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ON GENERAL SYSTEM OF VARIATIONAL INEQUALITIES IN BANACH SPACES*

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Abstract. Let X be either uniformly smooth or a reflexive Banach space which has a weakly continuous duality map J_φ with gauge φ. In this paper we propose an implicit iterative method and another explicit iterative method for solving a general system of valriational inequalities in X. These two methods are based on Korpelevich's extragradient method and Halpern's iterative method. Furthermore, we prove that under appropriate conditions the suggested algorithms converge strongly to some solutions of the considered general system of variational inequalities.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $B_1, B_2 : C \to H$ be two mappings. Recently, Ceng, Wang and Yao [9] introduced and considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\text{(1.1)} \qquad \begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases}$$

which is called a general system of variational inequalities (GSVI), which μ_1 and μ_2 are two positive constants. The set of solutions of problem (1.1) is denoted by GSVI(C, B_1, B_2). It is clearly to see that problem (1.1) covers as special case the following classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\text{(1.2)} \qquad \langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C.$$

The solution set of the VIP (1.2) is denoted by VI(C, A). Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. It is now well known that the variational inequalities are equivalent to the fixed point problems, the origin of which can be traced back to Lions and Stampacchia [13]. This alternative formulation has been used to suggest and analyze projection iterative method for solving variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous.

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Recently, Ceng, Wang and Yao [9] transformed problem (1.1) into a fixed point problem in the following way:

Lemma 1.1 (see [9])**.** *(For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of problem (1.1) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C, \tag{1.3}$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

In particular, if the mappings $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$.

Let X be a real Banach space whose dual space is denoted by X^*. Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \Rightarrow \quad \|x + y\|/2 \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strict convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of X is said to be the Frechet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. Also, we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \tau\right\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [17], no Banach space is q-uniformly smooth for $q > 2$.

Let X^* be the dual of X. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Moreover, it is known that J is single-valued if and only if X is smooth, whereas if X is uniformly smooth, then the mapping J is uniformly continuous on bounded subsets of X. Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence.

Definition 1.2. Let $A : C \to X$ be a mapping of C into X. Then A is said to be

(i) accretive if for each $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that
$$\langle Ax - Ay, j(x-y)\rangle \geq 0,$$
where J is the normalized duality mapping;

(ii) α-strongly accretive if for each $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that
$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2,$$
for some $\alpha \in (0,1)$;

(iii) β-inverse-strongly-accretive if for each $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that
$$\langle Ax - Ay, j(x-y)\rangle \geq \beta\|Ax-Ay\|^2,$$
for some $\beta > 0$;

(iv) λ-strictly pseudocontractive if for each $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that
$$\langle Ax - Ay, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x-y-(Fx-Fy)\|^2$$
for some $\lambda \in (0,1)$.

Very recently, Yao, Liou, Kang and Yu [20] studied the following general system of variational inequalities (GSVI) in a real smooth Banach space X, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x-x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x-y^*)\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.4}$$

where C is a nonempty, closed and convex subset of X, $B_1, B_2 : C \to X$ are two nonlinear mappings and μ_1 and μ_2 are two positive constants. Here the set of solutions of GSVI (1.4) is denoted by GSVI(C, B_1, B_2). In particular, if $X = H$, a real Hilbert space, then GSVI (1.4) reduces to GSVI (1.1) which was considered by Ceng, Wang and Yao [9].

In [20], Yao, Liou, Kang and Yu constructed two algorithms for solving GSVI (1.4) in a uniformly convex and 2-uniformly smooth Banach space: one implicit algorithm and another explicit algorithm. They proved the strong convergence of the proposed methods [20, Theorems 3.5 and 3.7], by vietue of the following inequality in a 2-uniformly smooth Banach space X.

Lemma 1.3 (see [18])**.** *Let X be a 2-uniformly smooth Banach space. Then*
$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2\|\kappa y\|^2, \quad \forall x, y \in X,$$
where κ is the 2-uniformly smooth constant of X and J is the normalized duality mapping from X into X^*.

Define the mapping $G : C \to C$ as follows
$$G(x) := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C.$$
The fixed point set of G is denoted by Ω. We remark that in [20, Theorems 3.5 and 3.7], the Banach space X is assumed to be both uniformly convex and 2-uniformly smooth. According to Lemma 1.3, the 2-uniform smoothness of X guarantees the nonexpansivity of the mapping $I - \mu_i B_i$ for α_i-inverse-strongly accretive mapping

$B_i : C \to X$ with $0 \le \mu_i \le \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$, and hence the composite mapping $G : C \to C$ is nonexpansive where $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$. However, the uniform convexity of X guarantees that there holds the demiclosedness principle for nonexpansive mappings. Naturally, it is interesting to know whether the uniform convexity and 2-uniformly smoothness of X can be replaced by the weaker geometrical property of X or not. The main purpose of this paper is to consider the above mentioned question and to give an affirmative answer. We will propose implicit and explicit algorithms based on Korpelevich's extragradient method [11] and Halpern's iterative method [15] to find approximate solutions of GSVI (1.4). Strong convergence results of these two methods will be established under very mild conditions. We observe that some recent results in this direction have been obtained in , e.g., [2, 3, 4, 5, 6, 7, 8, 14].

2. Preliminaries

The following lemmas will be used in the sequel. Lemma 2.1 can be found in [19] and Lemma 2.2 is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2.1. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \le 0$;
(iii) $\gamma_n \ge 0$ $(\forall n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.
Then $\limsup_{n\to\infty} s_n = 0$.

Lemma 2.2. *In a smooth Banach space X, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle, \quad \forall x, y \in X.$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in X.$$

Following Browder [1], we say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weak*ly to $J_\varphi(x)$). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(s)ds, \quad \forall t \ge 0.$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. The first part of the following lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [12].

Lemma 2.3. *Assume that X has a weakly continuous duality map J_φ with gauge φ.*

(i) *For all $x, y \in X$, there holds the inequality*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

(ii) *Assume a sequence $\{x_n\}$ in X is weakly convergent to a point x. Then there holds the identity*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y-x\|), \quad \forall y \in X.$$

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$ where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.4 ([16])**.** *Let C be a nonempty closed convex subset of a real smooth Banach space X, D be a nonempty subset of C and Π be a retraction from C onto D. Then Π is sunny and nonexpansive if and only if*

$$\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0,$$

for all $x \in C$ and $y \in D$.

