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MAPPINGS UNDER ASYMPTOTIC POINTWISE CONTRACTIVE TYPE CONDITIONS

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Abstract. In this paper, we introduce new asymptotic pointwise contractive type conditions and present fixed point theorems for mappings under such conditions in normed and Banach spaces.

1. Introduction

Let (M, d) be a metric space. A mapping $T : M \to M$ is called a *pointwise contraction* if there exists a function $\alpha : M \to [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha(x)d(x, y) \quad \text{for each } y \in M.$$

The notion of pointwise contractions was introduced in [2, 5]. Recently, Kirk [6] introduced an asymptotic version of the pointwise contraction as follows:

Definition 1.1. A mapping $T : M \to M$ is called an *asymptotic pointwise contraction* if there exists a function $\alpha : M \to [0, 1)$ such that, for each integer $n \geq 1$,

$$d(T^n x, T^n y) \leq \alpha_n(x)d(x, y) \quad \text{for each } x, y \in M,$$

where $\alpha_n \to \alpha$ pointwise on M.

The central fixed point result for such mappings is the following.

Theorem 1.2 ([7]). *Let C be a weakly compact convex subset of a Banach space E and let $T : C \to C$ be an asymptotic pointwise contraction. Then T has a unique fixed point $v \in C$, and for each $x \in C$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to v.*

Since then there have been numerous extensions of this fact; see, e.g., [1, 4, 8].

In this paper, we introduce new asymptotic pointwise contractive type conditions and present fixed point theorems for mappings under such conditions in normed and Banach spaces. In normed spaces, we discuss an asymptotic behavior of a mapping of asymptotic pointwise contraction type. Finally, we prove a fixed point theorem for mappings that are not necessarily continuous and that have an asymptotic contractive condition in a subset of a normed space. Our results extend and improve, for example, the corresponding result of Kirk and Xu [7] (Theorem 1.2, above).

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2. Asymptotic pointwise contraction types

In this section, motivated by the concept of asymptotic pointwise contraction (Definition 1.1), we introduce the following types of mappings with weaker assumptions.

Definition 2.1. Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be of *asymptotic pointwise contraction type* (resp. of *weak asymptotic pointwise contraction type*) if T^N is continuous for some integer $N \geq 1$ and there exists a function $\alpha : M \to [0, 1)$ such that, for each x in M,

$$\limsup_{n\to\infty} \sup_{y\in M} \{d(T^n x, T^n y) - \alpha_n(x)d(x, \tag{2.1}$$

$$(\text{resp. } \liminf_{n\to\infty} \sup_{y\in M} \{d(T^n x, T^n y) - \alpha_n(x)d(x, y)\} \leq 0), \tag{2.2}$$

where $\alpha_n \to \alpha$ pointwise on M.

Taking

$$r_n(x) = \sup_{y\in M} \{d(T^n x, T^n y) - \alpha_n(x)d(x, y)\} \in \mathbb{R}^+ \cup \{\infty\},$$

it can be easily seen from (2.1) (resp. (2.2)) that

$$\lim_{n\to\infty} r_n(x) = 0 \tag{2.3}$$

$$(\text{resp. } \liminf_{n\to\infty} r_n(x) \leq 0), \tag{2.4}$$

for all $x \in M$, and

$$d(T^n x, T^n y) \leq \alpha_n(x)d(x, y) + r_n(x). \tag{2.5}$$

It is easy to see that an asymptotic pointwise contraction is of asymptotic pointwise contraction type; but, the converse is not true:

Example 2.2. Let $M = \mathbb{R}^n$ for $n > 1$, equipped with the Euclidean norm. For each $(x_1, x_2, \ldots, x_n)$, define

$$T(x_1, x_2, \ldots, x_n) = (f(x_2), f(x_3), \ldots, f(x_n), 0),$$

where $f : \mathbb{R} \to \mathbb{R}$ is some discontinuous function with $f(0) = 0$. We deduce that T is discontinuous, and then, it would not an asymptotic pointwise contraction. But, we see that $T^n x = 0$, $\forall x \in \mathbb{R}^n$, and so, T is of asymptotic pointwise contraction type.

