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# NONSMOOTH WEIGHTED VARIATIONAL INEQUALITIES AND NONSMOOTH VECTOR OPTIMIZATION

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Abstract. In this paper, we use weighted sum method to study nonsmooth vector variational inequalities and nonsmooth vector optimization problem. In particular, we introduce nonsmooth weighted variational inequalities (in short, NWVI) and study some relationships among NWVI, nonsmooth vector variational inequalities (in short, NVVI), nonsmooth weighted optimization problem (in short, NWOP) and nonsmooth vector optimization problem (in short, NVOP). We establish some existence results for solutions of (NVVI) and (NWVI) under weighted pseudomonotonicity or densely weighted pseudomonotonicity. As applications of our results, some existence results for solutions of NWOP and NVOP for nondifferentiable functions by using the equivalence relations among NVVI, NWVI, NWOP and VOP can be easily derived.

## 1. Introduction

The theory of vector variational inequalities, initiated by Giannessi [10] in 1980, is one of the most elegant and power tools to study vector optimization problems (in short, VOP); See, for example, [1,2,4–6,9–12,17–19,22–24,26] and the references therein. The (vector) optimization problem may have a nonsmooth objective function. Therefore, Crespi et. al. [6] considered the Minty vector variational inequality defined by means of lower Dini directional derivative. They established the relations between a Minty vector variational inequality (MVVI) and the solutions of vector minimization problem (both ideal and weakly efficient but not efficient) solutions. Crespi et. al. [6] used the scalarization method to obtain their results. Lalitha and Mehta [17] considered a nonsmooth Stampacchia type vector variational inequality (in short, NSVVI) and established its equivalence with VOP. They established the existence of solutions of VOP under certain conditions. In [2], Ansari and Lee considered both the Minty and the Stampacchia type vector variational inequalities (MVVIs and SVVIs, respectively) defined by means of upper Dini directional derivative. By using the (MVVI), they provided a necessary and sufficient condition for an efficient solution of VOP for pseudoconvex functions involving upper Dini directional derivative. They established the relationship between the (MVVI) and the (SVVI) under upper sign continuity. Some relationships among efficient solutions,

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weakly efficient solutions, solutions of the (SVVI) and solutions of the (MVVI) are discussed. They also presented an existence result for the solutions of weak (MVVI) and the weak (SVVI). Their approach seems to be more direct than the one adopted in [6]. They extended the results of [11, 26] for pseudoconvex functions involving upper Dini directional derivative.

In this paper, we adopt the weighted sum method to study the nonsmooth vector variational inequalities and nonsmooth vector optimization problem. In particular, we introduce nonsmooth weighted variational inequalities (in short, NWVI) and study some relationships among nonsmooth weighted variational inequalities (in short, NWVI), nonsmooth vector variational inequalities (in short, NVVI) and nonsmooth vector optimization problem. We establish some existence results for solutions of (NVVI) and (NWVI) under weighted pseudomonotonicity or weighted pseudomonotonicity. As an application of our result, we derive an existence result for solutions of (WOP) for nondifferentiable functions by using the equivalence relations among (NVVI), (NWVI), (WOP) and (VOP). In the same way, several other existence results for solutions of (WOP) for nondifferentiable functions can be easily derived by using our results.

## 2. Formulations

Throughout the paper, we denote by $\mathbb{R}^n_+$ the non-negative orthant of $\mathbb{R}^n$, that is,

$$\mathbb{R}^n_+ = \{u = (u_1, \dots, u_n) \in \mathbb{R}^n : u_j \geq 0, \text{ for } j = 1, \dots, n\},$$

so that $\mathbb{R}^n_+$ has a nonempty interior with the topology induced in terms of convergence of vectors with respect to the Euclidean metric. That is,

$$\text{int } \mathbb{R}^n_+ = \{u = (u_1, \dots, u_n) \in \mathbb{R}^n : u_j > 0, \text{ for } j = 1, \dots, n\}.$$

We denote by $\mathbb{T}^n_+$ and int $\mathbb{T}^n_+$ the simplex of $\mathbb{R}^n_+$ and its interior, respectively, that is,

$$\mathbb{T}^n_+ = \left\{ u = (u_1, \dots, u_n) \in \mathbb{R}^n_+ : \sum_{j=1}^n u_j = 1 \right\}, \text{ and}$$

$$\text{int } \mathbb{T}^n_+ = \left\{ u = (u_1, \dots, u_n) \in \text{int } \mathbb{R}^n_+ : \sum_{j=1}^n u_j = 1 \right\}.$$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. The upper and lower Dini directional derivatives of $g$ at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ are defined as

$$\text{(upper Dini directional derivative)} \qquad g^D(x; d) = \limsup_{t \to 0^+} \frac{g(x + td) - g(x)}{t},$$

$$\text{(lower Dini directional derivative)} \qquad g_D(x; d) = \liminf_{t \to 0^+} \frac{g(x + td) - g(x)}{t}.$$

It is easy to see that $g_D(x; d) \leq g^D(x; d)$. If the function $g$ is convex, then the upper and lower Dini directional derivatives are equal to the directional derivative.

It can be easily seen that for all $r > 0$, $(rg)^D(x; d) = rg^D(x; d)$, and for all $r < 0$, $(rg)^D(x; d) = rg_D(x; d)$. Also, $g^D(x; d) \geq -g^D(x; -d)$ for all $x, d \in \mathbb{R}^n$.

We adopt the following ordering relations. We consider the cones $C_{\mathbf{0}} := \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$ and $\overset{\circ}{C} := \operatorname{int} \mathbb{R}^{\ell}_{+}$, where $\mathbb{R}^{\ell}_{+}$ is the nonnegative orthant of $\mathbb{R}^{\ell}$ and $\mathbf{0}$ is the origin of $\mathbb{R}^{\ell}$; let $D$ be a set of $\mathbb{R}^{\ell}$. Then for all $x, y \in D$,

$$x \geq_{C_{\mathbf{0}}} y \quad \Leftrightarrow \quad x - y \in C_{\mathbf{0}}; \qquad x \not\geq_{C_{\mathbf{0}}} y \quad \Leftrightarrow \quad x - y \notin C_{\mathbf{0}};$$

$$x \leq_{C_{\mathbf{0}}} y \quad \Leftrightarrow \quad y - x \in C_{\mathbf{0}}; \qquad x \not\leq_{C_{\mathbf{0}}} y \quad \Leftrightarrow \quad y - x \notin C_{\mathbf{0}};$$

$$x \geq_{\overset{\circ}{C}} y \quad \Leftrightarrow \quad x - y \in \overset{\circ}{C}; \qquad x \not\geq_{\overset{\circ}{C}} y \quad \Leftrightarrow \quad x - y \notin \overset{\circ}{C};$$

$$x \leq_{\overset{\circ}{C}} y \quad \Leftrightarrow \quad y - x \in \overset{\circ}{C}; \qquad x \not\leq_{\overset{\circ}{C}} y \quad \Leftrightarrow \quad y - x \notin \overset{\circ}{C}.$$

Let $K$ be a nonempty subset of $\mathbb{R}^n$ and $f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. The *vector optimization problem* (VOP) is defined as follows:

$$\text{(VOP)} \qquad \min f(x), \quad \text{subject to } x \in K,$$

where $f(x) = (f_1(x), \ldots, f_\ell(x))$.