It is well known that if $X = H$ a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$. Then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

3. Implicit Iterative Schemes

In this section, we introduce our implicit iterative schemes and show the strong convergence theorems. First, we give several useful lemmas. Lemmas 3.1 and 3.2 can be showed easily and therefore the proofs will be omitted.

Lemma 3.1. *Let C be a nonempty closed convex subset of a smooth Banach space X and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0, 1]$ we have*

$$\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\| \leq \left\{\sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\Big(1 + \frac{1}{\lambda_i}\Big)\right\}\|x - y\|, \quad \forall x, y \in C,$$

for $i = 1, 2$. In particular, if $1 - \frac{\lambda_i}{1+\lambda_i}\Big(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}\Big) \leq \mu_i \leq 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Lemma 3.2. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i+\lambda_i \geq 1$ for $i = 1, 2$. Let $G : C \to C$ be the mapping defined by*

$$G(x) = \Pi_C[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)], \quad \forall x \in C.$$

If $1 - \frac{\lambda_i}{1+\lambda_i}\left(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}\right) \leq \mu_i \leq 1$, then $G : C \to C$ is nonexpansive.

Lemma 3.3. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive for $i = 1, 2$. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (1.4) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$ where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof. We can rewrite GSVI (1.4) as

$$\begin{cases} \langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{3.1}$$

The conclusion then follows from Lemma 2.4. □

Remark 3.4. By Lemma 3.3, we observe that

$$x^* = \Pi_C[\Pi_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 \Pi_C(x^* - \mu_2 B_2 x^*)],$$

which implies that x^* is a fixed point of the mapping G. The set of fixed points of the mapping G will be denoted by Ω.

Now, in order to solve GSVI (1.4), we first introduce an implicit algorithm. Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. In what follows, we assume that $1 - \frac{\lambda_i}{1+\lambda_i}\left(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}\right) \leq \mu_i \leq 1$ for $i = 1, 2$. Let $F : C \to X$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha+\lambda \geq 1$. Now, take $t \in (0, 1)$. For given $\theta_t \in [0, 1)$, we define a mapping $T_t : C \to C$ by

$$T_t x = tu + (1 - t)\Pi_C(I - \theta_t F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C, \tag{3.2}$$

where $u \in C$ is a fixed element.

Define another mapping S_t

$$\begin{aligned} S_t x &= \Pi_C(I - \theta_t F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\ &= \Pi_C[(1 - \theta_t)I + \theta_t(I - F)]\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C. \end{aligned} \tag{3.3}$$

Then T_t is rewritten as

$$T_t x = tu + (1 - t)S_t x, \quad \forall x \in C. \tag{3.4}$$

Let us show that $S_t : C \to C$ is nonexpansive. As a matter of fact, utilizing the arguments similar to those in Lemma 3.1, we can derive

$$\|(I - F)x - (I - F)y\| \leq \sqrt{\frac{1 - \alpha}{\lambda}}\|x - y\|, \quad \forall x, y \in C.$$

Since $\alpha + \lambda \geq 1$, we get $\frac{1-\alpha}{\lambda} \leq 1$. It is clear that $I - F$ is nonexpansive and hence $I - \theta_t F = (1-\theta_t)I + \theta_t(I - F)$ is nonexpansive. So, $\Pi_C(I - \theta_t F)$ is nonexpansive. We note that by Lemma 3.1, $\Pi_C(I - \mu_i B_i)$ is nonexpansive for $i = 1, 2$. Thus, it follows from (3.3) that $S_t : C \to C$ is nonexpansive. This together with (3.4), implies that $T_t : C \to C$ is a contraction. Therefore, the Banach contraction principle guarantees that T_t has a unique fixed point in C, which we denote by x_t; that is,

$$x_t = tu + (1-t)S_t x_t = tu + (1-t)\Pi_C(I - \theta_t F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_t. \tag{3.5}$$

We now state and prove our first result.

Theorem 3.5. *Let C be a nonempty closed convex subset of a reflexive Banach space X which has a weakly continuous duality map J_φ with gauge φ. Let $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Assume that $1 - \frac{\lambda_i}{1+\lambda_i}\left(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}\right) \leq \mu_i \leq 1$ for $i = 1, 2$. Let $F : C \to X$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda \geq 1$. Fix $u \in X$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C to Eq. (3.5), where $\theta_t \in [0, 1), \forall t \in (0, 1)$ and $\lim_{t\to 0^+} \theta_t/t = 0$. Then $\Omega \neq \emptyset$ if and only if*

$$\limsup_{t\to 0^+} \|x_t\| < \infty, \tag{3.6}$$

and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to an element of Ω.

Proof. If $\Omega \neq \emptyset$, we can take $p \in \Omega$ to derive from (3.5) that, for $t \in (0, 1)$,

$$\begin{aligned} \|x_t - p\| &\leq t\|u - p\| + (1-t)\|S_t x_t - p\| \\ &\leq t\|u - p\| + (1-t)(\|S_t x_t - S_t p\| + \|S_t p - p\|) \\ &= t\|u - p\| + (1-t)(\|S_t x_t - S_t p\| + \|\Pi_C(I - \theta_t F)p - \Pi_C p\|) \\ &\leq t\|u - p\| + (1-t)\|x_t - p\| + \theta_t\|F(p)\|, \end{aligned}$$

which implies that

$$\|x_t - p\| \leq \|u - p\| + \frac{\theta_t}{t}\|F(p)\|. \tag{3.7}$$

Because $\lim_{t\to 0^+} \theta_t/t = 0$, we get from (3.7) that

$$\limsup_{t\to 0^+} \|x_t\| \leq \|p\| + \|u - p\| < \infty \tag{3.8}$$

and hence (3.6) holds.