Example 2.3. Let $M = \prod_{n\geq 1}[0, \frac{1}{n}] \subseteq C_0(\mathbb{N})$. For each $x = (x_1, x_2, x_3, \ldots)$ in M, define

$$T(x_1, x_2, x_3, \ldots) = (f(x_2), x_3, x_4, \ldots),$$

where $f : [0, 1] \to [0, 1]$ is a nonexpansive mapping. It is easy to see that T is a continuous nonlinear mapping from M to M which is of asymptotic pointwise contraction type. In fact, we notice that for every $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ in M,

$$\begin{aligned} \|T^n x - T^n y\| &= \|(f(x_{n+1}), x_{n+2}, x_{n+3}, \ldots) - (f(y_{n+1}), y_{n+2}, y_{n+3}, \ldots)\| \\ &\leq \sup\{|x_i - y_i| : i \geq n+1\} \leq \frac{1}{n+1}. \end{aligned}$$

Hence, for $\alpha_n(x) \to \alpha(x) < 1$, we have

$$\sup_{y\in M}(\|T^n x - T^n y\| - \alpha_n(x)\|x-y\|) \le \frac{1}{n+1} \to 0, \quad \text{as } n\to\infty.$$

But, T is not an asymptotic pointwise contraction. Indeed, for any $x = (x_1, x_2, x_3, \dots) \in M$ and $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| = \|x-y\|,$$

for every $y = (y_1, y_2, y_3, \dots) \in M$ for which $y_i = x_i$, $i = 1, 2, \dots, n+1$.

We now recall the concept of asymptotic center:

Let E be a Banach space, C a subset of E, and $\{x_n\}$ a bounded sequence in E. The asymptotic center of $\{x_n\}$ relative to C, denoted $A_C(x_n)$, is the set of minimizers in C (if any) of the function f given by

$$f(x) = \limsup_{n\to\infty}\|x_n - x\|;$$

i.e.,

$$A_C(x_n) = \{x \in C : f(x) = \inf_C f\}.$$

It is known that $f : E \to \mathbb{R}_+$ is convex, nonexpansive and hence weak lower semicontinuous. Moreover, if C is weakly compact, then $A_C(x_n)$ is nonempty (see [3, Lemma 9.1]).

Parallel to the proof of [7, Theorem 3.1] we employ the technique of asymptotic centers to prove the following extension of Theorem 1.2.

Theorem 2.4. *Let C be a nonempty weakly compact subset of a Banach space E and let $T : C \to C$ be a mapping of weak asymptotic pointwise contraction type. Then T has a unique fixed point $v \in C$ and for each $x \in C$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to v.*

Proof. Fix an $x \in C$ and define a function f by

$$f(u) = \limsup_{n\to\infty}\|T^n x - u\|, \quad u \in C.$$

Note that f satisfies the property

$$f(T^m u) \le \alpha_m(u) f(u) + r_m(u), \quad u \in C,\ m \ge 1. \tag{2.6}$$

Indeed, by (2.5), we have

$$\begin{aligned} f(T^m u) &= \limsup_{n\to\infty}\|T^n x - T^m u\| = \limsup_{n\to\infty}\|T^{m+n}x - T^m u\| \\ &\le \limsup_{n\to\infty}\alpha_m(u)\|T^n x - u\| + r_m(u) = \alpha_m(u) f(u) + r_m(u). \end{aligned}$$

Since C is weakly compact, $A_C(T^n x)$ is a nonempty subset of C. So, we may take a $u \in A_C(T^n x)$. Since T is of weak asymptotic pointwise contraction type, by (2.4), we have $\liminf_{m\to\infty} r_m(u) \le 0$. Thus, for a subsequence $\{r_{m_k}(u)\}$ of $\{r_m(u)\}$, we have

$$\lim_{k\to\infty} r_{m_k}(u) \le 0. \tag{2.7}$$

On the other hand, since C is weakly compact, there exists a subsequence of $\{T^{m_k}u\}$ converging weakly to some w in C. Without loss of generality, we may assume $T^{m_k}u \rightharpoonup w$. Now, by (2.6), (2.7) and the weak lower semicontinuity of f, we obtain

$$f(w) \leq \liminf_{k\to\infty} f(T^{m_k}u) \leq \liminf_{k\to\infty}[\alpha_{m_k}(u)f(u) + r_{m_k}(u)] = \alpha(u)f(u).$$