A point $\bar{x} \in K$ is said to be an *efficient solution* (respectively, *weak efficient solution*) of (VOP) if and only if

$$f(\bar{x}) \not\geq_{C_{\mathbf{0}}} f(y), \quad \text{for all } y \in K,$$

$$\left(\text{respectively, } f(\bar{x}) \not\geq_{\overset{\circ}{C}} f(y), \quad \text{for all } y \in K\right).$$

It is clear that every efficient solution is a weak efficient solution.

Let $K$ be a nonempty subset of $\mathbb{R}^n$, $f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function, and $f^D(x; d) = \left(f_1^D(x; d), \ldots, f_\ell^D(x; d)\right)$. The vector variational inequalities are two types, one is called Stampacchia type and another one is called Minty type. The *nonsmooth Stampacchia and Minty type vector variational inequality problems* are defined as follows:

(NSVVIP): Find $\bar{x} \in K$ such that

$$f^D(\bar{x}; y - \bar{x}) = \left(f_1^D(\bar{x}; y - \bar{x}), \ldots, f_\ell^D(\bar{x}; y - \bar{x})\right) \not\leq_{C_{\mathbf{0}}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.1}$$

(NMVVIP): Find $\bar{x} \in K$ such that

$$f^D(y; y - \bar{x}) = \left(f_1^D(y; y - \bar{x}), \ldots, f_\ell^D(y; y - \bar{x})\right) \not\leq_{C_{\mathbf{0}}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.2}$$

If we replace the order relation $\not\leq_{C_{\mathbf{0}}}$ by $\not\leq_{\overset{\circ}{C}}$ in (2.1) and $\not\leq_{C_{\mathbf{0}}}$ by $\not\leq_{\overset{\circ}{C}}$ in (2.2), then we get the following weak formulations of (NSVVIP) and (NMVVIP):

$(\text{NSVVIP})_w$: Find $\bar{x} \in K$ such that

$$f^D(\bar{x}; y - \bar{x}) = \left(f_1^D(\bar{x}; y - \bar{x}), \ldots, f_\ell^D(\bar{x}; y - \bar{x})\right) \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.3}$$

$(\text{NMVVIP})_w$: Find $\bar{x} \in K$ such that

$$f^D(y; y - \bar{x}) = \left(f_1^D(y; y - \bar{x}), \ldots, f_\ell^D(y; y - \bar{x})\right) \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.4}$$

Crespi et al. [6] considered these kinds of problems and studied the existence of their solutions. By using such existence results, they also studied the existence of an efficient or weak efficient solution of (VOP).

Let $h = (h_1, \dots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function such that, for each fixed $x \in K$, $h(x; d)$ is positively homogeneous in $d$. If we consider the (upper or lower) Dini directional derivative as a bifunction $h(x; d)$, with $x$ referring to a point in $\mathbb{R}^n$ and $d$ referring to a direction from $\mathbb{R}^n$, then (2.1), (2.2), (2.3) and (2.4) become the following nonsmooth vector variational inequality problems, namely, Stampacchia type vector variational inequality problems and Minty type vector variational inequality problems.
*Stampacchia vector variational inequality problem* (SVVIP): Find $\bar{x} \in K$ such that

$$h(\bar{x}; y - \bar{x}) = (h_1(\bar{x}; y - \bar{x}), \dots, h_\ell(\bar{x}; y - \bar{x})) \not\leq_{C_\mathbf{0}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.5}$$

*Minty vector variational inequality problem* (MVVIP): Find $\bar{x} \in K$ such that

$$h(y; y - \bar{x}) = \big(h_1(y; y - \bar{x}), \dots, h_\ell(y; y - \bar{x})\big) \not\leq_{C_\mathbf{0}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.6}$$

If we replace the order relation $\not\leq_{C_\mathbf{0}}$ by $\not\leq_{\overset{\circ}{C}}$ in (2.5) and $\not\leq_{C_\mathbf{0}}$ by $\not\leq_{\overset{\circ}{C}}$ in (2.5), then we get the following weak formulations of (SVVIP) and (MVVIP):
$(\text{SVVIP})_w$: Find $\bar{x} \in K$ such that

$$h(\bar{x}; y - \bar{x}) = (h_1(\bar{x}; y - \bar{x}), \dots, h_\ell(\bar{x}; y - \bar{x})) \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.7}$$

$(\text{MVVIP})_w$: Find $\bar{x} \in K$ such that

$$h(y; y - \bar{x}) = (h_1(y; y - \bar{x}), \dots, h_\ell(y; y - \bar{x})) \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{2.8}$$

We introduce the following weighted Stampacchia variational inequality problem and weighted Minty variational inequality problem.
*Weighted Stampacchia variational inequality problem* (WSVIP): Find $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, \dots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that

$$W \cdot h(\bar{x}; y - \bar{x}) = \sum_{i=1}^{\ell} W_i h_i(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K, \tag{2.9}$$

where $\cdot$ denotes the inner product on $\mathbb{R}^\ell$.

*Weighted Minty variational inequality problem* (WMVIP): Find $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, \dots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that

$$W \cdot h(y; y - \bar{x}) = \sum_{i=1}^{\ell} W_i h_i(y; y - \bar{x}) \geq 0, \quad \text{for all } y \in K. \tag{2.10}$$

If $W \in \mathbb{T}^\ell_+$, then the solution of (WSVIP) and (WMVIP) is called *normalized*.

If $h_i(x; d) = f_i^D(x; d)$ for each $i = 1, 2, \dots \ell$ and all $x, d \in \mathbb{R}^n$, then (2.9) and (2.10) are called *weighted nonsmooth Stampacchia variational inequality* (WNSVI) and *weighted nonsmooth Minty variational inequality* (WNMVI), respectively.

Rest of the paper, unless otherwise specified, we assume that $W = (W_1, \dots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ is a given weight vector.

We establish the following lemma which shows the relationship between (WSVIP) and (SVVIP).

**Lemma 2.1.** *Each normalized solution $\bar{x} \in K$ with the weight vector $W \in \mathbb{T}_+^\ell$ (respectively, $W \in$ int $\mathbb{T}_+^\ell$) of (WSVIP) is a solution of (SVVIP)$_w$ (respectively, (SVVIP)).*

*Proof.* Let $\bar{x} \in K$ be a normalized solution of (WSVIP) with the weight vector $W \in \mathbb{T}_+^\ell$ (respectively, $W \in \text{int } \mathbb{T}_+^\ell$). Assume contrary that $\bar{x} \in K$ is not a solution of (SVVIP)$_w$ (respectively, (SVVIP)). Then there exists some $y \in K$ such that

$$h(\bar{x}; y - \bar{x}) = (h_1(\bar{x}; y - \bar{x}), \ldots, h_\ell(\bar{x}; y - \bar{x})) \leq_{\overset{\circ}{C}} \mathbf{0}.$$

$$\Big(\text{respectively,} \quad h(\bar{x}; y - \bar{x}) = (h_1(\bar{x}; y - \bar{x}), \ldots, h_\ell(\bar{x}; y - \bar{x})) \leq_{C_0} \mathbf{0}.\Big)$$

Since $W \in \mathbb{T}_+^\ell$ (respectively, $W \in \text{int } \mathbb{T}_+^\ell$), we have

$$W \cdot h(\bar{x}; y - \bar{x}) < 0,$$

which contradicts to our assumption that $\bar{x} \in K$ is a normalized solution of (WSVIP). Hence $\bar{x} \in K$ is a solution of (SVVIP)$_w$ (respectively, (SVVIP)). □

In the same way as Lemma 2.1, we can easily establish the following lemma.