Conversely, assume (3.6); that is, $\{x_t\}$ remains bounded when $t \to 0^+$; hence $F(G(x_t))$ is bounded, where G is defined as in Lemma 3.2. Because, in terms of (3.5)

$$x_t - G(x_t) = \frac{t}{1-t}(u - x_t) + S_t x_t - G(x_t), \tag{3.9}$$

we obtain

$$\begin{aligned} \|x_t - G(x_t)\| &\leq \tfrac{t}{1-t}\|u - x_t\| + \|S_t x_t - G(x_t)\| \\ &= \tfrac{t}{1-t}\|u - x_t\| + \|\Pi_C(G(x_t) - \theta_t F(G(x_t))) - \Pi_C G(x_t)\| \\ &\leq \tfrac{t}{1-t}\|u - x_t\| + \theta_t\|F(G(x_t))\|, \end{aligned}$$

which hence yields

$$\lim_{t\to 0^+} \|x_t - G(x_t)\| = 0. \tag{3.10}$$

Now assume $t_n \to 0^+$. Because X is reflexive and $\{x_{t_n}\}$ is bounded, we may assume that $x_{t_n} \rightharpoonup z$ for some $z \in C$. Because J_φ is weakly continuous, we have by Lemma 2.3,

$$\limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - z\|) + \Phi(\|x - z\|), \quad \forall x \in X.$$

Put

$$f(x) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|), \quad \forall x \in X.$$

It follows that

$$f(x) = f(z) + \Phi(\|x - z\|), \quad \forall x \in X.$$

From (3.10), we obtain

$$\begin{aligned} f(G(z)) &= \limsup_{n\to\infty} \Phi(\|x_{t_n} - G(z)\|) \\ &= \limsup_{n\to\infty} \Phi(\|G(x_{t_n}) - G(z)\|) \\ &\le \limsup_{n\to\infty} \Phi(\|x_{t_n} - z\|) = f(z). \end{aligned} \tag{3.11}$$

On the other hand, however,

$$f(G(z)) = f(z) + \Phi(\|G(z) - z\|). \tag{3.12}$$

Combining Eqs. (3.11) and (3.12) yields

$$\Phi(\|G(z) - z\|) \le 0.$$

Hence, $G(z) = z$ and $z \in \Omega$; so Ω is nonempty and we further prove that the entire net $\{x_t\}$ actually strongly converges. Indeed, what has been shown above is that if $t_n \to 0^+$ and $s_m \to 0^+$ are chosen so that $x_{t_n} \rightharpoonup z$ and $x_{s_m} \rightharpoonup w$, then $z, w \in \Omega$ and $x_{t_n} \to z$ and $x_{s_m} \to w$. So it remains to show that $z = w$. Toward this, we observe that, for $t \in (0, 1)$ and $p \in \Omega$,

$$x_t - p = (1 - t)(S_t x_t - S_t p) + t(u - p) + (1 - t)(S_t p - p).$$

It follows that

$$\begin{aligned} \langle x_t - p, J_\varphi(x_t - p)\rangle &= (1 - t)\langle S_t x_t - S_t p, J_\varphi(x_t - p)\rangle \\ &\quad + t\langle u - p, J_\varphi(x_t - p)\rangle + (1 - t)\langle S_t p - p, J_\varphi(x_t - p)\rangle. \end{aligned}$$

Because $\langle x, J_\varphi(x)\rangle = \|x\|\varphi(\|x\|)$ for all $x \in X$, we deduce from the last equation that

$$\begin{aligned} \|x_t - p\|\varphi(\|x_t - p\|) &\le (1 - t)\|S_t x_t - S_t p\|\|J_\varphi(x_t - p)\| \\ &\quad + t\langle u - p, J_\varphi(x_t - p)\rangle + (1 - t)\|S_t p - p\|\|J_\varphi(x_t - p)\| \\ &\le (1 - t)\|x_t - p\|\varphi(\|x_t - p\|) + t\langle u - p, J_\varphi(x_t - p)\rangle \\ &\quad + (1 - t)\|\Pi_C(I - \theta_t F)p - \Pi_C p\|\varphi(\|x_t - p\|) \\ &\le (1 - t)\|x_t - p\|\varphi(\|x_t - p\|) \\ &\quad + t\langle u - p, J_\varphi(x_t - p)\rangle + \theta_t\|F(p)\|\varphi(\|x_t - p\|). \end{aligned}$$

Therefore,

$$\|x_t - p\|\varphi(\|x_t - p\|) \le \langle u - p, J_\varphi(x_t - p)\rangle + \frac{\theta_t}{t}\|F(p)\|\varphi(\|x_t - p\|). \tag{3.13}$$

Now taking the limit in (3.13) through $t_n \to 0$ and noting that $x_{t_n} \to z$ and $\lim_{t\to 0^+} \theta_t/t = 0$, we have

$$\|z-p\|\varphi(\|z-p\|) \le \langle u-p, J_\varphi(z-p)\rangle, \quad \forall p \in \Omega. \tag{3.14}$$

In particular,

$$\|z-w\|\varphi(\|z-w\|) \le \langle u-w, J_\varphi(z-w)\rangle. \tag{3.15}$$

Interchange z and w to attain

$$\|w-z\|\varphi(\|w-z\|) \le \langle u-z, J_\varphi(w-z)\rangle. \tag{3.16}$$

Adding up (3.15) and (3.16), we obtain

$$2\|z-w\|\varphi(\|z-w\|) \le \langle z-w, J_\varphi(z-w)\rangle = \|z-w\|\varphi(\|z-w\|).$$

Hence $\|z-w\|\varphi(\|z-w\|) = 0$, and we must have $z = w$. □

We next establish the version of Theorem 3.5 in a uniformly smooth Banach space.

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \ge 1$ for $i = 1, 2$. Assume that $1 - \frac{\lambda_i}{1+\lambda_i}\Big(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}\Big) \le \mu_i \le 1$ for $i = 1, 2$. Let $F : C \to X$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda \ge 1$. Fix $u \in X$ and $t \in (0,1)$. Let $x_t \in C$ be the unique solution in C to Eq. (3.5), where $\theta_t \in [0,1), \forall t \in (0,1)$ and $\lim_{t\to 0^+} \theta_t/t = 0$. Then $\Omega \ne \emptyset$ if and only if (3.6) holds and in this case $\{x_t\}$ converges as $t \to 0^+$ strongly to an element of Ω.*

Proof. The necessity of (3.6) follows from (3.8). To see the sufficiency, we first notice that both (3.9) and (3.10) hold. Let now $\{t_n\}$ be a sequence in $(0,1)$ such that $t_n \to 0$ as $n \to \infty$. Define a function g on C by

$$g(x) = \mathrm{LIM}_n \frac{1}{2}\|x_{t_n} - x\|^2, \quad \forall x \in C. \tag{3.17}$$

(Here LIM denotes a Banach limit on l^∞.)