But, since $w \in C$ and $u \in A_C(T^n x)$, we get

$$f(u) \leq f(w) \leq \alpha(u)f(u).$$

Hence $f(u) = 0$. This implies that $T^n x \to u$, in norm. From this and the continuity of T^N, for some $N \geq 1$, it follows that

$$T^N u = T^N(\lim_{n\to\infty} T^n x) = \lim_{n\to\infty} T^{n+N}x = u;$$

namely, u is a fixed point of T^N. Now, repeating the above proof for u instead of x, we deduce that $T^n u$ converges, in norm, to a member of C. But, $T^{kN}u = u$, for all $k \geq 1$. Hence, $T^n u \to u$, in norm. We show that $Tu = u$; for this purpose, consider an arbitrary $\epsilon > 0$. Then, there exists a $K_0 > 0$ such that $\|T^n u - u\| < \epsilon$, for all $n > K_0$. So, by choosing a natural number $k > K_0/N$, we obtain

$$\|Tu - u\| = \|T(T^{kN}u) - u\| = \|T^{kN+1}u - u\| < \epsilon.$$

Since the choice of $\epsilon > 0$ is arbitrary, we get $Tu = u$.

It is easy to verify that T can have only one fixed point. Indeed, if $v \in C$ is also a fixed point of T. Then, by (2.5), we have

$$\|u - v\| = \|T^n u - T^n v\| \leq \alpha_n(u)\|u - v\| + r_n(u), \quad \forall n \geq 1.$$

Taking liminf in the above inequality, it follows that

$$\|u - v\| \leq \alpha(u)\|u - v\|.$$

Since $\alpha(u) < 1$, we immediately get $u = v$. □

Next, we present a similar result for the class of mappings of asymptotic pointwise contraction type in a more general setting. In fact, we can replace the weak compactness by a weaker condition. In what follows, we will apply the notation

$$\omega_w(\{x_n\}) = \{y \in E : \ y = w - \lim_k x_{n_k}, \text{ for some } n_k \to \infty\},$$

for a sequence $\{x_n\}$ of a Banach space E.

Theorem 2.5. *Let C be an arbitrary subset of a Banach space E, $T : C \to C$ a mapping of asymptotic pointwise contraction type and D a nonempty weakly compact subset of C such that, for all $x \in D$, $\omega_w(\{T^n x\}) \cap D \neq \emptyset$. Assume also that some orbit of T is bounded. Then T has a unique fixed point $v \in C$ and for each $x \in C$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to v.*

Proof. Because an orbit of T is bounded and T is of asymptotic pointwise contraction type, it easily follows that, for every $x \in C$, $\{T^n x\}$ is bounded. Now, fix an $x \in C$ and define a function f by

$$f(u) = \limsup_{n\to\infty} \|T^n x - u\|, \quad u \in D.$$

Like that in proof of Theorem 2.4, we can show that f satisfies the property

$$f(T^m u) \le \alpha_m(u) f(u) + r_m(u), \quad u \in D,\ m \ge 1. \tag{2.8}$$

Since D is weakly compact, $A_D(T^n x)$ is a nonempty subset of D. So, we may take a $u \in A_D(T^n x) \subset D$. By the assumption, we have $\omega_w(\{T^n u\}) \cap D \neq \emptyset$. So, there exists a subsequence $\{T^{n_k} u\}$ of $\{T^n u\}$ converging weakly to some w in D. Note that, by (2.3), $\lim_{n\to\infty} r_n(u) = 0$. So, by (2.8) and the weak lower semicontinuity of f, we obtain

$$f(w) \le \liminf_{k\to\infty} f(T^{n_k} u) \le \liminf_{k\to\infty} [\alpha_{n_k}(u) f(u) + r_{n_k}(u)] \le \alpha(u) f(u).$$

But, since $w \in D$ and $u \in A_D(T^n x)$, we get

$$f(u) \le f(w) \le \alpha(u) f(u).$$

Hence $f(u) = 0$. This implies that $T^n x \to u$, in norm. Because T^N is continuous, for some $N \ge 1$, it follows that $T^N u = u$. Now, by (2.5), we obtain

$$\|u - Tu\| = \|T^{kN} u - T^{kN+1} u\| \le \alpha_{kN}(u) \|u - Tu\| + r_{kN}(u), \quad \forall k \in \mathbb{N}.$$

So, by (2.3), it follows that $\|u - Tu\| \le \alpha(u)\|u - Tu\|$. Hence, $Tu = u$. The rest of the proof is the same as that in Theorem 2.4. □

Corollary 2.6. *Let C be a nonempty weakly compact subset of a Banach space E and let $T : C \to C$ be an asymptotic pointwise contraction. Then T has a unique fixed point $v \in C$ and for each $x \in C$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to v.*

Corollary 2.7. *Let C be an arbitrary subset of a Banach space E and $T : C \to C$ a mapping of asymptotic pointwise contraction type. Assume that, for some $v \in C$ and some subsequence $\{T^{n_k} v\}$ of $\{T^n v\}$, $w - \lim_{k\to\infty} T^{n_k} v = v$. Assume also that some orbit of T is bounded. Then, v is a unique fixed point for T and, for each $x \in C$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to v.*

Proof. Taking $D = \{v\}$, it is easy to see from the assumption that $\omega_w(\{T^n v\}) \cap D \neq \emptyset$. Now, applying Theorem 2.5, the desired result follows. □

3. Mappings under conditions on orbits in normed spaces

In this section, we present some fixed point results in the context of normed spaces. We begin with the following lemma.

