ_1 being the first eigenvalue of $(-\triangle_p,W_0^{1,p}(\Omega))$. The energy functional related to $(P_{\lambda,\mu})$ is

$$\begin{aligned}I_{\lambda,\mu}(u)&=\int_\Omega G(x,Du)dx-\frac{\mu}{p}\int_\Omega|u|^pdx-\lambda\int_\Omega q(x)F(u)dx+\lambda j(u)\\&=\Phi_0(u)-\frac{\mu}{p}\|u\|_p^p-\lambda\Upsilon(u)+\lambda j(u)=\Phi(u)-\lambda\Upsilon(u)+\lambda j(u).\end{aligned}\tag{3.1}$$

Φ is C^1 and j is the indicator function of the set K introduced in Section 2. The locally Lipschitz part of $I_{\lambda,\mu}$ is $g=\Phi-\lambda\Upsilon$. We exploited basic properties of Υ. Taking into account Lemma 2.2 and the variational characterization

$$\mu_1=\inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\|\nabla u\|_p^p}{\|u\|_p^p}\,,$$

we deduce

$$\frac{\hat c\mu_1-\mu(p-1)}{\mu_1p(p-1)}\|u\|^p\le\Phi_0(u)-\frac{\mu}{p}\|u\|_p^p\le\frac{\tilde c\mu_1-\mu(p-1)}{\mu_1p(p-1)}\|u\|^p\tag{3.2}$$

$\forall u\in W_0^{1,p}(\Omega)$,

$$\frac{\hat c\mu_1-\mu(p-1)}{\mu_1(p-1)}\|u\|^p\le\langle Au,u\rangle-\mu\|u\|_p^p\le\frac{\tilde c\mu_1-\mu(p-1)}{\mu_1(p-1)}\|u\|^p\tag{3.3}$$

$\forall u\in W_0^{1,p}(\Omega)$, and $\Phi'_\mu(u)(w)=\langle Au,w\rangle-\mu\int_\Omega|u|^{p-2}uwdx$ is of type $(S)_+$. The Palais-Smale condition at a level c becomes now

$(PS)_c$ every sequence $\{u_n\}\subseteq X$ satisfying $I_{\lambda,\mu}(u_n)\to c$ and

$$(\Phi-\lambda\Upsilon)^0(u_n,v-u_n)+\lambda j(v)-\lambda j(u_n)\ge-\varepsilon_n\|v-u_n\|,\ \forall n\in\mathbb{N}\text{ and }\forall v\in X,$$

where $\varepsilon_n\to0^+$, has a convergent subsequence, and $u \in X$ is a critical point of $I_{\lambda,\mu}$ if it satisfies

$$(\Phi - \lambda\Upsilon)^0(u, v-u) + \lambda j(v) - \lambda j(u) \geq 0 \quad \forall v \in X\,,$$

that is

$$\langle Au, v-u\rangle - \mu \int_\Omega |u|^{p-2}u(v-u)dx + \lambda(-\Upsilon)^0(u; v-u) + \lambda j(v) - \lambda j(u) \geq 0\, \forall v \in W_0^{1,p}(\Omega).$$

Due to the structure of j, the inequality above is equivalent to

$$u \in K,\ \langle Au, v-u\rangle - \mu \int_\Omega |u|^{p-2}u(v-u)dx + \lambda(-\Upsilon)^0(u; v-u) \geq 0\, \forall v \in K.$$

Taking into account that $(-\Upsilon)^0(u; v) \leq \int_\Omega J^0(x, u; v-u)dx$ for all u, $v \in W_0^{1,p}(\Omega)$ it is immediately seen that any critical point of I_λ, besides belonging to K verifies

$$\langle Au, v-u\rangle - \mu \int_\Omega |u|^{p-2}u(v-u)dx + \lambda \int_\Omega J^0(x, u; v-u)dx \geq 0\, \forall v \in K, \tag{3.4}$$

so it turns out a solution to $(P_{\lambda,\mu})$.
Since we are interested in nonnegative solutions, we modify our problem. So, before stating our results, we define

$$\overline{f}(\xi) = \begin{cases} 0 & \text{if } \xi < 0, \\ f(\xi) & \text{if } \xi \geq 0, \end{cases} \quad \text{and} \quad \overline{F}(\xi) = \int_0^\xi \overline{f}(t)dt = \begin{cases} 0 & \text{if } \xi < 0, \\ F(\xi) & \text{if } \xi \geq 0. \end{cases}$$

If f satisfies $H(f)$, then the same properties hold true for $\overline{f}$, so we obtain the following energy functional, associated to the truncated problem

$$\overline{I}_{\lambda,\mu}(u) = \Phi(u) - \lambda\overline{\Upsilon}(u) + \lambda j(u)\,, \tag{3.5}$$

where j is always the indicator function of K, while $\overline{\Upsilon}(u) = 0$ when $u \leq 0$ a.e. Due to the structure of $\overline{\Upsilon}$ we can show the following Lemma, that guarantees the sequential weak upper semicontinuity of $\overline{\Upsilon} - j$.