Let K be the set of minimizers of g over C; that is,

$$K = \{x \in C : g(x) = \min_{y\in C} g(y)\}.$$

It is easily known that K is a closed bounded convex nonempty subset of C. Because of (3.10), K is also G-invariant (i.e., $G(K) \subset K$). Because a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, G admits a fixed point in K. Denote by v such a fixed point of G. Because v is a minimizer of g over C, it follows that, for $x \in C$,

$$\begin{aligned} 0 &\le [g(v + s(x-v)) - g(v)]/s \\ &= \mathrm{LIM}_n \tfrac{1}{2}(\|(x_{t_n} - v) + s(v-x)\|^2 - \|x_{t_n} - v\|^2)/s. \end{aligned} \tag{3.18}$$

Because the duality map J is uniformly continuous over bounded subsets of X, we can take the limit as $s \to 0$ under the Banach limit LIM to get

$$\mathrm{LIM}_n \langle x - v, J(x_{t_n} - v)\rangle \le 0, \quad \forall x \in C.$$

In particular,

$$\mathrm{LIM}_n\langle u-v, J(x_{t_n}-v)\rangle \le 0. \tag{3.19}$$

Because $J = J_\varphi$ with $\varphi(t) = t$ for all $t \in [0,\infty)$, it follows from (3.13) that

$$\|x_t - p\|^2 \le \langle u-v, J(x_t-p)\rangle + \frac{\theta_t}{t}\|F(p)\|\|x_t-p\|, \quad \forall p \in \Omega. \tag{3.20}$$

In particular,

$$\|x_{t_n} - v\|^2 \le \langle u-v, J(x_t-v)\rangle + \frac{\theta_{t_n}}{t_n}\|F(v)\|\|x_{t_n}-v\|. \tag{3.21}$$

Adding (3.21) to (3.19) and noting $\theta_{t_n}/t_n \to 0$, we obtain

$$\mathrm{LIM}_n\|x_{t_n}-v\| \le 0.$$

Hence there is a subsequence of $\{x_{t_n}\}$, still denoted $\{x_{t_n}\}$, converging strongly to v.

To see that the entire net $\{x_t\}$ actually converges strongly as $t \to 0$, we assume that there is another sequence $\{s_j\}$ in $(0,1)$, $s_j \to 0$ as $j \to \infty$, such that $x_{s_j} \to z$. Then we have $z \in \Omega$. From (3.20) we have

$$\|x_{t_n} - z\|^2 \le \langle u-z, J(x_{t_n}-z)\rangle + \frac{\theta_{t_n}}{t_n}\|F(z)\|\|x_{t_n}-z\|.$$

Letting $n \to \infty$ yields

$$\|v-z\|^2 \le \langle u-z, J(v-z)\rangle. \tag{3.22}$$

Similar argument gives us

$$\|z-v\|^2 \le \langle u-v, J(z-v)\rangle. \tag{3.23}$$

Adding up (3.22) and (3.23) yields

$$2\|z-v\|^2 \le \langle z-v, J(z-v)\rangle = \|z-v\|^2.$$

Hence $z = v$, and $\{x_t\}$ must be strongly convergent as $t \to 0^+$. □

Theorems 3.5 and 3.6 show that if X either is reflexive and has a weakly continuous duality map or is uniformly smooth, then in the case of $\Omega \ne \emptyset$, we can define a mapping $Q : C \to \Omega$ by setting

$$Q(u) = s - \lim_{t\to 0^+} x_t, \tag{3.24}$$

where x_t is the unique solution to the fixed point equation (3.5). The final result of this section verifies that Q is the sunny nonexpansive retraction from C onto Ω.

Theorem 3.7. *Let C be a nonempty closed convex subset of a reflexive Banach space X. Assume, in addition, X either has a weakly continuous duality map or is uniformly smooth. Assume $\Omega \ne \emptyset$. Then under the conditions of Theorems 3.5 (or of Theorems 3.6), Q is the sunny nonexpansive retraction from C onto Ω.*

Proof. We first show that Q is indeed a retraction from C onto Ω. As a matter of fact, assuming $p = G(p)$ implies via (3.5) (where $u = p$) that

$$\begin{aligned} \|x_t - p\| &= \|(1-t)(S_t x_t - S_t p) + (1-t)(S_t p - p)\| \\ &\le (1-t)\|S_t x_t - S_t p\| + (1-t)\|\Pi_C(I - \theta_t F)p - \Pi_C p\| \\ &\le (1-t)\|x_t - p\| + \theta_t \|F(p)\|, \end{aligned}$$

so we have

$$\|x_t - p\| \le \frac{\theta_t}{t}\|F(p)\|,$$

which together with $\lim_{t\to 0} \theta_t/t = 0$, implies that

$$\|Q(p) - p\| = \lim_{t\to 0^+} \|x_t - p\| = 0.$$

Hence $Q(p) = p$ for all $p \in \Omega$ and Q is a retraction onto Ω.

To show that Q is sunny nonexpansive, by [?] (see also [?]), it suffices to show that

$$\langle u - Q(u), J(p - Q(u))\rangle \le 0, \quad \forall u \in C,\ \forall p \in \Omega; \tag{3.25}$$

or equivalently,

$$\langle u - Q(u), J_\varphi(p - Q(u))\rangle \le 0, \quad \forall u \in C,\ \forall p \in \Omega. \tag{3.26}$$

In the case that X has a weakly continuous duality map J_φ, we compute that, for $p \in \Omega$,

$$\begin{aligned} \langle x_t - G(x_t), J_\varphi(x_t - p)\rangle &= \langle x_t - p, J_\varphi(x_t - p)\rangle + \langle p - G(x_t), J_\varphi(x_t - p)\rangle \\ &\ge (\|x_t - p\| - \|p - G(x_t)\|)\varphi(\|x_t - p\|) \\ &\ge 0. \end{aligned}$$

Next notice by (3.5)

$$x_t - u = -\frac{1-t}{t}[x_t - G(x_t) + G(x_t) - S_t x_t],$$

to deduce that

$$\begin{aligned} \langle x_t - u, J_\varphi(x_t - p)\rangle &= -\tfrac{1-t}{t}\langle x_t - G(x_t), J_\varphi(x_t - p)\rangle \\ &\quad -\tfrac{1-t}{t}\langle G(x_t) - S_t x_t, J_\varphi(x_t - p)\rangle \\ &\le \tfrac{1-t}{t}\|G(x_t) - S_t x_t\|\varphi(\|x_t - p\|) \\ &= \tfrac{1-t}{t}\|\Pi_C G(x_t) - \Pi_C(I - \theta_t F)G(x_t)\|\varphi(\|x_t - p\|) \\ &\le \tfrac{\theta_t}{t} F(G(x_t))\|\varphi(\|x_t - p\|). \end{aligned} \tag{3.27}$$

Because $\theta_t/t \to 0$ and $x_t \to Q(u)$ as $t \to 0^+$, taking the limit as $t \to 0^+$ in (3.27), we obtain (3.26).

