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MULTIVALUED F-CONTRACTIONS ON COMPLETE METRIC SPACES

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Abstract. In the present paper, we introduce the concept of multivalued F-contraction mappings and give some fixed point results, which generalize some multivalued fixed point theorems including Nadler's. Also, we give an illustrating example showing that our results are proper generalization of Nadler's.

1. Introduction and preliminaries

Fixed point theory contains many different fields of mathematics, such as nonlinear functional analysis, mathematical analysis, operator theory and general topology. Historically, the study of fixed point theory has developed in two major branches: the first is fixed point theory for contraction or contraction type mappings on complete metric spaces and the second is fixed point theory for continuous operators on compact and convex subsets of a normed space. The beginning of fixed point theory in normed space is attributed to the work of Brouwer in 1910, who proved that any continuous self-map of the closed unit ball of $\mathbb{R}^n$ has a fixed point. The beginning of fixed point theory on complete metric space is related to Banach Contraction Principle, published in 1922. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a contraction mapping if there exists a constant $L \in [0, 1)$, called a contraction factor, such that

$$d(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Banach Contraction Principle says that any contraction self-mappings on a complete metric space has a unique fixed point. It is one of a very powerful test for existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions (see[1, 2, 3, 4, 5, 10, 12, 15, 18, 21, 22, 25]). The most interesting generalization of this important theorem was given by Wardowski [24]. For the sake of completeness we recall the F-contraction, which was introduced by Wardowski [24], then we will mention his result.

Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

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Definition 1.1 ([24])**.** Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x, y \in X \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{1.2}$$

When Wardowki considered in (1.2) the different type of the mapping F then we obtain the variety of contractions, some of them are of a type known in the literature. We can examine the following examples:

Example 1.2 ([24])**.** Let $F_1 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_1(\alpha) = \ln \alpha$. It is clear that $F_1 \in \mathcal{F}$. Then each self mapping T on a metric space (X, d) satisfying (1.2) is an F_1-contraction such that

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, Tx \neq Ty. \tag{1.3}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Therefore T satisfies (1.1) with $L = e^{-\tau}$, thus T is a contraction.

Example 1.3 ([24])**.** Let $F_2 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_2(\alpha) = \alpha + \ln \alpha$. It is clear that $F_2 \in \mathcal{F}$. Then each self mapping T on a metric space (X, d) satisfying (1.2) is an F_2-contraction such that

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty) - d(x,y)} \leq e^{-\tau}, \text{ for all } x, y \in X, Tx \neq Ty. \tag{1.4}$$

We can find in [24] some different examples of the function F belonging to $\mathcal{F}$. In addition, Wardowski concluded that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous mapping.

Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, he obtained that every Banach contraction (1.3) satisfies the contractive condition (1.4). On the other side, Example 2.5 in [24] shows that the mapping T which is not F_1-contraction (Banach contraction), but still is an F_2-contraction. Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem 1.4 ([24])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

On the other hand, in 1969, using the concept of the Hausdorff metric, Nadler [17] introduced the notion of multivalued contraction mapping and proved a multivalued version of the well known Banach contraction principle. First we recall that Hausdorff metric H induced by a metric d on a set X. Let (X, d) be a metric space. Denote by $P(X)$ the family of all nonempty subsets of X, $CB(X)$ the family of all nonempty, closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. It is well known that, $H : CB(X) \times CB(X) \to \mathbb{R}$ defined by, for every $A, B \in CB(X)$,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

is a metric on $CB(X)$, which is called Hausdorff metric induced by d, where $d(x,B) = \inf\{d(x,y) : y \in B\}$. Let $T : X \to CB(X)$ be a map, then T is called multivalued contraction if for all $x, y \in X$ there exists $L \in [0,1)$ such that

$$H(Tx,Ty) \leq Ld(x,y).$$

Then Nadler [17] proved that every multivalued contraction mapping on complete metric space has a fixed point.

Inspired by his result, since then various fixed point results concerning multivalued contractions has been further developed in different directions by many authors. (see,[6, 7, 8, 11, 13, 14]). For generalizing the Nadler's result, Reich [20] presented the following problem: Let (X, d) be a complete metric space. Suppose that, $T : X \to CB(X)$ satisfies

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

for all $x, y \in X$, $x \neq y$, where $\alpha : (0,\infty) \to [0,1)$ and $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in (0,\infty)$. Does T have a fixed point? Reich [19] gives an affirmative answer to this problem when Tx is nonempty compact for $x \in X$. Another partial affirmative answer to the classical unsolved problem of Reich [20] was given by by Mizoguchi and Takahashi [16]. They consider the condition $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,\infty)$ on α. We can find both a simple proof of Mizoguchi-Takahashi fixed point theorem and an example showing that it is a real generalization of Nadler's in [23]. We can find some important results about this direction in [9].

The aim of this paper is to introduce the multivalued F-contractions, by combining the ideas of Wardowski and Nadler, and give a fixed point result for this type of mappings on a complete metric space.

2. The results

Definition 2.1. Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. Then T is said to be a multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x, y \in X\ [H(Tx,Ty) > 0) \Rightarrow \tau + F(H(Tx,Ty)) \leq F(d(x,y))]. \tag{2.1}$$

When we consider $F(\alpha) = \ln \alpha$,we can say that every multivalued contraction is also multivalued F-contraction.

Our main result is as follows:

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.*

Remark 2.3. Let A be a compact subset of a metric space (X, d) and $x \in X$, then there exists $a \in A$ such that $d(x, a) = d(x, A)$.