Lemma 3.1. *let C be a subset of a normed space E, $v \in C$, $\{n_k\}$ a strictly increasing sequence of natural numbers and $T : C \to C$ a mapping for which*

(a) $\limsup_{n\to\infty} \sup_{y\in O_T(v)} (\|T^n v - T^n y\| - \alpha_n \|v - y\|) \le 0$, *where* $\alpha_n \to \alpha \in [0, 1)$;
(b) $w - \lim_{k\to\infty} T^{n_k + l} v = v$, *for each integer* $l \ge 0$.

Then, $\lim_{k\to\infty} T^{n_k+l} v = v$, for each integer $l \ge 0$. Moreover, if, for some integer $N \ge 1$, T^N is continuous at v, then $Tv = v$.

Proof. Taking

$$r_n = \sup_{y\in O_T(v)} (\|T^n v - T^n y\| - \alpha_n \|v - y\|),$$

we get, by (a),

$$\limsup_{n\to\infty} r_n \le 0. \tag{3.1}$$

Now, for $y \in O_T(v)$, we have

$$\|T^n v - T^n y\| \le \alpha_n \|v - y\| + r_n. \tag{3.2}$$

On the other hand, (b) implies that

$$w - \lim_{k\to\infty} T^{n_k+2l+n_h} v = v, \tag{3.3}$$

for all $h \ge 1$ and $l \ge 0$. So, by (3.2), (3.3) and the weak lower semicontinuity of the norm, we have

$$\begin{aligned} \|T^{n_h+l}v - v\| &\le \liminf_{k\to\infty} \|T^{n_h+l}v - T^{n_k+2l+n_h}v\| \\ &\le \alpha_{n_h+l} \liminf_{k\to\infty} \|v - T^{n_k+l}v\| + r_{n_h+l}. \end{aligned}$$

Taking the limit superior as $h \to \infty$ and using (3.1), we obtain

$$\limsup_{h\to\infty} \|v - T^{n_h+l}v\| \le \alpha \liminf_{k\to\infty} \|v - T^{n_k+l}v\|.$$

Since $\alpha < 1$, it follows that

$$\lim_{k\to\infty} \|v - T^{n_k+l}v\| = 0;$$

that is,

$$\lim_{k\to\infty} T^{n_k+l}v = v, \quad \forall l \ge 0.$$

If T^N is also continuous,

$$T^N(v) = T^N(\lim_{k\to\infty} T^{n_k}v) = \lim_{k\to\infty} T^{n_k+N}v = v;$$

namely, v is a fixed point of T^N. We will show that $Tv = v$. Notice that $T^{kN}(v) = v$, $(\forall k \ge 1)$. Thus,

$$\|v - Tv\| = \|T^{kN}(v) - T^{kN}(Tv)\|, \quad \forall k \ge 1,$$

which implies, by (3.1) and (3.2), that

$$\|v - Tv\| = \limsup_{k\to\infty} \|T^{kN}(v) - T^{kN}(Tv)\| \le \alpha\|v - Tv\|.$$

Hence, $Tv = v$. □

Theorem 3.2. *Let C be a nonempty subset of a normed space E and $T : C \to C$ be a mapping of asymptotic pointwise contraction type. Assume that for some $v \in C$ and some subsequence $\{T^{n_k}v\}$ of $\{T^n v\}$ we have $w - \lim_{k\to\infty} T^{n_k+l}v = v$, for each integer $l \ge 0$. Then, v is a unique fixed point for T and, for each $x \in C$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to v.*