**Lemma 2.2.** *Each normalized solution $\bar{x} \in K$ with the weight vector $W \in \mathbb{T}_+^\ell$ (respectively, $W \in$ int $\mathbb{T}_+^\ell$) of (WMVIP) is a solution of (MVVIP)$_w$ (respectively, (MVVIP)).*

## 3. Preliminaries

Throughout the paper, unless otherwise specified, $K$ is a nonempty convex subset of $\mathbb{R}^n$.

A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be:

(a) *positively homogeneous* if for all $x \in \mathbb{R}^n$ and all $r \geq 0$, we have $g(rx) = rg(x)$;
(b) *subodd* if for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we have $g(x) \geq -g(-x)$.

**Definition 3.1.** A function $g : K \to \mathbb{R}$ is said to be:

(a) *quasiconvex* if for all $x, y \in K$ and all $\lambda \in ]0, 1[$,

$$g(x + \lambda(y - x)) \leq \max\{g(x), g(y)\};$$

(b) *semistrictly quasiconvex* if for all $x, y \in K$ with $g(y) < g(x)$,

$$g(x + \lambda(y - x)) < g(x), \quad \text{for all } \lambda \in ]0, 1[.$$

**Definition 3.2** ([15]). Let $q : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. A function $g : K \to \mathbb{R}$ is said to be:

(b) *q-convex* if for all $x, y \in K$,

$$q(x; y - x) \leq g(y) - g(x).$$

If strict inequality holds, then $g$ is called *strictly q-convex*;

(b) *q-quasiconvex* if for all $x, y \in K$,

$$g(y) < g(x) \quad \Rightarrow \quad q(x; y - x) \leq 0;$$

(c) *q-pseudoconvex* if for all $x, y \in K$, $x \neq y$,

$$g(y) < g(x) \quad \Rightarrow \quad q(x; y - x) < 0;$$

(d) *strictly q-pseudoconvex* if for all $x, y \in K$, $x \neq y$,

$$g(y) \leq g(x) \quad \Rightarrow \quad q(x; y - x) < 0.$$

If $q \equiv f^D$, where $f = (f_1, \ldots, f_\ell) : K \to \mathbb{R}^\ell$, the $q$-convexity, $q$-quasiconvexity, and so on are called $f^D$-convexity, $f^D$-quasiconvexity, and so on, respectively.

**Lemma 3.3** ([15, Theorem 4.1]). *Let $g : K \to \mathbb{R}$ be a function and $p, q : K \times \mathbb{R}^n \to \mathbb{R}$ be bifunctions such that for all $x \in K$ and all $d \in \mathbb{R}^n$, $p(x; d) \leq q(x; d)$. Then q-pseudoconvexity, and strict q-pseudoconvexity imply p-pseudoconvexity, and strict p-pseudoconvexity, respectively.*

**Definition 3.4** ([13,15]). A bifunction $q : K \times \mathbb{R}^n \to \mathbb{R}$ is said to be *pseudomonotone* if for every pair of distinct points $x, y \in K$, we have

$$q(x; y - x) \geq 0 \quad \Rightarrow \quad q(y; y - x) \geq 0. \tag{3.1}$$

**Definition 3.5.** A real-valued function $g : K \to \mathbb{R}$ is said to be *radially upper (lower) semicontinuous* (also known as *upper (lower) hemicontinuous* on $K$ if it is upper (lower) semicontinuous along the line segment in $K$.

If $g$ is radially upper as well as radially lower semicontinuous on $K$, then it is called *radially semicontinuous* on $K$.

**Lemma 3.6** ([25, Theorems 3.2 and 5.2]). *Let $g : K \to \mathbb{R}$ be a radially upper semicontinuous on $K$ and let the bifunction $q : K \times \mathbb{R}^n \to \mathbb{R}$ be subodd in the second argument such that for all $x \in K$, $q(x; \cdot) \leq g^D(x; \cdot)$. Then*

(a) *$g$ is q-pseudoconvex if and only if $q$ is pseudomonotone;*
(b) *$g$ is quasiconvex if and only if it is q-quasiconvex.*

**Lemma 3.7** ([7, Corollaries 15 and 16]). *Let $g : K \to \mathbb{R}$ be upper semicontinuous and $g^D$-pseudoconvex. Then, $g$ is quasiconvex and semistrictly quasiconvex.*

**Theorem 3.8** (Diewert Mean-Value Theorem). *[7, Theorem 1] Let $g : K \to \mathbb{R}$ be radially upper semicontinuous on $K$. Then, for any pair $x, y$ of distinct points of $K$, there exists $w \in [x, y[$ such that*

$$f(y) - f(x) \geq f^D(w; y - x),$$

*where $[x, y[$ denotes the line segment joining $x$ and $y$ including the endpoint $x$. In other words, there exists $\lambda \in [0, 1[$ such that*

$$f(y) - f(x) \geq f^D(w; y - x), \quad \text{where } w = x + \lambda(y - x).$$

Now we recall some definitions and results which will be used in the sequel.

For every nonempty set $A$, we denote by $2^A$ the family of all subsets of $A$. If $A$ is a nonempty subset of a vector space, then $coA$ denotes the convex hull of $A$.

The following result will be used to prove the existence of a solution of our problems.

**Theorem 3.9** ( [8, KKM-Fan Theorem]). *Let $K$ be a nonempty convex subset of a Hausdorff topological vector space $E$. Assume that $G : K \to 2^K \setminus \{\emptyset\}$ be a set-valued map satisfying the following conditions:*

(i) *For all $x \in K$, $G(x)$ is closed and is compact for at least one $x \in K$.*
(ii) *For any finite set $\{x_1, \ldots, x_m\}$ of $K$, $co\{x_1, \ldots, x_m\} \subseteq \bigcup_{i=1}^m G(x_i)$.*

*Then $\bigcap_{x\in K} G(x) \neq \emptyset$.*

A set-valued map $G : K \to 2^K$ is called a *KKM map* if it satisfies condition (ii) in Theorem 3.9.

We shall use the following fixed point theorem which is a particular form of Corollary 1 in [3].

