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FEASIBLE ITERATIVE ALGORITHMS FOR SPLIT COMMON SOLUTION PROBLEMS

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Abstract. In this paper, we introduce some new feasible iterative algorithms for the split common solution problems for equilibrium problems and fixed point problems of nonlinear mappings. Some examples illustrating our results are also given.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with zero vector θ, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty subset of H and T be a mapping from K into itself. The set of fixed points of T is denoted by $\mathcal{F}(T)$. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ weakly converges to x and $x_n \to x$ will symbolize strong convergence as usual.

Let K be a nonempty closed convex subset of H and let f be a bi-function from $K \times K$ into $\mathbb{R}$. The classical equilibrium problem is to find $x \in K$ such that

$$f(x,y) \geq 0, \ \ \forall\, y \in K. \tag{1.1}$$

Let $EP(f)$ denote the set of all solutions of the problem (1.1). Since a lot of problems in physics, optimization, and economics reduce to find a solution of (1.1) (see, for instance, [2, 12]), some authors have proposed some methods to find the solution of equilibrium problem (1.1); for instance, see [2, 7, 8, 12]. Some iterative algorithms for fixed point problems of nonexpansive mappings and the equilibrium problem (1.1) have been constructed; see, [4, 10, 11, 15, 16, 17].

Recently, some authors considered the common solution for a system of equilibrium problems and fixed point problems of nonlinear operators. Let I be an index set. For each $i \in I$, let f_i be a bi-function from $K \times K$ into $\mathbb{R}$. The system of equilibrium problem is to find $x \in K$ such that

$$f_i(x,y) \geq 0, \ \ \forall\, y \in K \ \text{ and } \forall\, i \in I. \tag{1.2}$$

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Let $\bigcap_{i\in I} EP(f_i)$ denote the set of all common solutions of the system of equilibrium problem (1.2).

For each $i \in I$, if $f_i(x,y) = \langle A_i x, y-x\rangle$, where $A_i : K \to K$ is a nonlinear operator, then the problem (1.2) becomes the following system of variational inequality problem:

$$\text{(1.3)} \qquad \text{Find an element } x \in K \text{ such that } \langle A_i x, y-x\rangle \geq 0, \ \forall\, y \in K.$$

As a generalization of nonexpansive mappings, some authors have constructed some iterative algorithms for fixed point problems of quasi-nonexpansive mappings and the equilibrium problem (1.1); see, [6, 14, 18, 19].

In this paper, we present a split common solution problem for fixed point problems of nonlinear mappings and equilibrium problems as follows.

Let E_1 and E_2 be two real Banach spaces. Let C be a closed convex subset of E_1, K a closed convex subset of E_2, $A : E_1 \to E_2$ a bounded linear operator, f a bi-function from $C \times C$ into $\mathbb{R}$ and $T : K \to K$ be nonlinear mappings with $\mathcal{F}(T) \neq \emptyset$. Suppose that $EP(f) \neq \emptyset$. We consider the mathematical model about the split common solution problem (**SCSP**, for short) as follows.

$$(\textbf{SCSP}) \qquad \text{Find an element } y \in EP(f) \text{ such that } Ay \in \mathcal{F}(T).$$

Let $\{p \in EP(f) : Ap \in \mathcal{F}(T)\}$ be the solution set of **SCSP**.

A simple example is given hereunder.

Example 1.1. Let $E_1 = E_2 = \mathbb{R}$, $C = [1, +\infty)$ and $K = (-\infty, -2]$. Let $f : C \times C \to \mathbb{R}$, $A : \mathbb{R} \to \mathbb{R}$ and $T : K \to K$ be define by $f(x,y) = y - x$, $A(x) = -2x$, $T(x) = x$, respectively. Clearly, A is a bounded linear operator, $EP(f) = \{1\}$ and $A(1) = -2 \in \mathcal{F}(T)$. So $1 \in \{p \in EP(f) : Ap \in \mathcal{F}(T)\} \neq \emptyset$.

Recently, the common solution problem for the equilibrium problem (1.1) and the fixed point problem of nonlinear operators have been studied by many authors in real Hilbert spaces or real Banach spaces and many strong or weak convergence theorems were established. However, the equilibrium problem (1.1) and the fixed point problem of nonlinear operators always belong to difference subsets of spaces in general. These show that **SCSP** is very important and it is an essence of the development of the common solution problem for the equilibrium problem (1.1) and the fixed point problem of nonlinear operators. In this paper, we introduce some new feasible iterative algorithms for the split common solution problems for equilibrium problems and fixed point problems of nonlinear mappings. Some examples illustrating our results are also given.