Lemma 3.1. *Let q and f verify $H(q)$ and $H(f)$, respectively. Then the functional $\overline{\Upsilon} - j : W_0^{1,p}(\Omega) \to \mathbb{R}$ is sequentially weakly upper semicontinuous.*

Proof. Let $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega)$ weakly converging to u in $W_0^{1,p}(\Omega)$. Then $\{u_n\} \rightharpoonup u$ in $L^{s+1}(\Omega)$ and a.e. If $u \notin K$ then, being K weakly closed, $u_n \notin K$ for $n \geq \overline{n}$, so

$$\limsup_{n\to+\infty}(\overline{\Upsilon} - j)(u_n) = -\infty = (\overline{\Upsilon} - j)(u)\,. \tag{3.6}$$

If $u \in K$ then $(\overline{\Upsilon} - j)(u) = \overline{\Upsilon}(u)$ and two situations may occur: only for a finite $n \in \mathbb{N}$ one has $u_n \in K$, or there is a countable subset of $\mathbb{N}$, say $\mathbb{N}_1$, such that $u_n \in K$ for $n \in \mathbb{N}_1$. In the first case

$$\limsup_{n\to+\infty}(\overline{\Upsilon} - j)(u_n) = -\infty < \overline{\Upsilon}(u)\,, \tag{3.7}$$

while when $u_n \in K$ for $n \in N_1$, then for such n, due to the continuity of $\overline{F}$ we have $\overline{F}(u_n(x)) \to \overline{F}(u(x))$ a.e., while from (f_2) we deduce

$$|\overline{F}(u_n(x)| \leq a_1\psi(x) + \frac{a_2}{s+1}\psi^{s+1}(x)\,,$$

so, by the Lebesgue's theorem, we obtain

$$\lim_{n\to+\infty, n\in\mathbb{N}_1} \int_\Omega q(x)\overline{F}(u_n(x))dx = \int_\Omega q(x)\overline{F}(u(x))dx,$$

that is

$$\limsup_{n\to+\infty, n\in\mathbb{N}_1} (\overline{\Upsilon} - j)(u_n) = (\overline{\Upsilon} - j)(u)\,. \tag{3.8}$$

The conclusion follows from (3.6), (3.7) and (3.8). □

Remark 3.2. *The same conclusion of Lemma 3.1 holds also for Υ, except that when $s = p^* - 1$. In fact, when $s < p^* - 1$, we use the compactness of the embedding of $W_0^{1,p}(\Omega)$ in $L^s(\Omega)$ to obtain the weak continuity of Υ (and so, a fortiori the weak sequential upper semicontinuity), but it is well known that the compactness of such embedding is no longer true when $s = p^* - 1$.*

The functions φ_1 and φ_2 of the abstract result of [3] can be written as

$$\varphi_1(r) = \inf_{\substack{\Phi(v) < r,\\ v\in K}} \frac{\sup_{\substack{\Phi(u) < r,\\ u\in K}} \overline{\Upsilon}(u) - \overline{\Upsilon}(v)}{r - \Phi(v)}$$

and

$$\varphi_2(r) = \sup_{\Phi(v)>r, v\in K} \frac{\overline{\Upsilon}(v) - \sup_{\Phi(u)\le r, u\in K}\overline{\Upsilon}(u)}{\Phi(v) - r}\,.$$

Theorem 3.3. *Let a, q and f verify $H(a)$, $H(q)$ and $H(f)$, respectively. Assume further that $\mu < \frac{\hat{c}\mu_1}{p-1}$ and $\exists \phi \in K$ and $R > 0$ such that*

(i) $\|\phi\| > R$;

(ii) $\frac{\overline{\Upsilon}(\phi)}{\Phi(\phi)} > \frac{\mu_1 p(p-1)\|q\|_\infty R^{1-p}}{\mu_1\hat{c} - \mu(p-1)} \left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s \right)$

Then there exists $r > 0$ such that for each $\lambda \in \Lambda := \left] \frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)} \right[$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions.

Proof. Consider the functional $\overline{I}_{\lambda,\mu}$ defined in (3.5). From (3.3) we deduce

$$\Phi(\phi) > \frac{(\mu_1\hat{c} - \mu(p-1))R^p}{\mu_1 p(p-1)} = r;\ \Phi(u) < r \Rightarrow \|u\|^p < \frac{r\mu_1 p(p-1)}{\mu_1\hat{c} - \mu(p-1)} = R^p\,. \tag{3.9}$$

Owing to (f_1) and (3.9), we can write

$$\varphi_1(r) \le \frac{\sup_{\Phi(u)<r,\, u\in K}\overline{\Upsilon}(u) - \overline{\Upsilon}(0)}{r - \Phi(0)} = \frac{\sup_{\Phi(u)<r,\, u\in K}\overline{\Upsilon}(u)}{r} \tag{3.10}$$

$$\le \frac{\sup_{\|u\|<R,\, u\in K}\|q\|_\infty \left(a_1 c_1\|u\| + \frac{a_2 c_{s+1}^{s+1}}{s+1}\|u\|^{s+1}\right)}{r} \le \frac{R\|q\|_\infty \left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s \right)}{r}\,.$$

Using (f_1), (3.9), (3.10) and (ii) we deduce

$$\varphi_2(r) \geq \frac{\overline{\Upsilon}(\phi) - R\|q\|_\infty \left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s\right)}{\Phi(\phi) - r} > \tag{3.11}$$

$$\|q\|_\infty R \left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s\right) \frac{\frac{\Phi(\phi)}{r} - 1}{\Phi(\phi) - r} = \frac{R\|q\|_\infty \left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s\right)}{r} > \varphi_1(r)\,.$$