Proof. It follows from Lemma 3.1 that $\lim_{k\to\infty} T^{n_k+l}v = v$, for each integer $l \ge 0$. Because T is of asymptotic pointwise contraction type,

$$\begin{aligned} \|T^m v - T^{n_k+m}v\| &\le \alpha_m(v)\|v - T^{n_k}v\| + r_m(v) \\ &= \alpha_m(v) \lim_{h\to\infty} \|T^{n_h+n_k}v - T^{n_k}v\| + r_m(v) \end{aligned}$$

$$
\begin{aligned}
&= \alpha_m(v) \lim_{h\to\infty} \|T^{n_k-m}(T^m v) - T^{n_k-m}(T^{m+n_h}v)\| + r_m(v) \\
&\leq \alpha_m(v)\alpha_{n_k-m}(T^m v) \lim_{h\to\infty} \|T^m v - T^{m+n_h}v\| \\
&\quad + r_{n_k-m}(T^m v) + r_m(v), \\
&= \alpha_m(v)\alpha_{n_k-m}(T^m v)\|T^m v - v\| + r_{n_k-m}(T^m v) + r_m(v),
\end{aligned}
$$

for all $m \in \mathbb{N}$ and $n_k > m$. Taking the limit as $k \to \infty$, we obtain

$$
\begin{aligned}
\|T^m v - v\| &\leq \alpha_m(v)\alpha(T^m v)\|T^m v - v\| + r_m(v) \\
&\leq \alpha_m(v)\|T^m v - v\| + r_m(v), \quad \forall m \in \mathbb{N}.
\end{aligned}
$$

That is,

$$
(1 - \alpha_m(v))\|T^m v - v\| \leq r_m(v), \quad \forall m \in \mathbb{N}.
$$

Since $\alpha_m(v) \to \alpha(v) < 1$ and $r_m(v) \to 0$, we get $\|T^m v - v\| \to 0$, as $m \to \infty$.

Because T is of asymptotic pointwise contraction type, we have, for each $x \in C$,

$$
\|T^n v - T^{n+m}x\| \leq \alpha_n(v)\|v - T^m x\| + r_n(v).
$$

From this and $T^n v \to v$, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} \|v - T^n x\| &= \limsup_{n\to\infty} \|v - T^{n+m}x\| = \limsup_{n\to\infty} \|T^n v - T^{n+m}x\| \\
&\leq \alpha(v)\|v - T^m x\|.
\end{aligned}
$$

Consequently,

$$
\limsup_{n\to\infty} \|v - T^n x\| \leq \alpha(v) \liminf_{m\to\infty} \|v - T^m x\|,
$$

which implies that

$$
\limsup_{n\to\infty} \|v - T^n x\| = 0.
$$

This, concludes the desired result. □

Finally, we state a result for noncontinuous mappings. It is worth mentioning that existence results for fixed point of noncontinuous contractive mappings of asymptotic types are rather rare.

In the following, we make the convention $\sup \varnothing = -\infty$.

Theorem 3.3. *let C be a subset of a normed space E, $v \in C$, $\{n_k\}$ a strictly increasing sequence of natural numbers and $T : C \to C$ a mapping for which*

(a) $\forall x \in \mathcal{O}(v)$, $\limsup_{n\to\infty} \sup(\|T^n x - T^n y\| - \|x - y\| : y \in \mathcal{O}(x),\ y \neq x) < 0$;

(b) $w - \lim_{k\to\infty} T^{n_k+l}v = v$, *for each integer* $l \geq 0$.

Then, $Tv = v$.

Proof. Note that (b) implies

$$
w - lim_{k\to\infty} T(T^{n_k+l}v) = w - lim_{k\to\infty} T^{n_k+l}v = v.
$$

So, if we show that for some $l \in \{0, 1, 2, \dots\}$, $\{T^{n_k+l}v\}$ has a constant subsequence, then we have proved that $Tv = v$. Suppose, for contradiction, that $\{T^{n_k+l}v\}$ does not have any constant subsequence, for every l in $\{0, 1, 2, \dots\}$; i.e., for each l in $\{0, 1, 2, \dots\}$ and $z \in C$, there exists a natural number $M(l, z)$ such that

$$
T^{n_k+l}v \neq z, \ \ \forall k \geq M(l, z). \tag{3.4}
$$

By (a), there exists a natural number N_0 such that

$$\sup_{y\in\mathcal{O}(v),y\neq v}(\|T^{N_0}v-T^{N_0}y\|-\|v-y\|)<0. \tag{3.5}$$

Furthermore, we can choose a natural number N such that for every $n\geq N$,

$$\sup_{y\in\mathcal{O}(T^{N_0}v),y\neq T^{N_0}v}(\|T^n(T^{N_0}v)-T^ny\|-\|T^{N_0}v-y\|)<0. \tag{3.6}$$