In the case that X is uniformly smooth, by repeating the above argument with the gauge $\varphi(t) = t$ for all $t \in [0, \infty)$, we obtain (3.25). □

4. Explicit Iterative Schemes

In this section, we introduce our explicit iterative schemes which are the discretization of the implicit iterative schemes (3.5), and show the strong convergence theorems.

Algorithm 4.1. Let C be a nonempty closed convex subset of a real smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $B_1, B_2 : C \to X$ be two nonlinear mappings. Let $F : C \to X$ be α-strongly accretive and λ-strictly pseudocontractive. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \beta_n u + (1-\beta_n)\Pi_C(I-\gamma_n F)\Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)x_n, \quad \forall n \geq 0, \tag{4.1}$$

where $\{\beta_n\} \subset (0,1)$, $\{\gamma_n\} \subset [0,1)$, $u \in C$ is a fixed element and μ_1, μ_2 are two positive numbers.

In particular, if $B_1 = B_2 = A$, then (4.1) reduces to the following:

$$x_{n+1} = \beta_n u + (1-\beta_n)\Pi_C(I-\gamma_n F)\Pi_C(I-\mu_1 A)\Pi_C(I-\mu_2 A)x_n, \quad \forall n \geq 0. \tag{4.2}$$

Theorem 4.2. *Let C be a nonempty closed convex subset of a reflexive Banach space X which has a weakly continuous duality map J_φ with gauge φ. Let $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Assume that $1 - \frac{\lambda_i}{1+\lambda_i}\left(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}\right) \leq \mu_i \leq 1$ for $i = 1, 2$. Let $F : C \to X$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda \geq 1$. Let $\Omega \neq \emptyset$ and assume that*

(i) $\beta_n \to 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $\lim_{n\to\infty} \gamma_n/\beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$;
(iv) $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\gamma_n - \gamma_{n-1}|/\beta_n = 0$.
Then the sequence $\{x_n\}$ generated by scheme (4.1) converges strongly to an element of Ω.

Proof. For each $n \geq 0$, let S_n be defined by

$$S_n x = \Pi_C(I-\gamma_n F)\Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)x, \quad \forall x \in C.$$

Then we know that

(i) the scheme (4.1) is rewritten as

$$x_{n+1} = \beta_n u + (1-\beta_n)S_n x_n, \quad \forall n \geq 0; \tag{4.3}$$

(ii) S_n is nonexpansive by the similar argument to that of the nonexpansivity of S_t in (3.5);

(iii) $S_n p = \Pi_C(I-\gamma_n F)p$ for all $p \in \Omega$.

Thus, we deduce that for $p \in \Omega$,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n(u-p) + (1-\beta_n)(S_n x_n - p)\| \\ &\leq \beta_n\|u-p\| + (1-\beta_n)\|S_n x_n - p\| \\ &\leq \beta_n\|u-p\| + (1-\beta_n)(\|S_n x_n - S_n p\| + \|S_n p - p\|) \\ &= \beta_n\|u-p\| + (1-\beta_n)(\|S_n x_n - S_n p\| + \|\Pi_C(I-\gamma_n F)p - \Pi_C p\|) \\ &\leq \beta_n\|u-p\| + (1-\beta_n)\|x_n - p\| + \gamma_n\|F(p)\|. \end{aligned} \tag{4.4}$$

Because $\lim_{n\to\infty}\gamma_n/\beta_n = 0$, we may assume without loss of generality that $\gamma_n \le \beta_n$ for all $n \ge 0$. Hence, from (4.4) we get

$$\|x_{n+1}-p\| \le \beta_n(\|u-p\|+\|F(p)\|)+(1-\beta_n)\|x_n-p\|, \quad \forall n \ge 0.$$

By induction, we conclude that

$$\|x_n-p\| \le \max\{\|u-p\|+\|F(p)\|, \|x_0-p\|\}, \quad \forall n \ge 0. \tag{4.5}$$

Therefore, $\{x_n\}$ is bounded, so are the sequences $\{G(x_n)\}$ and $\{F(G(x_n))\}$. Also, from (4.1), we have

$$\begin{aligned}\|x_{n+1}-G(x_n)\| &\le \beta_n\|u-G(x_n)\|+(1-\beta_n)\|S_nx_n-G(x_n)\| \\ &= \beta_n\|u-G(x_n)\|+(1-\beta_n)\|\Pi_C(I-\gamma_nF)G(x_n)-\Pi_CG(x_n)\| \\ &\le \beta_n\|u-G(x_n)\|+(1-\beta_n)\gamma_n\|F(G(x_n))\| \\ &\le \beta_n\|u-G(x_n)\|+\gamma_n\|F(G(x_n))\|,\end{aligned}$$

which together with $\beta_n \to 0$ and $\gamma_n \to 0$, implies that

$$\lim_{n\to\infty}\|x_{n+1}-G(x_n)\| = 0. \tag{4.6}$$

Now we note that

$$\begin{aligned}x_{n+1}-x_n &= \beta_nu+(1-\beta_n)S_nx_n-\beta_{n-1}u-(1-\beta_{n-1})S_{n-1}x_{n-1} \\ &= (\beta_n-\beta_{n-1})(u-S_{n-1}x_{n-1})+(1-\beta_n)(S_nx_n-S_{n-1}x_{n-1}).\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (\beta_n-\beta_{n-1})\|u-S_{n-1}x_{n-1}\|+(1-\beta_n)\|S_nx_n-S_{n-1}x_{n-1}\| \\ &\le (1-\beta_n)[\|S_nx_n-S_nx_{n-1}\|+\|S_nx_{n-1}-S_{n-1}x_{n-1}\|] \\ &\quad +|\beta_n-\beta_{n-1}|\|u-S_{n-1}x_{n-1}\| \\ &= (1-\beta_n)[\|S_nx_n-S_nx_{n-1}\|+\|\Pi_C(I-\gamma_nF)G(x_{n-1}) \\ &\qquad -\Pi_C(I-\gamma_{n-1}F)G(x_{n-1})\|] \\ &\quad +|\beta_n-\beta_{n-1}|\|u-S_{n-1}x_{n-1}\| \\ &\le (1-\beta_n)[\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|F(G(x_{n-1}))\|] \\ &\quad +|\beta_n-\beta_{n-1}|\|u-S_{n-1}x_{n-1}\| \\ &\le (1-\beta_n)\|x_n-x_{n-1}\|+M(|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|),\end{aligned}$$