Clearly Φ and $\overline{\Upsilon}$ satisfy the assumptions of Theorem 2.5 (see also Lemma 3.1). Now, we prove that $\overline{I}_{\lambda,\mu}$ is coercive. When $u \notin K$ then $\overline{I}_\lambda(u) = +\infty$, while for $u \in K$ we deduce the following estimate

$$\overline{I}_{\lambda,\mu}(u) = \Phi(u) - \lambda\Upsilon(u^+) \geq \Phi(u) - \lambda\|q\|_\infty \left(a_1\|\psi\|_1 + \frac{a_2}{s+1}\|\psi\|_{s+1}^{s+1}\right)\,, \tag{3.12}$$

that shows that $\overline{I}_{\lambda,\mu}$ is coercive. Now, we show that $\overline{I}_{\lambda,\mu}$ satisfies the Palais-Smale condition. So, let $\{u_n\}$ be a $(PS)_c$ sequence for $\overline{I}_{\lambda,\mu}$. Then $\overline{I}_{\lambda,\mu}(u_n) \to c$ and

$(\Phi - \lambda\overline{\Upsilon})^0(u_n, v - u_n) + \lambda j(v) - \lambda j(u_n) \geq -\varepsilon_n\|v - u_n\|$, $\forall n \in \mathbb{N}$ and $\forall v \in W_0^{1,p}(\Omega)$.

Clearly $\{u_n\} \subseteq K$, and the properties of $(\Phi - \lambda\overline{\Upsilon})^0$ guarantee that

$$\begin{aligned}\langle Au_n, v - u_n\rangle - \mu\int_\Omega |u_n|^{p-2}u_n(v - u_n)dx + \lambda\int_\Omega \overline{J}^0(x, u_n; v - u_n)\\ \geq -\varepsilon_n\|v - u_n\|,\ \forall n \in \mathbb{N}\,\forall v \in W_0^{1,p}(\Omega).\end{aligned} \tag{3.13}$$

Since $\overline{I}_{\lambda,\mu}(u_n) \to c$, from (3.9) and (3.12) we deduce that $\{u_n\}$ is bounded, hence we can estract a subsequence, say still $\{u_n\}$, converging to u, weakly in $W_0^{1,p}(\Omega)$ and in $L^{s+1}(\Omega)$ and strongly in $L^p(\Omega)$. Hence

$$\lim_{n\to+\infty}\int_\Omega |u_n(x)|^{p-2}u_n(x)(u(x) - u_n(x))dx = 0\,. \tag{3.14}$$

Taking into account that $a_1 + a_2\psi^s(x) \in L^{(s+1)'}(\Omega)$ we have

$$\lim_{n\to+\infty}\int_\Omega (a_1 + a_2\psi^s(x))|u(x) - u_n(x)|dx = 0\,,$$

that forces

$$\begin{aligned}\lim_{n\to+\infty}\left|\int_\Omega \overline{J}^0(x, u_n; u - u_n)dx\right| &= \lim_{n\to+\infty}\left|\int_{u_n>0} J^0(x, u_n; u - u_n)dx\right|\\ &\leq \|q\|_\infty \cdot \int_\Omega (a_1 + a_2\psi^s(x))|u(x) - u_n(x)|dx = 0\,.\end{aligned} \tag{3.15}$$

So if we choose $v = u$ in (3.13), and take advantage of (3.14) and (3.15), then we obtain

$$\lim_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0\,.$$

Since A is of type $(S)_+$ (see Lemma 2.2(c)), we have $u_n \to u$ and the $(PS)_c$-condition is totally proved. Therefore all the assumptions of Theorem 2.5 are fulfilled, so it applies with $r = \frac{(\mu_1 \hat{c} - \mu(p-1))R^p}{\mu_1 p(p-1)}$, and $\overline{I}_{\lambda,\mu}$ has at least three distinct critical points. Any critical point u of $\overline{I}_{\lambda,\mu}$, satisfies

$$\langle Au, v-u\rangle - \mu \int_\Omega |u|^{p-2}u(v-u)dx + \lambda \int_\Omega \overline{J}^0(x,u;v-u)dx \geq 0 \ \forall v \in K\,,$$

that is

$$\langle Au, v-u\rangle - \mu \int_\Omega |u|^{p-2}u(v-u)dx + \lambda \int_{u\geq 0} J^0(x,u;v-u)dx \geq 0\, \forall v \in K\,. \tag{3.16}$$

If we choose $v = u^+ \in K$ in (3.16) then we obtain

$$\int_\Omega \big(a(x,Du), Du^-\big)_N\, dx + \mu \int_{u\leq 0} |u|^p dx + \lambda \int_{u\geq 0} J^0(x,u^+;u^-)dx \geq 0\,,\ \text{hence}$$

$$-(\langle A(-u^-), -u^-\rangle - \mu \|u^-\|_p^p) \geq 0.$$

From the inequality above and taking into account (3.3) we deduce

$$-u^- = 0 \text{ a.e. in } \Omega,\ \text{so } u \geq 0 \text{ a.e.},,$$

so (3.16) is equivalent to (3.4), and we have obtained three distinct nonnegative solutions to $(P_{\lambda,\mu})$. □

Remark 3.4. *The assumptions in Theorem 3.3 involve the behaviour of $\overline{\Upsilon}$ and Φ at a point ϕ, so they are not global growth conditions.*

Remark 3.5. *When $s < p^* - 1$ then the coercivity of the functional $\overline{I}_{\lambda,\mu}$ guaranees that it satisfies the $(PS)_c$ (see* [10]*, Proposition 2.3).*