Now, we claim that the following inequality holds:

$$\liminf_{k\to\infty}\|T^mv-T^{m+n_k}v\|\leq\liminf_{k\to\infty}\|T^{N_0}v-T^{n_k+N_0}v\|,\ \ \forall m>N+N_0. \tag{3.7}$$

Indeed, we notice that, by (3.4), we have $T^{n_k+N_0}v\neq T^{N_0}v$, for all $k\geq M(N_0,T^{N_0}v)$. From this and (3.6), it follows that, for every $n\geq N$,

$$\sup_{k\geq M(N_0,T^{N_0}v)}(\|T^n(T^{N_0}v)-T^n(T^{n_k+N_0}v)\|-\|T^{N_0}v-T^{n_k+N_0}v\|)<0,$$

i.e.,

$$\|T^n(T^{N_0}v)-T^n(T^{n_k+N_0}v)\|<\|T^{N_0}v-T^{n_k+N_0}v\|,$$

for every $n\geq N$ and $k\geq M(N_0,T^{N_0}v)$. Hence,

$$\liminf_{k\to\infty}\|T^{n+N_0}v-T^{n+N_0+n_k}v\|\leq\liminf_{k\to\infty}\|T^{N_0}v-T^{n_k+N_0}v\|,$$

for every $n\geq N$, and this is equivalent to (3.7).

In this stage, considering (3.5), take

$$r=\sup_{y\in\mathcal{O}(v),y\neq v}(\|T^{N_0}v-T^{N_0}y\|-\|v-y\|)<0, \tag{3.8}$$

and choose $K_0>M(0,v)$ such that

$$n_k>N+N_0,\quad\forall k>K_0. \tag{3.9}$$

From (3.4), we have $T^{n_k}v\neq v$, for all $k>K_0$. So, considering (3.8), we obtain

$$\|T^{N_0}v-T^{n_k+N_0}v\|-\|v-T^{n_k}v\|=\|T^{N_0}v-T^{N_0}(T^{n_k}v)\|-\|v-T^{n_k}v\|\leq r,$$

for every integer $k>K_0$.Thus, for a fixed $k>K_0$, we have, by (b) and the lower semicontinuity of norm,

$$\begin{aligned}\|T^{N_0}v-T^{n_k+N_0}v\| &\leq \|v-T^{n_k}v\|+r\leq\liminf_{h\to\infty}\|T^{n_h+n_k}v-T^{n_k}v\|+r\\ &= \liminf_{h\to\infty}\|T^{n_k}v-T^{n_k+n_h}v\|+r\\ &\leq \liminf_{h\to\infty}\|T^{N_0}v-T^{N_0+n_h}v\|+r,\end{aligned}$$

where, the last inequality follows by combining (3.9) and (3.7). Therefore, we have obtained the following:

$$\|T^{N_0}v-T^{n_k+N_0}v\|\leq\liminf_{h\to\infty}\|T^{N_0}v-T^{N_0+n_h}v\|+r,\quad\forall k>K_0.$$

From this, it follows that

$$\limsup_{k\to\infty}\|T^{N_0}v-T^{n_k+N_0}v\|\leq\liminf_{h\to\infty}\|T^{N_0}v-T^{N_0+n_h}v\|+r.$$

Equivalently,

$$0 \leq \limsup_{k\to\infty} \|T^{N_0}v - T^{n_k+N_0}v\| - \liminf_{h\to\infty} \|T^{N_0}v - T^{N_0+n_h}v\| \leq r,$$

a contradiction of (3.8). Therefore, we have proved that $Tv = v$. □

Example 3.4. Let $E = \ell^\infty$, the space of all bounded real-valued functions defined on $\mathbb{N}$ with supremum norm. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary mapping such that $f(\frac{1}{n}) = \frac{1}{n+1}$, for all $n \in \mathbb{N}$. For each $x = (x_1, x_2, x_3, \dots)$ in E, define

$$T(x_1, x_2, x_3, \dots) = (f(x_2), x_3, x_4, \dots).$$

It is worth noticing that, depending on the various choices of the function f, T is not necessarily continuous. It is easy to verify that, for $v = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$, the assertion (a) of Theorem 3.3 (also Lemma 3.1) holds. But, $Tv \neq v$. So, applying Theorem 3.3, we deduce that for any strictly increasing sequence $\{n_k\}$ of natural numbers, there exists an integer $l \geq 0$ such that the assertion $w{-}\lim_{k\to\infty} T^{n_k+l}v = v$ doesn't hold.

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