where $\sup_{n\ge0}\{\|F(G(x_n))\|+\|u-S_nx_n\|\} \le M$ for some $M > 0$. So, utilizing Lemma 2.1, from conditions (i), (iii) and (iv) we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0,$$

which together with (4.6), implies that

$$\lim_{n\to\infty}\|x_n-G(x_n)\| = 0. \tag{4.7}$$

Because Ω is the fixed point set of the nonexpansive mapping G, we see from Theorem 3.7 that there exists a unique sunny nonexpansive retraction Q from C onto Ω. Let $q = Q(u)$. We then claim that

$$\limsup_{n\to\infty}\langle u-q, J_\varphi(x_n-q)\rangle \le 0. \tag{4.8}$$

Indeed, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-q, J_\varphi(x_n-q)\rangle = \lim_{k\to\infty}\langle u-q, J_\varphi(x_{n_k}-q)\rangle. \tag{4.9}$$

Because X is reflexive and $\{x_n\}$ is bounded, we may further assume that $x_{n_k} \rightharpoonup \tilde{x} \in C$. Because J_φ is weakly continuous, we have by Lemma 2.3,

$$\limsup_{k\to\infty} \Phi(\|x_{n_k} - x\|) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - \tilde{x}\|) + \Phi(\|x - \tilde{x}\|), \quad \forall x \in X.$$

Put

$$f(x) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - x\|), \quad \forall x \in X.$$

It follows that

$$f(x) = f(\tilde{x}) + \Phi(\|x - \tilde{x}\|), \quad \forall x \in X.$$

From (4.7), we obtain

$$\begin{aligned} f(G(\tilde{x})) &= \limsup_{k\to\infty} \Phi(\|x_{n_k} - G(\tilde{x})\|) \\ &= \limsup_{k\to\infty} \Phi(\|G(x_{n_k}) - G(\tilde{x})\|) \\ &\le \limsup_{k\to\infty} \Phi(\|x_{n_k} - \tilde{x}\|) = f(\tilde{x}). \end{aligned} \tag{4.10}$$

On the other hand, however,

$$f(G(\tilde{x})) = f(\tilde{x}) + \Phi(\|G(\tilde{x}) - \tilde{x}\|). \tag{4.11}$$

Combining Eqs. (4.10) and (4.11) yields

$$\Phi(\|G(\tilde{x}) - \tilde{x}\|) \le 0.$$

Hence, $G(\tilde{x}) = \tilde{x}$; i.e., $\tilde{x} \in \Omega$. So, from (4.9) and (3.25), we have

$$\limsup_{n\to\infty} \langle u - q, J_\varphi(x_n - q)\rangle = \langle u - q, J_\varphi(\tilde{x} - q)\rangle \le 0.$$

That is, (4.8) holds. Finally to prove that $x_n \to q$, we apply Lemma 2.3 to get

$$\begin{aligned} \Phi(\|x_{n+1} - q\|) &= \Phi(\|(1-\beta_n)(S_n x_n - q) + \beta_n(u - q)\|) \\ &= \Phi(\|(1-\beta_n)(S_n x_n - S_n q) + \beta_n(u - q) + (1-\beta_n)(S_n q - q)\|) \\ &\le (1-\beta_n)\Phi(\|S_n x_n - S_n q\|) + \beta_n\langle u - q, J_\varphi(x_{n+1} - q)\rangle \\ &\quad + (1-\beta_n)\langle S_n q - q, J_\varphi(x_{n+1} - q)\rangle \\ &\le (1-\beta_n)\Phi(\|x_n - q\|) + \beta_n\langle u - q, J_\varphi(x_{n+1} - q)\rangle \\ &\quad + (1-\beta_n)\|S_n q - q\|\varphi(\|x_{n+1} - q\|) \\ &= (1-\beta_n)\Phi(\|x_n - q\|) + \beta_n\langle u - q, J_\varphi(x_{n+1} - q)\rangle \\ &\quad + (1-\beta_n)\|\Pi_C(I - \gamma_n F)q - \Pi_C q\|\varphi(\|x_{n+1} - q\|) \\ &\le (1-\beta_n)\Phi(\|x_n - q\|) + \beta_n\langle u - q, J_\varphi(x_{n+1} - q)\rangle \\ &\quad + \gamma_n\|F(q)\|\varphi(\|x_{n+1} - q\|) \\ &= (1-\beta_n)\Phi(\|x_n - q\|) + \beta_n[\langle u - q, J_\varphi(x_{n+1} - q)\rangle \\ &\quad + (\gamma_n/\beta_n)\|F(q)\|\varphi(\|x_{n+1} - q\|)] \\ &\le (1-\beta_n)\Phi(\|x_n - q\|) + \beta_n[\langle u - q, J_\varphi(x_{n+1} - q)\rangle + (\gamma_n/\beta_n)\beta], \end{aligned} \tag{4.12}$$

where $\sup_{n\ge 0}\|F(q)\|\varphi(\|x_n - q\| \le \beta$ for some $\beta > 0$.

By virtue of condition (ii) and (4.8), we apply Lemma 2.1 to (4.12) to conclude that $\|x_n - q\| \to 0$ as $n \to \infty$. □

Remark 4.3. Because for $1 < p < \infty$ and $p \neq 2$, the space L^p, which is uniformly smooth, fails to have a weakly continuous duality (see [10], [21]), there is no doubt that Theorem 4.2 is not applicable to L^p. So it is useful to consider the convergence of scheme (4.1) in the case where the underlying space X is uniformly smooth.