In the next two results we show that also under a condition involving the growth of F in a point b we can guarantee the multiplicity of nonnegative solutions. To this end, from now on, we use the following notations:

$$\Omega_+ := \{x \in \Omega : q(x) > 0\},\ \Omega_- := \{x \in \Omega : q(x) < 0\},\ \Omega_0 := \{x \in \Omega : q(x) = 0\},$$

$$q_+ = essinf q^+,\ q_- = essinf q^-\,.$$

Since $\|q^+\|_\infty > 0$ (see (q_2)) we can find a point $x_0 \in \Omega_+$ and a positive number D_+ such that $B(x_0, D_+) \subseteq int\overline{\Omega}_+$ and $q > 0$ a.e. on $B(x_0, D_+)$. Analogously, if $\|q^-\|_\infty > 0$ then we can find a point $x_1 \in \Omega_-$ and a positive number D_- such that $B(x_1, D_-) \subseteq int\overline{\Omega}_-$ and $q < 0$ a.e. on $B(x_1, D_-)$. Finally, we put

$$A_{r,\mu} = \sup_{\Phi(u)\leq r,\ u\in K} \overline{\Upsilon}(u)\,,$$

$$k_{+,\mu} = \left[\frac{\mu_1 p(p-1)\Gamma(1+\frac{N}{2})}{(\mu_1\hat{c} - \mu(p-1))\pi^{\frac{N}{2}}\left(D_+^N - \left(\frac{(p-1)D_+}{p}\right)^N\right)}\right]^{\frac{1}{p}} \frac{D_+}{p}\,,$$

$$k_{-,\mu} = \left[\frac{\mu_1 p(p-1)\Gamma(1+\frac{N}{2})}{(\mu_1\hat{c} - \mu(p-1))\pi^{\frac{N}{2}}\left(D_-^N - \left(\frac{(p-1)D_-}{p}\right)^N\right)}\right]^{\frac{1}{p}} \frac{D_-}{p}\,.$$

Theorem 3.6. *Let a, q and f verify $H(a)$, $H(q)$ and $H(f)$, respectively. Assume further that $\mu < \frac{\hat{c}\mu_1}{p-1}$, $q_+ > 0$ and there exist $r > 0$ and $b > 0$ such that $b > k_{+,\mu}(r)^{\frac{1}{p}}$, $F(t) \geq 0$ in $[0, b]$, $\frac{F(b)}{b^p} \cdot \frac{q_+\mu_1(p-1)}{(\mu_1\tilde{c}-\mu(p-1))p^{p-1}} \cdot \frac{D_+^p(p-1)^N}{p^N-(p-1)^N} > \frac{A_{r,\mu}}{r}$, $\psi(x) \geq b$ in $B(x_0, D_+)$. Then for each $\lambda \in \Lambda := \left]\frac{(\mu_1\tilde{c}-\mu(p-1))p^{p-1}}{\mu_1(p-1)} \frac{p^N-(p-1)^N}{D_+^p(p-1)^N} \frac{b^p}{q_+F(b)}, \frac{r}{A_{r,\mu}}\right[$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions.*

Proof. Fix $\lambda \in \Lambda$ and observe that (see also (3.10)):

$$\varphi_1(r) \leq \frac{A_{r,\mu}}{r}\,. \tag{3.17}$$

Now, we consider the function

$$u(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, D_+), \\ \frac{pb}{D_+}(D_+ - |x - x_0|) & \text{if } x \in B(x_0, D_+) \setminus B(x_0, \frac{p-1}{p}D_+)\ , \\ b & \text{if } x \in B(x_0, \frac{p-1}{p}D_+)\ . \end{cases}$$

$u \in K$ and

$$\|u\|^p = \left(\frac{pb}{D_+}\right)^p \frac{\pi^{\frac{N}{2}}\left(D_+^N - \left(\frac{(p-1)D_+}{p}\right)^N\right)}{\Gamma(1+\frac{N}{2})},$$

so, from (3.3) we deduce

$$\begin{aligned} r < \frac{b^p}{k_{+,\mu}^p} &= \frac{b^p(\mu_1\hat{c} - \mu(p-1))p^p}{\mu_1 p(p-1)D_+^p} \cdot \frac{\pi^{\frac{N}{2}}D_+^N}{\Gamma(1+\frac{N}{2})}\left(1 - \left(\frac{p-1}{p}\right)^N\right) \\ &= \frac{\mu_1\hat{c} - \mu(p-1)}{\mu_1 p(p-1)}\|u\|^p \leq \Phi(u) \leq \frac{(\mu_1\tilde{c} - \mu(p-1))b^p}{(\mu_1\hat{c} - \mu(p-1))k_{+,\mu}^p}\,. \end{aligned} \tag{3.18}$$

On the other hand, for $\overline{\Upsilon}(u)$ we have the following estimate

$$\begin{aligned} \overline{\Upsilon}(u) &= \int_{B(x_0,D_+)\setminus B(x_0,\frac{p-1}{p}D_+)} q(x)F(u(x))dx + \int_{B(x_0,\frac{p-1}{p}D_+)} q(x)F(u(x))dx \\ &\geq q_+F(b)\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\left(\frac{(p-1)D_+}{p}\right)^N \end{aligned} \tag{3.19}$$