**Theorem 3.10** ( [3]). *Let $K$ be a nonempty convex subset of a Hausdorff topological vector space $E$ and $S, T : K \to 2^K$ be set-valued maps. Assume that the following conditions hold:*

(i) *For all $x \in K$, $S(x)$ is nonempty and $coS(x) \subseteq T(x)$.*
(ii) *For all $y \in K$, $S^{-1}(y) = \{x \in K : y \in S(x)\}$ is open.*
(iii) *There exist a nonempty compact convex subset $C$ of $K$ and a nonempty compact subset $D$ of $K$ such that for each $x \in K \setminus D$, there exists $\tilde{y} \in C$ satisfying $x \in S^{-1}(\tilde{y})$.*

*Then there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.*

## 4. Necessary and sufficient conditions

Throughout the paper, unless otherwise specified, $h = (h_1, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is a vector-valued bifunction.

**Definition 4.1.** A vector-valued function $f = (f_1, \ldots, f_\ell) : K \to \mathbb{R}^\ell$ is said to be:

(a) *weighted quasiconvex* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for all $x, y \in K$ and all $\lambda \in ]0, 1[$,
$$W \cdot f(x + \lambda(y - x)) \leq \max\{W \cdot f(x), W \cdot f(y)\};$$
(b) *weighted semistrictly quasiconvex* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for all $x, y \in K$,
$$f(y) < f(x) \quad \Rightarrow \quad W \cdot f(x + \lambda(y - x)) < W \cdot f(x), \quad \text{for all } \lambda \in ]0, 1[.$$

**Definition 4.2.** A vector-valued function $f = (f_1, \ldots, f_\ell) : K \to \mathbb{R}^\ell$ is said to be:

(a) *weighted $h$-pseudoconvex* (respectively, *strictly weighted $h$-pseudoconvex*) w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for all $x, y \in K$,
$$W \cdot h(x; y - x) \geq 0 \quad \Rightarrow \quad W \cdot f(x) \leq W \cdot f(y)$$
$$\Big(\text{respectively,} \quad W \cdot h(x; y - x) \geq 0 \quad \Rightarrow \quad W \cdot f(x) < W \cdot f(y)\Big);$$
(b) *weighted $h$-quasiconvex* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for all $x, y \in K$,
$$W \cdot f(y) < W \cdot f(x) \quad \Rightarrow \quad W \cdot h(x; y - x) \leq 0.$$

If $h \equiv f^D$, then the weighted $h$-pseudoconvexity (respectively, strict weighted $h$-pseudoconvexity and weighted $h$-quasiconvexity) w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ is called weighted $f^D$-pseudoconvexity (respectively, strict weighted $f^D$-pseudoconvexity and weighted $f^D$-quasiconvexity) w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.

**Lemma 4.3.** *Let $f = (f_1, \dots, f_\ell) : K \to \mathbb{R}^\ell$ be a vector-valued function such that for each $f_i$ is radially upper semicontinuous on $K$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function and $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that $d \mapsto W \cdot h(x; d)$ is subodd and $W \cdot h(x; d) \le (W \cdot f)^D(x; d)$ for all $x \in K$ and $d \in \mathbb{R}^n$.*

(a) *$f$ is weighted quasiconvex if and only if it is weighted $h$-quasiconvex w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.*
(b) *$f$ is weighted $h$-pseudoconvex if and only if $h$ is weighted pseudomonotone w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.*

*Proof.* Consider $g(x) = W \cdot f(x)$ and $q(x; d) = W \cdot h(x; d)$ for all $x \in K$ and $d \in \mathbb{R}^n$. Then by Lemma 3.6, we get the conclusion. $\square$

**Lemma 4.4.** *Let $f = (f_1, \dots, f_\ell) : K \to \mathbb{R}^\ell$ be weighted $f^D$-pseudoconvex w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ and for each $i = 1, \dots, \ell$, $f_i$ be upper semicontinuous. Then $f$ is weighted quasiconvex and weighted semistrictly quasiconvex w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.*

*Proof.* Define $g : K \to \mathbb{R}$ by

$$g(x) = \sum_{i=1}^{\ell} W_i f_i(x), \quad \text{for all } x \in K.$$

Then for all $x \in K$ and all $d \in \mathbb{R}^n$, we have

$$\begin{aligned} g^D(x; d) &= (W_1 f_1 + \cdots + W_\ell f_\ell)^D (x; d) \\ &\le W_1 f_1^D(x; d) + \cdots + W_\ell f_\ell^D(x; d) \\ &= W \cdot f^D(x; d). \end{aligned}$$

We claim that $g$ is $g^D$-pseudoconvex, that is, for all $x, y \in K$ with $x \ne y$, we have $g(y) < g(x)$ implies $g^D(x; y - x) < 0$.

Let $g(y) < g(x)$. Then, $W \cdot f(y) < W \cdot f(x)$. Since $f$ is weighted $f^D$-pseudoconvex w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$, we have $W \cdot f^D(x; y - x) < 0$, that is, $g^D(x; y - x) < 0$. Hence $g$ is $g^D$-pseudomonotone. Since each $f_i$ is upper semicontinuous, $g$ is also upper semicontinuous. Hence, by Lemma 3.7, $g$ is quasiconvex and semistrictly quasiconvex, that is, $f$ is weighted quasiconvex and weighted semistrictly quasiconvex w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. $\square$

Let $K$ be a nonempty subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ be a vector-valued function. We consider the following *weighted optimization problem:*

(WOP) $$\min W \cdot f(x) \quad \text{subject to } x \in K,$$

where $W = (W_1, \dots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.

If $W \in \mathbb{T}^\ell_+$, then the solution of (WOP) is called *normalized.*

The following lemma is well-known.

**Lemma 4.5.** (See, for example, [21, Theorem 3.1.1 and Theorem 3.1.2]) *Each normalized solution $\bar{x} \in K$ with weight vector $W \in \mathbb{T}_+^\ell$ (respectively, $W \in int\ \mathbb{T}_+^\ell$) of (WOP) is a weak efficient solution of (VOP) (respectively, an efficient solution of (VOP)).*

The following result provides the relation among the solutions of (WOP), (WNSVIP), (WSVIP), and efficient solution of (VOP).

**Proposition 4.6.** *Let $K$ be a nonempty convex subset of $\mathbb{R}^n$.*

(a) *If $\bar{x} \in K$ is a solution of (WOP), then it is a solution of (WNSVIP).*
(b) *If $-f$ is strictly weighted $h$-pseudoconvex w. r. t. the weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$, then every solution of (WOP) is a solution of (WSVIP).*
(c) *If $f$ is weighted $h$-pseudoconvex w. r. t. the weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$, then every solution of (WSVIP) is a solution of (WOP).*
(d) *If $f$ is strictly weighted $h$-pseudoconvex w. r. t. the weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$, then the solution (if there is any) of (WSVIP) is unique. Furthermore, this unique solution of (WSVIP) is an efficient solution of (VOP).*

*Proof.* (a) Let $\bar{x} \in K$ be a solution of (WOP). Then for all $y \in K$,

$$W \cdot f(y) - W \cdot f(\bar{x}) \geq 0.$$

Since $K$ is convex, $\bar{x} + \lambda(y - \bar{x}) \in K$ for all $\lambda \in [0, 1]$, and hence,

$$\frac{1}{\lambda}\left(W \cdot f(\bar{x} + \lambda(y - \bar{x})) - W \cdot f(y)\right) \geq 0, \quad \text{for all } \lambda \in \,]0, 1].$$

Therefore,

$$(W \cdot f)^D\,(\bar{x}; y - \bar{x}) = \limsup_{\lambda \to 0^+} \frac{W \cdot f(\bar{x} + \lambda(y - \bar{x})) - W \cdot f(y)}{\lambda} \geq 0.$$

Thus,

$$0 \leq (W \cdot f)^D\,(\bar{x}; y - \bar{x}) \leq W \cdot f^D(\bar{x}; y - \bar{x}),$$

and hence, $\bar{x} \in K$ is a solution of (WNSVIP).