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Assume that $1 - \frac{\lambda_i}{1+\lambda_i}\Big(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}\Big) \leq \mu_i \leq 1$ for $i = 1, 2$. Let $F : C \to X$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda \geq 1$. Let $\Omega \neq \emptyset$ and assume that*

(i) $\beta_n \to 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $\lim_{n\to\infty} \gamma_n/\beta_n = 0$;
(iii) $\frac{1}{\beta_n}\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \leq \tau,\ \forall n \geq 1$ *for some* $\tau > 0$;
(iv) $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\beta_n^2} = 0$;
(v) $\lim_{n\to\infty} \frac{|\gamma_n - \gamma_{n-1}|}{\beta_n^2} = 0$.

Then the sequence $\{x_n\}$ generated by scheme (4.1) converges strongly to an element of Ω.

Proof. For each $n \geq 0$, let S_n be defined by

$$S_n x = \Pi_C(I - \gamma_n F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C.$$

Then we know that

(i) the scheme (4.1) is rewritten as

$$x_{n+1} = \beta_n u + (1 - \beta_n)S_n x_n, \quad \forall n \geq 0;$$

(ii) S_n is nonexpansive by the similar argument to that of the nonexpansivity of S_t in (3.5);

(iii) $S_n p = \Pi_C(I - \gamma_n F)p$ for all $p \in \Omega$.

Repeating the same argument as in (4.5), we can deduce that for $p \in \Omega$,

$$\|x_n - p\| \leq \max\{\|u - p\| + \|F(p)\|, \|x_0 - p\|\}, \quad \forall n \geq 0.$$

Therefore, $\{x_n\}$ is bounded, so are the sequences $\{G(x_n)\}$ and $\{F(G(x_n))\}$. Repeating the same arguments as in the proof of Theorem 4.2, we can conclude that

$$\lim_{n\to\infty} \|x_{n+1} - G(x_n)\| = 0$$

and

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

which hence yield

$$\lim_{n\to\infty} \|x_n - G(x_n)\| = 0. \tag{4.13}$$

Observe that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (\beta_n - \beta_{n-1})\|u - S_{n-1}x_{n-1}\| + (1 - \beta_n)\|S_n x_n - S_{n-1}x_{n-1}\| \\ &\leq (1 - \beta_n)[\|S_n x_n - S_n x_{n-1}\| + \|\Pi_C(I - \gamma_n F)G(x_{n-1}) \\ &\qquad - \Pi_C(I - \gamma_{n-1}F)G(x_{n-1})\|] \\ &\quad + |\beta_n - \beta_{n-1}|\|u - S_{n-1}x_{n-1}\| \\ &\leq (1 - \beta_n)\|x_n - x_{n-1}\| + M(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|),\end{aligned}$$

where $\sup_{n\geq 0}\{\|F(G(x_n))\|+\|u-S_nx_n\|\} \leq M$ for some $M > 0$. Then it immediately follows from condition (iii) that

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\beta_n} &\leq (1-\beta_n)\frac{\|x_n-x_{n-1}\|}{\beta_n} + M\frac{|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|}{\beta_n} \\
&= (1-\beta_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + (1-\beta_n)\|x_n-x_{n-1}\|\Big(\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\Big) \\
&\quad + M\frac{|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|}{\beta_n} \\
&\leq (1-\beta_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \|x_n-x_{n-1}\|\Big|\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\Big| \\
&\quad + M\Big(\frac{|\beta_n-\beta_{n-1}|}{\beta_n}+\frac{|\gamma_n-\gamma_{n-1}|}{\beta_n}\Big) \\
&= (1-\beta_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \beta_n\|x_n-x_{n-1}\|\frac{1}{\beta_n}\Big|\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\Big| \\
&\quad + M\Big(\frac{|\beta_n-\beta_{n-1}|}{\beta_n}+\frac{|\gamma_n-\gamma_{n-1}|}{\beta_n}\Big) \\
&\leq (1-\beta_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \beta_n\tau\|x_n-x_{n-1}\| \\
&\quad + M\Big(\frac{|\beta_n-\beta_{n-1}|}{\beta_n}+\frac{|\gamma_n-\gamma_{n-1}|}{\beta_n}\Big) \\
&= (1-\beta_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} \\
&\quad + \beta_n\Big[\tau\|x_n-x_{n-1}\| + M\Big(\frac{|\beta_n-\beta_{n-1}|}{\beta_n^2}+\frac{|\gamma_n-\gamma_{n-1}|}{\beta_n^2}\Big)\Big].
\end{aligned}$$

Utilizing Lemma 2.1, from conditions (i), (iv) and (v) we conclude that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n} = 0.$$

Because Ω is the fixed point set of the nonexpansive mapping G, we see from Theorem 3.7 that there exists a unique sunny nonexpansive retraction Q from C onto Ω. Let $q = Q(u)$. We then claim that

$$\limsup_{n\to\infty}\langle u-q, J(x_n-q)\rangle \leq 0. \tag{4.14}$$

Indeed, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-q, J(x_n-q)\rangle = \lim_{k\to\infty}\langle u-q, J(x_{n_k}-q)\rangle. \tag{4.15}$$

Define a function g on C by

$$g(x) = \mathrm{LIM}_k\frac{1}{2}\|x_{n_k}-x\|^2, \quad \forall x\in C. \tag{4.16}$$

(Here LIM denotes a Banach limit on l^∞.)

Let K be the set of minimizers of g over C; that is,

$$K = \{x \in C : g(x) = \min_{y \in C} g(y)\}.$$

It is easily known that K is a closed bounded convex nonempty subset of C. Because of (4.13), K is also G-invariant (i.e., $G(K) \subset K$). Because a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, G admits a fixed point in K. Denote by v such a fixed point of G. Because v is a minimizer of g over C, it follows that, for $x \in C$,

$$\begin{aligned} 0 &\le [g(v + s(x - v)) - g(v)]/s \\ &= \mathrm{LIM}_k \tfrac{1}{2}(\|(x_{n_k} - v) + s(v - x)\|^2 - \|x_{n_k} - v\|^2)/s. \end{aligned} \tag{4.17}$$

Because the duality map J is uniformly continuous over bounded subsets of X, we can take the limit as $s \to 0$ under the Banach limit LIM to get

$$\mathrm{LIM}_k \langle x - v, J(x_{n_k} - v)\rangle \le 0, \quad \forall x \in C.$$

In particular,

$$\mathrm{LIM}_k \langle u - v, J(x_{n_k} - v)\rangle \le 0. \tag{4.18}$$

On the other hand, observe that for $p \in \Omega$,

$$x_{n+1} - p = (1 - \beta_n)(S_n x_n - S_n p) + \beta_n(u - p) + (1 - \beta_n)(S_n p - p).$$