Combining (3.19) and the last inequality on the right of (3.18) we obtain

$$\begin{aligned} \frac{\overline{\Upsilon}(u)}{\Phi(u)} &\geq \frac{F(b)q_+(\mu_1\hat{c} - \mu(p-1))k_+^p\pi^{\frac{N}{2}}(p-1)^N D_+^N}{b^p(\mu_1\tilde{c} - \mu(p-1))\Gamma(1+\frac{N}{2})p^N} \\ &= \frac{F(b)}{b^p} \cdot \frac{q_+(p-1)}{(\mu_1\tilde{c} - \mu(p-1))p^{p-1}} \cdot \frac{D_+^p(p-1)^N}{p^N - (p-1)^N} > \frac{A_{r,\mu}}{r}\,. \end{aligned} \tag{3.20}$$

Owing to (3.17) and (3.20) we deduce

$$\varphi_2(r) \geq \frac{\overline{\Upsilon}(u) - A_{r,\mu}}{\Phi(u) - r} > \frac{\frac{A_{r,\mu}}{r}\Phi(u) - A_{r,\mu}}{\Phi(u) - r} = \frac{A_{r,\mu}}{r} \geq \varphi_1(r)\,.$$

The rest of the proof is the same of previous result. $\square$

Theorem 3.7. *Let a, q and f verify $H(a)$, $H(q)$ and $H(f)$, respectively. Assume further that $\mu < \frac{\hat{c}\mu_1}{p-1}$, $q_- > 0$ and there exist $r > 0$ and $b > 0$ such that $b > k_{-,\mu}(r)^{\frac{1}{p}} > 0$, $F(t) \leq 0$ in $[0, b]$, $\frac{F(b)}{b^p} \frac{q_-\mu_1(p-1)}{(\mu_1\tilde{c}-\mu(p-1))p^{p-1}} \frac{D_-^p(p-1)^N}{p^N-(p-1)^N} > \frac{A_{r,\mu}}{r}$, $\psi(x) \geq b$ in $B(x_1, D_-)$. Then for each $\lambda \in \Lambda := \left]\frac{(\mu_1\tilde{c}-\mu(p-1))p^{p-1}}{\mu_1(p-1)} \frac{p^N-(p-1)^N}{D_-^p(p-1)^N} \frac{b^p}{q_-F(b)}, \frac{r}{A_{r,\mu}}\right[$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions.*

Remark 3.8. *In previous results we avoided any assumption on the behaviour of f at 0. Anyway, the inequalities in Theorems 3.6 and 3.7 involve the behaviour of ψ and q on a subset of Ω. If we can estimate $\frac{A_{r,\mu}}{r}$ for some $r > 0$ then this subset can be small provided $\frac{F(\psi(x))}{\psi(x)^p}$ is big on it.*

In the last existence result for $(P_{\lambda,\mu})$, we obtain an unbounded interval Λ, such that for each $\lambda \in \Lambda$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions. We need some additional assumptions on f:

(f_3) $s > p - 1$;
(f_4) $\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi^{p-1}} = 0$;
(f_5) there exist a nonnegative function $v \in K$, such that
$$\int_\Omega q(x)F(v(x))dx > 0\,.$$

Theorem 3.9. *Let a, p and f verify $H(a)$, $H(q)$, $H(f)$ and (f_3), (f_4), (f_5), respectively. Assume further that $\mu < \frac{\hat{c}\mu_1}{p-1}$. Then for each $\lambda \in \Lambda := \left[\frac{\Phi(v)}{\Upsilon(v)}, +\infty\right[$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions.*

Proof. Our assumptions (f_1) and (f_4) guarantee that corresponding to $\varepsilon > 0$ we can find a positive number $A(\varepsilon)$ such that
$$|F(\xi)| \leq \varepsilon\xi^p + A(\varepsilon)\xi^{s+1} \;\forall\; \xi \geq 0\,,$$
so
$$\overline{\Upsilon}(u) \leq \|q\|_\infty \left(\varepsilon c_1^p\|u\|^p + A(\varepsilon)c_{s+1}^{s+1}\|u\|^{s+1}\right) \;\forall u \in K\,. \tag{3.21}$$
Using (3.9) and (3.21) we obtain the upper bound:
$$\begin{aligned}\varphi_1(r) &\leq \frac{\sup_{\Phi(u)\,<\,r}\overline{\Upsilon}(u)}{r}\\ &\leq \frac{\|q\|_\infty\left(\frac{\mu_1p(p-1)}{\mu_1\hat{c}-\mu(p-1)}\varepsilon c_1^p r + A(\varepsilon)c_{s+1}^{s+1}\left(\frac{\mu_1p(p-1)}{\mu_1\hat{c}-\mu(p-1)}\right)^{s+1/p} r^{s+1/p}\right)}{r} \qquad (3.22)\\ &= \|q\|_\infty\left(\frac{\mu_1p(p-1)}{\mu_1\hat{c}-\mu(p-1)}\varepsilon c_1^p + A(\varepsilon)c_{s+1}^{s+1}\left(\frac{\mu_1p(p-1)}{\mu_1\hat{c}-\mu(p-1)}\right)^{\frac{s+1}{p}} r^{\frac{s+1-p}{p}}\right)\\ &\qquad\qquad \forall\varepsilon,\, r > 0\,.\end{aligned}$$

Since $s+1-p > 0$, from (3.22) we deduce that, corresponding to $\sigma > 0$ we can find $\rho > 0$ such that
$$\varphi_1(r) \leq \frac{\sup_{\Phi(u)\,<\,r}\overline{\Upsilon}(u)}{r} < \sigma \;\forall r < \rho\,. \tag{3.23}$$