(b) Suppose that $\bar{x}$ is a solution of (WOP), but not a solution of (WSVIP). Then, there exists $y \in K$ such that

$$W \cdot h(\bar{x}; y - \bar{x}) < 0. \tag{4.1}$$

Since $-f$ is strictly weighted $h$-pseudoconvex, we have

$$W \cdot f(\bar{x}) > W \cdot f(y),$$

a contradiction to our supposition that $\bar{x}$ is a solution of (WOP). Hence, $\bar{x}$ is a solution of (WSVIP).

(c) Assume that $\bar{x} \in K$ is a solution of (WSVIP) w. r. t. the weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$. Then,

$$W \cdot h(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K.$$

By weighted $h$-pseudoconvexity of $f$ w. r. t. the weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$, we obtain the conclusion.

(d) Suppose that there are two distinct solutions $\bar{x}$ and $\hat{x}$ of (WSVIP). Then,

$$W \cdot h(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K, \tag{4.2}$$

and

$$W \cdot h(\hat{x}; y - \hat{x}) \geq 0, \quad \text{for all } y \in K. \tag{4.3}$$

By strict weighted $h$-pseudoconvexity of $f$, we obtain

$$W \cdot f(\bar{x}) < W \cdot f(y), \quad \text{for all } y \in K,$$

and

$$W \cdot f(\hat{x}) < W \cdot f(y), \quad \text{for all } y \in K.$$

In particular, we have

$$W \cdot f(\bar{x}) < W \cdot f(\hat{x}) < W \cdot f(\bar{x}),$$

which is a contradiction. Hence, the solution of (WSVIP) is unique.

Furthermore, suppose that $\bar{x}$ is a unique solution of (WSVIP) but not an efficient solution of (VOP). Then, there exists $\tilde{y} \in K$ such that

$$f(\bar{x}) \geq_{C_{\mathbf{0}}} f(\tilde{y}) \quad \Leftrightarrow \quad f(\bar{x}) - f(\tilde{y}) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

Therefore, $f_i(\bar{x}) \geq f_i(\tilde{y})$ for all $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$ and $f_j(\bar{x}) > f_j(\tilde{y})$ for some $j \in \mathscr{I}$. Since $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$, we have $W \cdot f(\bar{x}) \geq W \cdot f(\tilde{y})$. By strict weighted $h$-pseudoconvexity of $f$, we obtain

$$W \cdot h(\bar{x}; \tilde{y} - \bar{x}) < 0,$$

a contradiction to the fact that $\bar{x}$ is a solution of (WSVIP). Hence, $\bar{x}$ is an efficient solution of (VOP). □

In view of Lemma 4.5 and Proposition 4.6 (c), we conclude the following result.

**Corollary 4.7.** *If $f = (f_1, \ldots, f_\ell) : K \to \mathbb{R}^\ell$ is weighted $h$-pseudoconvex w. r. t. the weight vector $W \in \mathbb{T}^\ell_+$ (respectively, $W \in \operatorname{int} \mathbb{T}^\ell_+$), then every solution of (WSVIP) is a weak efficient solution of (VOP) (respectively, an efficient solution of (VOP)).*

We present the relation between a solution of (WMVIP) and a solution of (WOP).

**Proposition 4.8.** *Let $K$ be a nonempty convex subset of $\mathbb{R}^n$. For each $i \in \mathscr{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be upper semicontinuous and, for all $x \in K$, let $h_i(x; \cdot)$ be positively homogeneous and subodd. Let $f$ be weighted $f^D$-pseudoconvex w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that $W \cdot h(x; \cdot) \leq (W \cdot f)^D(x; \cdot)$ for all $x, d \in \mathbb{R}^n$. Then, $\bar{x} \in K$ is a solution of (WMVIP) if and only if it is a solution of (WOP) w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.*

*Proof.* Suppose that $\bar{x} \in K$ is a solution of (WMVIP), but not a solution of (WOP) w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. Then, there exists $y \in K$ such that

$$W \cdot f(x) > W \cdot f(y).$$

Let $y(\lambda) = \bar{x} + \lambda(y - \bar{x})$ for all $\lambda \in [0, 1]$. Then $y(\lambda) \in K$ for all $\lambda \in [0, 1]$ because $K$ is convex. Since $f$ is weighted $f^D$-pseudoconvex w. r. t. the weight vector

$W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$, by Lemma 4.4, $f$ is weighted quasiconvex and weighted semistrictly quasiconvex w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. Therefore,

$$W \cdot f(y(\lambda)) < W \cdot f(\bar{x}), \quad \text{for all } \lambda \in ]0,1[. \tag{4.4}$$

Consider a real-valued function $g : K \to \mathbb{R}$ defined by

$$g(x) = \sum_{i=1}^{\ell} W_i f_i(x), \quad \text{for all } x \in K.$$

Then $g^D(x;d) \leq \sum_{i=1}^{\ell} W_i f_i^D(x;d)$ for all $x \in K$ and all $d \in \mathbb{R}^n$. It is easy to see that $g^D(x;\cdot)$ positively homogeneous. Also, $g$ is upper semicontinuous because each $f_i$ is upper semicontinuous. By the Diewert Mean-Value Theorem 3.8, there exists $\alpha \in ]0,1[$ such that

$$g(y(\lambda)) - g(\bar{x}) \geq g^D(y(\alpha); y(\lambda) - \bar{x}), \quad \text{for all } \lambda \in ]0,1[. \tag{4.5}$$

The inequality (4.4) can be re-written as

$$g(y(\lambda)) = g(\bar{x} + \lambda(y - \bar{x})) < g(\bar{x}). \tag{4.6}$$

By combining (4.5) and (4.6), we obtain

$$g^D(y(\alpha); y(\lambda) - \bar{x}) < 0.$$

Since $g^D(x;\cdot)$ is positively homogeneous in the second argument, we have

$$g^D(y(\alpha); y - \bar{x}) < 0.$$

Since $y(\alpha) - \bar{x} = \alpha(y - \bar{x})$, we have

$$g^D(y(\alpha); y(\alpha) - \bar{x}) < 0.$$

By suboddness of $g^D(\cdot;\cdot)$ in the second argument, we obtain $g^D(y(\alpha); \bar{x} - y(\alpha)) > 0$. Since $g^D(x;d) \leq \sum_{i=1}^{\ell} W_i f_i^D(x;d)$ for all $x \in K$ and $d \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{\ell} W_i f_i^D(y(\alpha); y(\alpha) - \bar{x}) > 0,$$

a contradiction to the fact that $\bar{x}$ is a solution of (WMVIP). Hence, $\bar{x} \in K$ is a solution of (WOP).