2. Preliminaries

A Banach space $(X, \|\cdot\|)$ is said to satisfy *Opial's condition*, if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall\, y \in X, y \neq x.$$

It is well known that any Hilbert space satisfies Opial's condition.

Let K be a nonempty subset of a Banach space $(X, \|\cdot\|)$. Recall that a mapping $T : K \to K$ is said to be

(1) *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$;
(2) *quasi-nonexpansive* if $\mathcal{F}(T) \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in K$ and $p \in \mathcal{F}(T)$.

Definition 2.1 (see [13])**.** Let K be a nonempty closed convex subset of a real Hilbert space H and T a mapping from K into K. The mapping T is said to be *demiclosed* if, for any sequence $\{x_n\}$ which weakly converges to y, and if the sequence $\{Tx_n\}$ strongly converges to z, then $Ty = z$.

Remark 2.2. In Definition 2.1, the particular case of demiclosedness at zero is frequently used in some iterative convergence algorithms, which is the particular case when $z = \theta$, the zero vector of H; for more detail, one can refer to [13].

Now, we first introduce the concept of zero-demiclosedness.

Definition 2.3. Let K be a nonempty closed convex subset of a real Hilbert space and T a mapping from K into K. The mapping T is called *zero-demiclosed* if $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \to 0$ and $x_n \rightharpoonup z \in K$ implies $Tz = z$.

Proposition 2.4. *Let K be a nonempty closed convex subset of a real Hilbert space with zero vector θ. Then the following statements hold.*

(a) *Let T be a mapping from K into K. Then T is zero-demiclosed if and only if $I - T$ is demiclosed at θ;*
(b) *Let T be a nonexpansive mapping from H into itself. If there is a bounded sequence $\{x_n\} \subset H$ such that $\|x_n - Tx_n\| \to 0$ as $n \to 0$, then T is zero-demiclosed.*

Proof. Obviously, the conclusion (a) holds. To see (b), since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $z \in H$ such that $x_{n_k} \rightharpoonup z$. One can claim $Tz = z$. Indeed, if $Tz \neq z$, it follows from the Opial's condition that

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - z\| &< \liminf_{k\to\infty} \|x_{n_k} - Tz\| \\
&\le \liminf_{k\to\infty} \{\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\|\} \\
&= \liminf_{k\to\infty} \|Tx_{n_k} - Tz\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - z\|,
\end{aligned}$$

which is a contradiction. So $Tz = z$ and hence T is zero-demiclosed. □

Example 2.5. Let $H = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C := [0, +\infty)$. Let T be a mapping from C into C defined by

$$Tx = \begin{cases} \frac{1}{x}, & x \in (1, +\infty), \\ 0, & x \in [0, 1]. \end{cases}$$

Then T is a quasi-nonexpansive mapping but not zero-demiclosed.

Proof. It is easy to verify that $\mathcal{F}(T) = \{0\}$ and T is a quasi-nonexpansive mapping. We claim that T is not zero-demiclosed. Let $\{x_n\}$ be a sequence defined by $x_n = 1 + \frac{1}{n}$, $n \in \mathbb{N}$. Clearly, $x_n \to 1$ and $x_n - Tx_n \to 0$ as $n \to \infty$ and $1 \notin \mathcal{F}(T)$. So T is not zero-demiclosed. □

Example 2.6. Let $H = \mathbb{R}$ with the inner product defined by $\langle x, y \rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C := [0, 1]$. Let T_1, T_2 be two mappings from C into C defined by

$$T_1x = \begin{cases} \frac{7}{8}, & x = 1/5, \\ 1, & \text{otherwise} \end{cases}$$

and

$$T_2x = \begin{cases} \frac{5}{6}, & x = 1/5, \\ 1, & \text{otherwise} \end{cases}.$$

Then T_1 and T_2 are all zero-demiclosed quasi-nonexpansive mappings.

Proof. It is easy to verify that $\mathcal{F}(T_1) = \mathcal{F}(T_2) = \{1\}$ and T_1, T_2 are all quasi-nonexpansive mappings, so it suffices to prove that T_1 and T_2 are all zero-demiclosed.