Fix $\lambda \geq \frac{\Phi(v)}{\overline{\Upsilon}(v)}$ and choose σ such that $0 < \sigma < \frac{1}{\lambda} \leq \frac{\overline{\Upsilon}(v)}{\Phi(v)}$; if we take $r < min\{\Phi(v), \rho\}$, then (3.23) and our choice lead to

$$\varphi_2(r) \geq \frac{\overline{\Upsilon}(v) - \sigma r}{\Phi(v) - r} \geq \frac{\frac{\Phi(v)}{\lambda} - \sigma r}{\Phi(v) - r} > \frac{\frac{\Phi(v)-r}{\lambda}}{\Phi(v) - r} = \frac{1}{\lambda} > \sigma > \varphi_1(r)\,.$$

Clearly all the other assumptions of Theorem 2.5 are satisfied, so for each $\lambda \in \Lambda := \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions. The conclusion follows from the inequalities $\frac{1}{\varphi_2(r)} < \lambda < \frac{1}{\varphi_1(r)}$. □

Remark 3.10. *When a is the $p-$Laplacian then $\hat{c} = \tilde{c} = p - 1$. So (ii) becomes $\frac{\overline{\Upsilon}(\phi)}{\Phi(\phi)} > \frac{\mu_1 p \|q\|_\infty R^{1-p}}{\mu_1 - \mu}\left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s\right)$ and also the constants involved in theorems 3.6 and 3.7 take a simpler form.*

Remark 3.11. *When $\mu = 0$ then we have a non perturbed problem and (ii) is $\frac{\overline{\Upsilon}(\phi)}{\Phi_0(\phi)} > \frac{p(p-1)\|q\|_\infty R^{1-p}}{\hat{c}}\left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s\right)$ and also the constants involved in theorems 3.6 and 3.7 take a simpler form. Finally, for the non perturbed problem with the $p-$Laplacian then we have $\frac{\overline{\Upsilon}(\phi)}{\|\phi\|^p} > \|q\|_\infty R^{1-p}\left(a_1 c_1 + \frac{a_2 c_{s+1}^{s+1}}{s+1} R^s\right)$.*

Corollary 3.12. *Let a, q verify $H(a)$ and $H(q)$ respectively. Let $s \in]p-1, p^* - 1]$ and $f(\xi) = |\xi|^{s-1}\xi$, for all $\xi \in \mathbb{R}$. Assume that there exist a nonnegative function $v \in K$, such that $B = \int_\Omega q(x) v^{s+1}(x)dx > 0$. Then for each $\lambda \in \Lambda := \left[\frac{(\mu_1\tilde{c} - \mu(p-1))(s+1)\|v\|^p}{\mu_1 p(p-1)B}, +\infty\right[$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions.*

Proof. Our choice of f forces $\overline{\Upsilon}(u) = \frac{1}{s+1}\int_\Omega q(x) u^{s+1}(x)dx$ for all $u \in W_0^{1,p}(\Omega)$ and

$$\frac{\Phi(v)}{\Upsilon(v)} \leq \frac{(\mu_1\tilde{c} - \mu(p-1))(s+1)\|v\|^p}{\mu_1 p(p-1)B}\,.$$

Since all the assumptions of Theorem 3.9 are satisfied, the conclusion now follows by that result. □

Corollary 3.13. *Under the same assumptions of Corollary 3.12, let A be the $p-$Laplacian. Then for each $\lambda \in \Lambda := \left[\frac{(\mu_1 - \mu)(s+1)\|v\|^p}{\mu_1 p B}, +\infty\right[$, $(P_{\lambda,\mu})$ has three distinct nonnegative solutions.*

4. Existence results for $(P_{\lambda,\nu})$

In this Section we discuss the existence of solutions to $(P_{\lambda,\nu})$, involving a slow perturbation of the homogeneous operator B introduced in Section 2. We recall that now we weaken the regularity assumptions in $H(a)$ (see $H(b)$ in Section 2), and the perturbation depends on the first eigenvalue of the new operator.
So, let ν_1 be the first eigenvalue for the Dirichlet problem related to B. If $\nu < \nu_1$ then the quantity

$$\|u\|_0 = \left(\langle Bu\,, u\rangle - \nu \int_\Omega u^2(x)dx\right)^{\frac{1}{2}}$$

is a norm on $H_0^1(\Omega)$, equivalent to $\|u\| = \langle Bu\,, u\rangle^{\frac{1}{2}}$ previously adopted:

$$\left(1 - \frac{\nu}{\nu_1}\right)\|u\|^2 \le \|u\|_0^2 \le \|u\|^2 \quad \forall u \in H_0^1(\Omega). \tag{4.1}$$

If we consider the problem

$$(P_{\lambda,\nu})\begin{cases} \text{Find } u \in H_0^1(\Omega),\ u \le \psi \text{ in } \Omega, \text{ satisfying} \\ \langle Bu, v-u\rangle - \nu \int_\Omega u(v-u)dx + \lambda \int_\Omega J^0(x, u(x); (v-u)(x))\, dx \ge 0 \\ \text{for all } v \in H_0^1(\Omega),\, v \le \psi \text{ in } \Omega, \end{cases}$$

and truncate f exactly as in previous section, then the energy functional related to the truncated problem is

$$\overline{I}_{\lambda,\nu}(u) = \frac{1}{2}\|u\|_0^2 - \lambda \int_\Omega q(x)\overline{F}(u)dx + \lambda j(u) = \Phi_\nu(u) - \lambda\overline{\Upsilon}(u) + \lambda j(u).$$