Conversely, assume that $\bar{x} \in K$ is a solution of (WOP), then $W \cdot f(\bar{x}) \leq W \cot f(y)$ for all $y \in K$. By weighted $f^D$-pseudoconvexity of $f$ and Lemma 3.6, $f$ is weighted $f^D$-quasiconvex, and hence

$$W \cdot h(y; \bar{x} - y) \leq 0.$$

Since $W \cdot h(x;d) \leq W \cdot f^D(x;d) \leq (W \cdot f)^D(x;d)$, we have

$$W \cdot f^D(y; \bar{x} - y) \leq 0.$$

By suboddness of $h$ in the second argument, we obtain

$$W \cdot h(y; \bar{x} - y) \geq 0.$$

Thus, $\bar{x}$ is a solution of (WMVIP). □

## 5. Existence Theory

**Definition 5.1.** A vector-valued bifunction $h = (h_1, \dots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be:

(a) *weighted pseudomonotone* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for all $x, y \in K$,
$$W \cdot h(x; y - x) \geq 0 \quad \Rightarrow \quad W \cdot h(y; y - x) \geq 0;$$

(b) *strictly weighted pseudomonotone* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for all $x, y \in K$ with $x \neq y$,
$$W \cdot h(x; y - x) \geq 0 \quad \Rightarrow \quad W \cdot h(y; y - x) > 0;$$

(c) *weighted subodd* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if $W \cdot h(x; d) \geq -W \cdot h(x; -d)$;

(d) *weighted proper subodd* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if
$$W \cdot h(x; d) = W_1 h_1(x; d_1) + W_2 h_2(x; d_2) + \cdots + W_\ell h_\ell(x; d_\ell) \geq 0,$$
for every $d_i \in \mathbb{R}^n$ with $\sum_{i=1}^{\ell} d_i = 0$ and $x \in K$.

We introduce the notion of weighted upper sign continuity for a bifunction $h$, which extends the concept of upper sign continuity introduced by Hadjisavvas [14].

**Definition 5.2.** A vector-valued bifunction $h = (h_1, \dots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be *weighted upper sign continuous* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for all $x, y \in K$ and $\lambda \in ]0, 1[$,
$$W \cdot h(x + \lambda(y - x); y - x) \geq 0 \quad \Rightarrow \quad W \cdot h(x; y - x) \geq 0.$$

Of course, if the function $x \mapsto W \cdot h(x; d)$ is radially upper semicontinuous, then $h$ is weighted upper sign continuous.

The following proposition can be easily proved by using the definition of weighted pseudomonotonicity and weighted upper sign continuity, and therefore, we omit the proof.

**Lemma 5.3.** *Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted pseudomonotone and weighted upper sign continuous w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that for each fixed $x \in K$, $h(x; \cdot)$ is positively homogeneous. Then, $\bar{x} \in K$ is a solution of (WSVIP) if and only if it is a solution of (WMVIP).*

**Corollary 5.4.** *Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted pseudomonotone w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that $W \cdot h(x; d)$ is radially upper semicontinuous and positively homogeneous in $d$. Then, $\bar{x} \in K$ is a solution of (WSVIP) if and only if it is a solution of (WMVIP).*

**Definition 5.5** ([20]). A subset $K_0$ of $K$ is said to be *segment-dense in* $K$ if for all $x \in K$, there can be found $x_0 \in K_0$ such that $x$ is a cluster point of the set $[x, x_0] \cap K_0$, where $[x, x_0]$ denotes the line segment joining $x$ and $x_0$ including end points.

We define densely weighted pseudomonotonicity, which generalize the notion of densely pseudomonotonicity considered by Luc [20].

**Definition 5.6.** The vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be *densely weighted pseudomonotone (*respectively*, densely strict weighted pseudomonotone)* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ on $K$ if there exists a segment-dense subset $K_0 \subseteq K$ such that $h$ is weighted pseudomonotone (respectively, strictly weighted pseudomonotone) w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ on $K_0$.

We consider the following weighted Minty variational inequality problem defined over the set $K_0$, where $K_0$ is same as in the definition of densely weighted pseudomonotone map. (WMVIP)$_0$: Find $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, \ldots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that

$$W \cdot h(y; y - \bar{x}) \geq 0, \quad \text{for all } y \in K_0. \tag{5.1}$$

Obviously, (WMVIP) $\Rightarrow$ (WMVIP)$_0$.

**Lemma 5.7.** *If $W \cdot h(\cdot;\cdot)$ is upper semicontinuous in both the argument, then (WMVIP)$_0$ $\Rightarrow$ (WMVIP).*

*Proof.* Let $\bar{x} \in K$ be a solution of (WMVIP)$_0$. Then

$$W \cdot h(y; y - \bar{x}) \geq 0, \quad \text{for all } y \in K_0. \tag{5.2}$$

Since $K_0$ is segment-dense, for all $y \in K$, we can find $y_0 \in K_0$ and $y_m \in [y, y_0] \cap K_0$ for all $m \in \mathbb{N}$ such that $\lim_{m\to\infty} y_m = y$. Then from (5.2), we get

$$W \cdot h(y_m; y_m - \bar{x}) \geq 0, \quad \text{for all } m \in \mathbb{N}.$$

Since $\lim_{m\to\infty} y_m = y$ and $h$ is upper semicontinuous in both the arguments, we have

$$W \cdot h(y; y - \bar{x}) \geq \limsup_{m\to\infty} W \cdot h(y_m; y_m - \bar{x}) \geq 0, \quad \text{for all } y \in K.$$

Hence $\bar{x}$ is a solution of (WMVIP). □

We prove the existence of a solution of (WSVIP) under the weighted pseudomonotonicity.

**Theorem 5.8.** *Let $K$ be a nonempty, compact and convex subset of $\mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted upper sign continuous, weighted proper subodd and weighted pseudomonotone w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that for each $x \in K$, $W \cdot h(x; \cdot)$ is positively homogeneous and upper semicontinuous. Then there exists a solution $\bar{x} \in K$ of (WMVIP), and hence, it is a solution of (WSVIP). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}$ then there exists a normalized solution $\bar{x} \in K$ of (WSVIP), and hence, it is a solution of (SVVIP)$_w$. Moreover, if $W \in \prod_{i=1}^n (int\, \mathbb{T})$, then $\bar{x} \in K$ is a solution of (SVVIP).*