Then it follows that

$$\begin{aligned} &\|x_n - p\|^2 + \langle x_{n+1} - x_n, J(x_n - p)\rangle \\ &= \langle x_n - p, J(x_n - p)\rangle + \langle x_{n+1} - x_n, J(x_n - p)\rangle \\ &= \langle x_{n+1} - p, J(x_n - p)\rangle \\ &= (1 - \beta_n)\langle S_n x_n - S_n p, J(x_n - p)\rangle \\ &\quad + \beta_n \langle u - p, J(x_n - p)\rangle + (1 - \beta_n)\langle S_n p - p, J(x_n - p)\rangle \\ &\le (1 - \beta_n)\|S_n x_n - S_n p\|\|x_n - p\| \\ &\quad + \beta_n \langle u - p, J(x_n - p)\rangle + (1 - \beta_n)\|S_n p - p\|\|x_n - p\| \\ &\le (1 - \beta_n)\|x_n - p\|^2 \\ &\quad + \beta_n \langle u - p, J(x_n - p)\rangle + (1 - \beta_n)\|\Pi_C(I - \gamma_n F)p - \Pi_C p\|\|x_n - p\| \\ &\le (1 - \beta_n)\|x_n - p\|^2 \\ &\quad + \beta_n \langle u - p, J(x_n - p)\rangle + \gamma_n \|F(p)\|\|x_n - p\|, \end{aligned}$$

which hence yields

$$\begin{aligned} \|x_n - p\|^2 &\le \tfrac{1}{\beta_n}\langle x_n - x_{n+1}, J(x_n - p)\rangle + \langle u - p, J(x_n - p)\rangle \\ &\quad + \tfrac{\gamma_n}{\beta_n}\|F(p)\|\|x_n - p\| \\ &\le \tfrac{\|x_{n+1} - x_n\|}{\beta_n}\|x_n - p\| + \langle u - p, J(x_n - p)\rangle + \tfrac{\gamma_n}{\beta_n}\|F(p)\|\|x_n - p\|. \end{aligned}$$

In particular,

$$\|x_{n_k} - v\|^2 \le \frac{\|x_{n_k+1} - x_{n_k}\|}{\beta_{n_k}}\|x_{n_k} - v\| + \langle u - v, J(x_{n_k} - v)\rangle + \frac{\gamma_{n_k}}{\beta_{n_k}}\|F(v)\|\|x_{n_k} - v\|.$$

Because $\frac{\|x_{n_k+1} - x_{n_k}\|}{\beta_{n_k}} \to 0$ and $\frac{\gamma_{n_k}}{\beta_{n_k}} \to 0$, it follows from (4.18) that

$$\mathrm{LIM}_k \|x_{n_k} - v\| \le 0.$$

So there exists a subsequence of $\{x_{n_k}\}$, still denoted $\{x_{n_k}\}$, converging strongly to v. Consequently, from (4.15) it immediately follows that

$$\limsup_{n\to\infty}\langle u-q, J(x_n-q)\rangle = \langle u-q, J(v-q)\rangle \le 0.$$

This shows that (4.14) holds. Finally to prove that $x_n \to q$, we apply Lemma 2.2 to get

(4.19)

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\beta_n)(S_nx_n-q)+\beta_n(u-q)\|^2\\
&= \|(1-\beta_n)(S_nx_n-S_nq)+\beta_n(u-q)+(1-\beta_n)(S_nq-q)\|^2\\
&\le (1-\beta_n)\|S_nx_n-S_nq\|^2+2\beta_n\langle u-q, J(x_{n+1}-q)\rangle\\
&\quad +2(1-\beta_n)\langle S_nq-q, J(x_{n+1}-q)\rangle\\
&\le (1-\beta_n)\|x_n-q\|^2+2\beta_n\langle u-q, J(x_{n+1}-q)\rangle\\
&\quad +2(1-\beta_n)\|S_nq-q\|\|x_{n+1}-q\|\\
&= (1-\beta_n)\|x_n-q\|^2+2\beta_n\langle u-q, J(x_{n+1}-q)\rangle\\
&\quad +2(1-\beta_n)\|\Pi_C(I-\gamma_nF)q-\Pi_Cq\|\|x_{n+1}-q\|\\
&\le (1-\beta_n)\|x_n-q\|^2+2\beta_n\langle u-q, J(x_{n+1}-q)\rangle\\
&\quad +2\gamma_n\|F(q)\|\|x_{n+1}-q\|\\
&= (1-\beta_n)\|x_n-q\|^2+2\beta_n[\langle u-q, J(x_{n+1}-q)\rangle\\
&\quad +(\gamma_n/\beta_n)\|F(q)\|\|x_{n+1}-q\|]\\
&\le (1-\beta_n)\|x_n-q\|^2+2\beta_n[\langle u-q, J(x_{n+1}-q)\rangle+(\gamma_n/\beta_n)\beta],
\end{aligned}$$

where $\sup_{n\ge 0}\|F(q)\|\|x_n-q\| \le \beta$ for some $\beta > 0$. Note that the uniform continuity of J over bounded subsets of X together with (4.14), leads to

$$\begin{aligned}
&\limsup_{n\to\infty}\langle u-q, J(x_{n+1}-q)\rangle\\
&= \limsup_{n\to\infty}(\langle u-q, J(x_n-q)\rangle + \langle u-q, J(x_{n+1}-q)-J(x_n-q)\rangle)\\
&= \limsup_{n\to\infty}\langle u-q, J(x_n-q)\rangle \le 0.
\end{aligned}$$

By virtue of condition (ii), we apply Lemma 2.1 to (4.19) to conclude that $\|x_n-q\| \to 0$ as $n \to \infty$. □

Remark 4.5. As an example, we consider the following sequences:

(i) for given $s \in (0, \frac{1}{4}]$, $\{\beta_n\}$ is chosen as

$$\beta_n = \frac{1}{(n+1)^s}, \quad \forall n \ge 0;$$

(ii) for given $t \in (\frac{1}{4}, \frac{1}{3}]$, $\{\gamma_n\}$ is chosen as

$$\gamma_n = \frac{1}{(n+1)^t}, \quad \forall n \ge 0.$$

Then conditions (i)-(v) of Theorem 4.4 are satisfied.

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