Proof of Theorem 2.2. Let $x_0 \in X$. As Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is complete. Let $x_1 \notin Tx_1$. Then, since Tx_1 is closed, $d(x_1, Tx_1) > 0$. On the other hand, from $d(x_1, Tx_1) \leq H(Tx_0, Tx_1)$ and (F1)

$$F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1))$$

From (2.1), we can write that

$$F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)) \leq F(d(x_1, x_0)) - \tau \tag{2.2}$$

Since Tx_1 is compact, we obtain that there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$. Then, from (2.2)

$$F(d(x_1, x_2)) \leq F(H(Tx_0, Tx_1)) \leq F(d(x_1, x_0)) - \tau$$

If we continue recursively, then we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n-1})) - \tau \tag{2.3}$$

for all $n = 1, 2, \cdots$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of T and so the proof is complete. Thus, suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote $a_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \cdots$. Then $a_n > 0$ for all $n \in \mathbb{N}$ and, using (2.3), the following holds:

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{2.4}$$

From (2.4), we get $\lim_{n\to\infty} F(a_n) = -\infty$.Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0.$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

By (2.4), the following holds for all $n \in \mathbb{N}$

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0. \tag{2.5}$$

Letting $n \to \infty$ in (2.5), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{2.6}$$

From (2.6), there exits $n_1 \in \mathbb{N}$ such that $n a_n^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$a_n \leq \frac{1}{n^{1/k}}. \tag{2.7}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$.Using the triangular inequality for the metric and from (2.7), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \end{aligned}$$

$$\leq \sum_{i=n}^{\infty} a_i$$
$$\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, passing to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

From (2.1), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$H(Tx, Ty) < d(x, y)$$

and so

$$H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$. Then

$$d(x_{n+1}, Tz) \leq H(Tx_n, Tz) \leq d(x_n, z)$$

Passing to limit $n \to \infty$, we obtain $d(z, Tz) = 0$.Thus, we get $z \in \overline{Tz} = Tz$. This completes the proof. □

Remark 2.4. Note that in Theorem 2.2, Tx is compact for all $x \in X$. Thus, we can present the following problem: Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction. Does T has a fixed point? By adding a condition on F, we can give a partial answer for this problem as follows:

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction. Suppose that, F also satisfies*

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

Then T has a fixed point.

Proof. Let $x_0 \in X$. As Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is complete. Let $x_1 \notin Tx_1$. Then, since Tx_1 is closed, $d(x_1, Tx_1) > 0$. On the other hand, from $d(x_1, Tx_1) \leq H(Tx_0, Tx_1)$ and (F1)

$$F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)).$$

From (2.1), we can write that

$$F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)) \leq F(d(x_1, x_0)) - \tau. \tag{2.8}$$

From (F4) we can write (note that $d(x_1, Tx_1) > 0$)

$$F(d(x_1, Tx_1)) = \inf_{y \in Tx_1} F(d(x_1, y)),$$

and so from (2.8) we have

$$\inf_{y \in Tx_1} F(d(x_1, y)) \leq F(d(x_1, x_0)) - \tau < F(d(x_1, x_0)) - \frac{\tau}{2}. \tag{2.9}$$

Then, from (2.9) there exists $x_2 \in Tx_1$ such that

$$F(d(x_1,x_2)) \leq F(d(x_1,x_0)) - \frac{\tau}{2}.$$

If $x_2 \in Tx_2$ we are finished. Otherwise, by the same way we can find $x_3 \in Tx_2$ such that

$$F(d(x_2,x_3)) \leq F(d(x_2,x_1)) - \frac{\tau}{2}.$$

We continue recursively, then we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$F(d(x_n,x_{n+1})) \leq F(d(x_n,x_{n-1})) - \frac{\tau}{2}$$

for all $n = 1, 2, \cdots$. The rest of the proof can be completed as in the proof of Theorem 2.2. □

Remark 2.6. Note that if F is right-continuous and satisfies (F1), then it satisfies (F4).

In the light of the Example 2.5 of [24], we can give the following example. This example shows that T is a multivalued F-contraction but it is not multivalued contraction.

Example 2.7. Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x,y) = |x-y|, x, y \in X$. Then (X,d) is a complete metric space. Define the mapping $T : X \to K(X)$ by the formulae:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases}$$

We claim that T is a multivalued F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$. To see this, we consider the following cases.

First, observe that

$$\forall m, n \in \mathbb{N} \ [H(Tx_m, Tx_n) > 0 \Leftrightarrow ((m > 2 \text{ and } n = 1) \text{ or } (m > n > 1))]$$

Case 1. For $m > 2$ and $n = 1$, we have

$$\begin{aligned} \frac{H(Tx_m,Tx_1)}{d(x_m,x_1)} e^{H(Tx_m,Tx_1)-d(x_m,x_1)} &= \frac{x_{m-1}-x_1}{x_m-x_1} e^{x_{m-1}-x_m} \\ &= \frac{m^2-m-2}{m^2+m-2} e^{-m} < e^{-m} < e^{-1} \end{aligned}$$

Case 2. For $m > n > 1$, we have

$$\begin{aligned} \frac{H(Tx_m,Tx_n)}{d(x_m,x_n)} e^{H(Tx_m,Tx_n)-d(x_m,x_n)} &= \frac{x_{m-1}-x_{n-1}}{x_m-x_n} e^{x_{m-1}-x_{n-1}-x_m+x_n} \\ &= \frac{m+n-1}{m+n+1} e^{n-m} < e^{n-m} \leq e^{-1} \end{aligned}$$

This shows that T is multivalued F-contraction (see (1.4)), therefore, all conditions of Theorem 2.2 (or Theorem 2.5) are satisfied and so T has a fixed point in X.

On the other hand, since

$$\lim_{n\to\infty} \frac{H(Tx_n,Tx_1)}{d(x_n,x_1)} = \lim_{n\to\infty} \frac{x_{n-1}-1}{x_n-1} = 1,$$

then T is not a multivalued contraction.

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