The assumptions $H(b)$ are weaker than $H(a)$, but the structure of the problem is as $(P_{\lambda,\mu})$, the functional Φ_ν has the same regularity properties of Φ (see (3.3), (3.2), (4.1)) and $\overline{I}_{\lambda,\nu}$ satisfies the $(PS)_c$-condition too. Hence we obtain the following existence results:

Theorem 4.1. *Let b, q and f verify $H(b)$, $H(q)$ and $H(f)$, respectively. Assume further that $\nu < \nu_1$ and $\exists \phi \in K$ and $R > 0$ such that*

(i) $\|\phi\|_0 > R$;

(ii) $\frac{\overline{\Upsilon}(\phi)}{\|\phi\|_0^2} > \frac{\nu_1\|q\|_\infty R^{1-p}}{\nu_1-\nu}\left(a_1c_1 + \frac{a_2c_{s+1}^{s+1}}{s+1}R^s\right)$

Then there exists $r > 0$ such that for each $\lambda \in \Lambda := \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$, $(P_{\lambda,\nu})$ has three distinct nonnegative solutions.

Let

$$A_{r,\nu} = \sup_{\Phi_\nu(u)\,\le\, r,\ u\,\in\, K} \overline{\Upsilon}(u),$$

$$k_{+,\nu} = \left[\frac{2^{N-2}\Gamma(1+\frac{N}{2})}{(\nu_1-\nu)\pi^{\frac{N}{2}}\,(2^N-1)\,D_+^{N-2}}\right]^{\frac{1}{2}},\ k_{-,\nu} = \left[\frac{2^{N-2}\Gamma(1+\frac{N}{2})}{(\nu_1-\nu)\pi^{\frac{N}{2}}\,(2^N-1)\,D_-^{N-2}}\right]^{\frac{1}{2}}.$$

Theorem 4.2. *Let b, q and f verify $H(b)$, $H(q)$ and $H(f)$, respectively. Assume further that $\nu < \nu_1$, $q_+ > 0$ and there exist $r > 0$ and $b > 0$ such that $b > k_{+,\nu}\sqrt{r}$, $F(t) \ge 0$ in $[0, b]$, $\frac{F(b)}{b^2}\cdot\frac{q_+}{2(\nu_1-\nu)}\cdot\frac{D_+^2}{2^N-1} > \frac{A_{r,\nu}}{r}$, $\psi(x) \ge b$ in $B(x_0, D_+)$. Then for each $\lambda \in \Lambda := \left]\frac{2(\nu_1-\nu)(2^N-1)}{D_+^2}\frac{b^2}{q_+F(b)}, \frac{r}{A_{r,\nu}}\right[$, $(P_{\lambda,\nu})$ has three distinct nonnegative solutions.*

Theorem 4.3. *Let b, q and f verify $H(b)$, $H(q)$ and $H(f)$, respectively. Assume further that $\nu < \nu_1$, $q_- > 0$ and there exist $\sigma > 0$ and $b > 0$ such that $b > k_{-,\nu}\sqrt{r}$, $F(t) \le 0$ in $[0, b]$, $\frac{F(b)}{b^2}\cdot\frac{q_-}{2(\nu_1-\nu)}\cdot\frac{D_-^2}{2^N-1} > \frac{A_{r,\nu}}{r}$, $\psi(x) \ge b$ in $B(x_1, D_-)$. Then for each $\lambda \in \Lambda := \left]\frac{2(\nu_1-\nu)(2^N-1)}{D_-^2}\frac{b^2}{q_-F(b)}, \frac{r}{A_{r,\nu}}\right[$, $(P_{\lambda,\nu})$ has three distinct nonnegative solutions.*

Theorem 4.4. *Let b, q and f verify $H(b)$, $H(q)$, $H(f)$, (f_3), (f_4) and (f_5) with $p = 2$, respectively. Assume further that $\nu < \nu_1$. Then for each $\lambda \in \Lambda := \left[\frac{\Phi_\nu(v)}{\Upsilon(v)}, +\infty\right[$, $(P_{\lambda,\nu})$ has three distinct nonnegative solutions.*

Remark 4.5. *Theorem 4.4 is a generalization of Theorem 3.1 of* [1] *and Theorem 1 of* [13]*. In fact both results deals with $\lambda = 1$ and they require the condition $(MP2)$: there exists a nonnegative function $v \in K$, such that*

$$\int_\Omega q(x)F(v(x))dx \geq \frac{\|v\|_0^2}{2},$$

leading to $\frac{\Phi_\nu(v)}{\Upsilon(v)} \leq 1$, so the results hold true by choosing $\lambda = 1$ in our Theorem 4.4. We point out that in both results f has subcritical growth, while in our theorems we can choose also $s = 2^* - 1$.

5. Examples

In this section we give some examples to which we can apply one of the previous results.