*Proof.* For each $y \in K$, define two set-valued maps $P, Q : K \to 2^K$ by

$$P(y) = \{x \in K : W \cdot h(x; y - x) \geq 0\}$$

and

$$Q(y) = \{x \in K : W \cdot h(y, y - x) \geq 0\}.$$

By upper semicontinuity of $W \cdot h(x; d)$ in $d$, $Q(y)$ is a closed subset of a compact set $K$, and hence, $Q(y)$ is compact for each $y \in K$. By weighted pseudomonotonicity of $h$, we have $P(y) \subseteq Q(y)$. By using the standard argument and weighted proper subodddness of $h$, it is easy to see that for every finite set $\{x_1, \ldots, x_m\}$ of $K$ one has $\text{co}\{x_1, \ldots, x_m\} \subseteq \bigcup_{k=1}^{m} P(x_k)$ (see for example, proof of Theorem 2.2 in [17]). Since for all $y \in K$, $P(y) \subseteq Q(y)$, we also have, $co\{x_1, \ldots, x_m\} \subseteq \bigcup_{k=1}^{m} Q(x_k)$. By applying Theorem 3.9, we have $\bigcap_{y \in K} Q(y) \neq \emptyset$, that is, there exists $\bar{x} \in K$ such that

$$W \cdot h(y; y - \bar{x}) \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x} \in K$ is a solution of (WMVIP). By Lemma 5.3, $\bar{x}$ is a solution of (WSVIP).

If $W \in \prod_{i \in I} \mathbb{T}$, then $\bar{x} \in K$ is a normalized solution of (WSVIP), and hence, by Lemma 2.1 it is a solution of $(\text{SVVIP})_w$. Further, if $W \in \prod_{i \in I} (\text{int } \mathbb{T})$ then again by Lemma 2.1 $\bar{x} \in K$ is a solution of (SVVIP). □

**Corollary 5.9.** *Let $K$ be a nonempty, compact and convex subset of $\mathbb{R}^n$. Let $f = (f_1, \ldots, f_\ell) : K \to \mathbb{R}^\ell$ be a vector-valued function such that for each $f_i$ is radially upper semicontinuous on $K$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted upper sign continuous and weighted proper subodd w. r. t. the same weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$ such that $d \mapsto W \cdot h(x; d)$ is positively homogeneous, upper semicontinuous and $W \cdot h(x; d) \leq (W \cdot f)^D(x; d)$ for all $x \in K$ and $d \in \mathbb{R}^n$. Then there exists a solution $\bar{x} \in K$ of (WOP). Furthermore, if $W \in \prod_{i \in I} \mathbb{T}_+^\ell$ (respectively, $W \in \text{int } \mathbb{T}_+^\ell$), then there exists a weak efficient solution $\bar{x} \in K$ of (VOP) (respectively, efficient solution $\bar{x} \in K$ of (VOP)).*

**Theorem 5.10.** *Let $K$ be a nonempty, compact and convex subset of $\mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted upper sign continuous, weighted proper subodd and strictly weighted pseudomonotone w. r. t. the same weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$ such that for each $x \in K$, $W \cdot h(x; \cdot)$ is positively homogeneous and upper semicontinuous. Then there exists a unique solution $\bar{x} \in K$ of (WMVIP), and hence, it is a unique solution of (WSVIP). Furthermore, if $W \in \prod_{i \in I} \mathbb{T}$ then there exists a normalized unique solution $\bar{x} \in K$ of (WSVIP), and hence, it is unique solution of $(SVVIP)_w$. Moreover, if $W \in \prod_{i=1}^{n} (int\ \mathbb{T})$, then $\bar{x} \in K$ is a unique solution of (SVVIP).*

*Proof.* In view of Theorem 5.8, it is sufficient to show that (WSVIP) has at most one solution. Suppose there exist two solutions $x'$ and $x''$ of (WSVIP), then we have

$$W \cdot h(x''; x' - x'') \geq 0.$$

By the weighted strictly pseudomonotonicity of $h$, we have

$$W \cdot h(x'; x' - x'') > 0.$$

By weighted proper suboddness of $h$, we have $W \cdot h(x'; x'' - x') < 0$. that is, $x'$ is not a solution of (WSVIP), a contradiction. □

When $K$ is not necessarily compact, we have the following results.

**Theorem 5.11.** *Let $K$ be a nonempty, closed and convex subset of $\mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted upper sign continuous, weighted pseudomonotone and weighted proper subodd w. r. t. weight vector $W \in \mathbb{R}_+^\ell \setminus \{\mathbf{0}\}$ on $K$ such that for each $x \in K$, $W \cdot h(x; \cdot)$ is positively homogeneous and upper semicontinuous. Assume that there exist a compact subset $C$ of $\mathbb{R}^n$ and $\tilde{y} \in C \cap K$ such that*

$$W \cdot h(x; \tilde{y} - x) < 0, \quad \textit{for all } x \in K \setminus C. \tag{5.3}$$

*Then there exists a solution $\bar{x} \in K$ of (WMVIP), and hence, it is a solution of (WSVIP). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}$ then there exists a normalized solution $\bar{x} \in K$ of (WSVIP), and hence, it is a solution of (SVVIP)$_w$. Moreover, if $W \in \prod_{i=1}^{n}(int\ \mathbb{T})$, then $\bar{x} \in K$ is a solution of (SVVIP).*

*Proof.* Let the set-valued maps $P, Q : K \to 2^K$ be the same as in the proof of Theorem 5.8. Let $\tilde{y} \in K$ and the set $C$ be the same as in the hypothesis. Then, we want to show that $P(\tilde{y})$ is compact. If $P(\tilde{y}) \nsubseteq C$, then there exists $x \in P(\tilde{y})$ such that $x \in K \setminus C$. It follows that

$$W \cdot h(x; \tilde{y} - x) \geq 0,$$

which contradicts (5.3). Therefore, we have $P(\tilde{y}) \subseteq C$. Then $P(\tilde{y})$ is a closed subset of a compact set $C$, and hence, compact. Since for each $y \in K$, $Q(y)$ is closed and $P(y) \subseteq Q(y)$. Therefore, $Q(\tilde{y})$ is compact. Then by using the same argument as in the proof of Theorem 5.8, there exists a solution of (WMVIP). Rest of the proof follows from the proof of Theorem 5.8. □

**Theorem 5.12.** *Let $K$ be a nonempty, closed and convex subset of $\mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted upper sign continuous, weighted pseudomonotone and weighted proper subodd w. r. t. weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ on $K$ such that for each $x \in K$, let $h(x; \cdot)$ be positively homogeneous and upper semicontinuous. Assume that there exist a compact subset $D$ of $\mathbb{R}^n$ and $\tilde{y} \in D \cap K$ such that*

$$W \cdot h(\tilde{y}; \tilde{y} - x) < 0, \quad \textit{for all } x \in K \setminus D. \tag{5.4}$$

*Then there exists a solution $\bar{x} \in K$ of (WMVIP), and hence, it is a solution of (WSVIP). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}$ then there exists a normalized solution $\bar{x} \in K$ of (WSVIP), and hence, it is a solution of (SVVIP)$_w$. Moreover, if $W \in \prod_{i=1}^{n}(int\ \mathbb{T})$, then $\bar{x} \in K$ is a solution of (SVVIP).*

*Proof.* Let the set-valued maps $P, Q : K \to 2^K$ be the same as in the proof of Theorem 5.8. Let $\tilde{y} \in K$ and the set $D$ be the same as in the hypothesis. By the same argument as in the proof of Theorem 5.8, we derive that $Q$ is a KKM map and, for each $y \in K$, $Q(y)$ is closed. It can be easily seen that $Q(\tilde{y}) \subseteq (K \cap D)$ is a compact subset of $K$. Then, by Theorem 3.9, $\bigcap_{y\in K} Q(y) \neq \emptyset$. Rest of the proof follows on the lines of the proof of Theorem 5.8. □

We use Theorem 3.10 to establish the following existence result for a solution of (WMVIP) and (WSVIP). Some conditions in this result are different from the conditions in Theorems 5.11 and 5.12.