Example 5.1. *Let us consider the functions: $f : \mathbb{R} \to \mathbb{R}$ and $\psi : [-4, 4] \to \mathbb{R}$ defined as*

$$f(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ 1 + \xi^3 & \text{if } \xi > 0, \end{cases} \quad , \ \psi(x) = (4 - |x|)^5,$$

and $q : (-4, 4) \to \mathbb{R}$ satisfying $H(q)$, and

$$q(x) \geq \ q_+ = \frac{1}{100} \quad \text{a.e. } x \in \ [-1, 1], \ \|q\|_\infty = 1.$$

Under the assumptions above the problem

$$\begin{cases} \text{Find } u \in H_0^1((-4,4)), \ u \leq \psi \text{ in } (-4,4), \text{ satisfying} \\ \\ \int_{-4}^4 u'(v' - u')dx \geq 2^{-6} \int_{-4}^4 q(x)f(u(x))(v - u)(x)dx \\ \\ \text{for all } v \in H_0^1((-4,4)), \ v \leq \psi \text{ in } (-4,4), \end{cases}$$

as three distinct nontrivial nonnegative solutions.
We are going to show that the functions involved satisfy the assumptions of Theorem 3.6 (or of Theorem 4.2). To this end we observe that $\mu = 0$, $B(x_0, D_+) = (-1, 1)$, $D_+ = 1$, $k_{+,0} = \frac{1}{2}$, and

$$F(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ \xi + \frac{\xi^4}{4} & \text{if } \xi > 0. \end{cases}$$

Pick $r = \frac{\sqrt[3]{2}}{2}$ and $b = \psi(1) = 3^5 > \frac{1}{2}\sqrt{r}$, in order to obtain

$$\frac{F(b)}{b^2} \cdot \frac{q_+}{2} > \frac{b^2 q_+}{2^3} = \frac{3^{10}}{800}.$$

Taking into account the inequalities

$$\|u\|_\infty \leq \sqrt{2}\|u\| \ \ \forall u \in H_0^1((-4,4)), \text{ and } \Phi_0(u) = \frac{1}{2}\|u\|^2 \leq r, \tag{5.1}$$

our choice of r forces

$$\frac{A_{r,0}}{r} \leq \frac{\int_{-4}^{4}\left(|u(x)| + \frac{|u(x)|^4}{4}\right) dx}{r} \leq 32\sqrt[3]{2} < \frac{3^{10}}{800}\,.$$

Hence, we can apply Theorem 3.6. Since $2^{-6} \in]\frac{800}{3^{10}}, \frac{1}{32\sqrt[3]{2}}[\subseteq \Lambda$, *where* Λ *is the interval obtained in the Theorem 3.6, our problem has three distinct nonnegative solutions. It remains to show that any solution* u *is nontrivial. In fact if* $u \equiv 0$ *where a solution then*

$$0 \geq 2^{-6}\int_{-4}^{4} q(x)v(x)dx \text{ for all } v \in H_0^1((-4,4)),\ v \leq \psi \text{ in } (-4,4)\,. \tag{5.2}$$

If we take $v \in H_0^1((-4,4))$, *with* $\mathrm{supp} v \subseteq [-1,1]$ *and* $v > 0$ *a.e. in* $[-1,1]$, *then we obtain* $\int_{-4}^{4} q(x)v(x)dx > 0$, *and this contradicts (5.2), so* $u \equiv 0$ *is not a solution.*

Remark 5.2. *The techniques used in previous example allow to handle many problems where* F *has constant sign and* f *has growth greater than* $p-1$ *for* ξ *sufficiently large (not necessarily continuous for* $\xi > 0$*), but fails to satisfy the assumptions of Theorem 3.9. In fact when* ψ *is big enough on a subset of* Ω *where* $q(x)$ *has constant sign, then the inequalities assumed in one of Theorems 3.6, 3.7, 4.2 and 4.3 can be easily verified.*

Example 5.3. *Now, we give an application of Corollary 3.13 for a problem with critical growth. Take* $p = 3$, $N = 6$ *(hence* $p^* = 6$*),* $\Omega = B(0,10)$ *and consider the following problem*

$$(P_1)\begin{cases} \text{Find } u \in W_0^{1,3}(\Omega),\ u \leq \psi \text{ in } \Omega, \text{ satisfying} \\ \int_\Omega |\nabla u|\nabla u \nabla(v-u)dx \geq \int_\Omega q(x)|u(x)|^4 u(x)(v-u)(x)dx \\ \text{for all } v \in W_0^{1,3}(\Omega),\ v \leq \psi \text{ in } \Omega, \end{cases}$$

where ψ *is a nonnegative function of* $W_0^{1,3}(\Omega)$, $q(x)$ *satisfies* $H(q)$, *and*

$$\psi \geq 1, q(x) \geq 0 \text{ a.e. in } B(0,1), \text{ and also } q(x) > 2^{13} \text{ a.e. in } B\left(0,\frac{1}{2}\right).$$

In order to apply Corollary 3.13, we must only verify the condition $B > 0$, *because the other assumptions are satisfied. To this end, we take*

$$v(x) = \begin{cases} 1-|x| & \text{if } |x| \leq 1\,, \\ 0 & \text{if } 1 < |x| < 10\,. \end{cases}$$

v *is a nonnegative function belonging in* K, $\|v\|_3^3 = |B(0,1)|$ *and*

$$\frac{1}{6}B = \frac{1}{6}\int_\Omega q(x)|v(x)|^6 dx > \frac{2^{13}}{6}\int_{B(0,\frac{1}{2})}\frac{1}{2^6}dx = \frac{2^6}{3}\left|B\left(0,\frac{1}{2}\right)\right| = \frac{1}{3}\|v\|_3^3\,. \tag{5.3}$$

Since $1 \in \Lambda := \left[\frac{6\|v\|^3}{3B}, +\infty\right[$, (P_1) *has three distinct nonnegative solutions.*

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