**Theorem 5.13.** *Let $K$ be a nonempty convex subset of $\mathbb{R}^n$. Let $h = (h_1, \ldots, h_\ell) : K \to \mathbb{R}^\ell$ be weighted pseudomonotone, weighted upper sign continuous w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. For each $x \in K$, let $h(x; \cdot)$ be weighted quasiconvex w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that $W \cdot h(x; \cdot)$ is upper semicontinuous and positively homogeneous with $W \cdot h(x, \mathbf{0}) = 0$. Assume that there exist a nonempty compact convex subset $C$ of $K$ and a nonempty compact subset $D$ of $K$ such that for each $x \in K \setminus D$, there exists $\tilde{y} \in C$ such that $W \cdot h(\tilde{y}; \tilde{y} - x) < 0$. Then there exists a solution $\bar{x} \in K$ of (WMVIP), and hence, it is a solution of (WSVIP). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}$ then there exists a normalized solution $\bar{x} \in K$ of*

*(WSVIP), and hence, it is a solution of (SVVIP)$_w$. Moreover, if* $W \in \prod_{i=1}^n (int\, \mathbb{T})$, *then* $\bar{x} \in K$ *is a solution of (SVVIP).*

*Proof.* For each $x \in K$, define set-valued maps $S, T : K \to 2^K$ by

$$S(x) = \{y \in K : W \cdot h(y; y - x) < 0\}$$

and

$$T(x) = \{y \in K : W \cdot h(x; y - x) < 0\}.$$

By the weighted quasiconvexity of $h$ in the second argument, $T(x)$ is convex, for each $x \in K$. From the weighted pseudomonotonicity of $h$, we have $S(x) \subseteq T(x)$ for all $x \in K$. Since $T(x)$ is convex, $coS(x) \subseteq T(x)$ for all $x \in K$.

For each $y \in K$, the complement of $S^{-1}(y)$ in $K$ is

$$[S^{-1}(y)]^c = \{x \in K : W \cdot h(y; y - x) \geq 0\}$$

is closed in $K$ because the upper semicontinuity of $W \cdot h(x; d)$ in $d$. Hence, $S^{-1}(y)$ is open in $K$.

Assume that for all $x \in K$, $S(x)$ is nonempty. Then all the conditions of Theorem 3.10 are satisfied, and therefore, there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$. It follows that

$$0 = W \cdot h(\hat{x}; \hat{x} - \hat{x}) < 0,$$

a contradiction. Hence, there exists $\bar{x} \in K$ such that $S(\bar{x}) = \emptyset$. This implies that for all $y \in K$,

$$W \cdot h(y; y - \bar{x}) \geq 0.$$

Thus, $\bar{x}$ is a solution of (WMVIP). By Lemma 5.3, $\bar{x} \in K$ is a solution of (WSVIP). Rest of the proof is same as in the last part of the proof of Theorem 5.8 □

We introduce weighted properly quasiconvexity which generalizes the notion of 0-diagonally quasiconcavity [27].

**Definition 5.14.** Let $K$ be a nonempty subset of $\mathbb{R}^n$. A vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be *weighted properly quasimonotone* w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ if for every $x_1, x_2, \ldots, x_m \in K$ and every $y \in co\{x_1, x_2, \ldots, x_m\}$, there exists $i \in \{1, 2, \ldots, m\}$ such that

$$W \cdot h(x_i; y - x_i) \leq 0.$$

**Theorem 5.15.** *Let* $K$ *be a nonempty compact convex subset of* $\mathbb{R}^n$ *and let* $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ *be weighted properly quasimonotone w. r. t. the weight vector* $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ *such that* $W \cdot h(\cdot; \cdot)$ *is upper semicontinuous. Then, (WMVIP) has a solution.*

*Proof.* Define the set-valued mapping $P : K \to 2^K$ by

$$P(y) = \{x \in K : W \cdot h(y; y - x) \geq 0\}, \quad \text{for all } y \in K.$$

For any $y_1, y_2, \ldots, y_m \in K$ and $\tilde{y} \in co\{y_1, y_2, \ldots, y_m\}$, weighted proper quasimonotonicity implies that $\tilde{y} \in \bigcap_{i=1}^m P(y_i)$. Also, for each $y \in K$, $P(y)$ is a closed subset of a compact set $K$, and hence, compact. Therefore, by Theorem 3.9, it follows that $\bigcap_{y \in K} P(y) \neq \emptyset$. Thus, any $\bar{x} \in \bigcap_{y \in K} P(y)$ is a solution of (WMVIP). □

Finally, we establish an existence result under densely weighted pseudomonotonicity.

**Theorem 5.16.** *Let $K$ be a nonempty, compact and convex subset of $\mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted proper subodd and densely weighted pseudomonotone w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. Assume that $W \cdot h(\cdot;\cdot)$ is upper semicontinuous, and for each $x \in K$, let $W \cdot h(x;\cdot)$ be positively homogeneous. Then there exists a solution $\bar{x} \in K$ of (WMVIP)$_0$, and hence, it is a solution of (WMVIP). Furthermore, $\bar{x}$ is also a solution of (WSVIP).*

*Proof.* Let $K_0$ be the same as in the definition of a densely weighted pseudomonotone map. For each $y \in K_0$, define two set-valued maps $P, Q : K_0 \to 2^K$ by

$$P(y) = \{x \in K : W \cdot h(x; y - x) \geq 0\}$$

and

$$Q(y) = \{x \in K : W \cdot h(y, y - x) \geq 0\}.$$

Following the same argument as in the proof of Theorem 5.8, we obtain that (WMVIP)$_0$ has a solution $\bar{x}$. By Lemma 5.7, $\bar{x}$ is a solution of (WMVIP). Therefore, by Lemma 5.3, $\bar{x} \in K$ is a solution of (WSVIP). □

**Corollary 5.17.** *Let $K$ be a nonempty, compact and convex subset of $\mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weighted proper subodd and densely strict weighted pseudomonotone w. r. t. the same weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. Assume that $W \cdot h(\cdot;\cdot)$ is upper semicontinuous, and for each $x \in K$, let $W \cdot h(x;\cdot)$ be positively homogeneous. Then there exists a solution $\bar{x} \in K$ of (WMVIP)$_0$, and hence, it is a solution (WMVIP). Furthermore, $\bar{x} \in K$ is a solution of (WSVIP), and it is unique if $\bar{x} \in K_0$.*

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