Let $\{x_n\} \subset C$ is a sequence satisfying $x_n - T_1x_n \to 0$ and $x_n \to z$ as $n \to \infty$. We want to prove $z \in \mathcal{F}(T_1)$ or, to be more precise, $z = 1$. In fact, since $x_n - T_1x_n \to 0$, without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $x_{n_i} \neq 1/5$ for all $i \in \mathbb{N}$. Since

$$|z - 1| \leq |z - x_{n_i}| + |x_{n_i} - T_1x_{n_i}| + |T_1x_{n_i} - 1| \to 0 \quad \text{as } n_i \to \infty,$$

which implies $z = 1$. This shows T_1 is a zero-demiclosed mapping. Similarly, we can prove T_2 is also a zero-demiclosed mapping. □

Let K be a closed convex subset of a real Hilbert space H. For each point $x \in H$, there exists a unique nearest point in K, denoted by P_Kx, such that

$$\|x - P_Kx\| \leq \|x - y\|, \ \forall\, y \in K.$$

The mapping P_K is called the *metric projection* from H onto K. It is well known that P_K satisfies

$$\langle x - y, P_Kx - P_Ky \rangle \geq \|P_Kx - P_Ky\|^2$$

for every $x, y \in H$. Moreover, P_Kx is characterized by the property: for $x \in H$, and $z \in K$,

$$z = P_K(x) \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \ \ \forall\, y \in K.$$

The following results are crucial to our main results.

Lemma 2.7 (see, e.g., [4])**.** *Let H be a real Hilbert space. Then the following hold.*

(a) $\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y \rangle$ *for all* $x, y \in H$;

(b) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$ *for all* $x, y \in H$.

Lemma 2.8 (see [2])**.** *Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions.*

(A1) $F(x, x) = 0$ *for all* $x \in K$;

(A2) F *is monotone, that is,* $F(x, y) + F(y, x) \leq 0$ *for all* x, $y \in K$;

(A3) *for each* $x, y, z \in K$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) *for each* $x \in K, y \mapsto F(x, y)$ *is convex and lower semi-continuous.*

Let $r > 0$ *and* $x \in H$*. Then, there exists* $z \in K$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \text{ for all } y \in K.$$

Lemma 2.9 (see [7])**.** *Let* K *be a nonempty closed convex subset of* H *and let* F *be a bi-function of* $K \times K$ *into* $\mathbb{R}$ *satisfying (A1)-(A4). For* $r > 0$ *and* $x \in H$*, define a mapping* $T_r : H \to K$ *as follows:*

$$T_r(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall\, y \in K \right\} \tag{2.1}$$

for all $x \in H$*. Then the following hold:*

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(iii) $\mathcal{F}(T_r) = EP(F)$*;*

(iv) $EP(F)$ *is closed and convex.*

In 1999, Atsushiba and Takahashi [1] introduced the concept of the W-mapping as follows:

$$\begin{aligned} U_1 &= \beta_1 T_1 + (1-\beta_1)I, \\ U_2 &= \beta_2 T_2 U_1 + (1-\beta_2)I, \\ &\vdots \\ U_{N-1} &= \beta_{N-1} T_{N-1} U_{N-2} + (1-\beta_{N-1})I, \\ W = U_N &= \beta_N T_N U_{N-1} + (1-\beta_N)I. \end{aligned} \tag{2.2}$$

where $\{T_i\}_i^N$ is a finite family of mappings of K into itself and $\beta_i \in [0, 1]$ for all $i = 1, 2, \ldots, N$ with $\sum_{i=1}^N \beta_i = 1$. Such a mapping W is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\beta_1, \beta_2, \ldots, \beta_N$; see also [16].

Lemma 2.10 (see [5])**.** *Let* K *be a nonempty closed convex subset of a strictly convex Banach space* X*. Let* $\{T_i\}_{i=1}^N$ *be a finite family of quasi-nonexpansive and L-Lipschitz mappings of* K *into itself such that* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *. and let* $\beta_1, \beta_2, \ldots, \beta_N$ *be real numbers such that* $0 < \beta_i < 1$ *for all* $i = 1, 2, \ldots, N-1$*,* $0 < \beta_N \leq 1$*, and* $\sum_{i=1}^N \beta_i = 1$*. Let* W *be the* W*-mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\beta_1, \beta_2, \ldots, \beta_N$*. Then, the following conclusions hold:*

(i) W *is quasi-nonexpansive and Lipschitz;*

(ii) $\mathcal{F}(W) = \bigcap_{i=1}^N \mathcal{F}(T_i)$*.*

Remark 2.11. (i) Under the same assumptions as Lemma 2.10, if $\{T_i\}_{i=1}^N$ is a finite family of quasi-nonexpansive mappings of K into itself, then, from the proof of [5, Lemma 3.1], we see that W is quasi-nonexpansive;

(ii) It is well-known that any real Hilbert space is a strictly convex Banach space. So Lemma 2.10 is also true in a real Hilbert space.

Example 2.12. Let H, C, T_1 and T_2 be the same as Example 2.6. Let $U_1x = \frac{1}{2}T_1x + \frac{1}{2}x$ for all $x \in C$. Define a W-mappings as follows:

$$Wx = \frac{1}{2}T_2U_1x + \frac{1}{2}x \text{ for all } x \in C.$$

Then the following hold.

(i) $\mathcal{F}(W) = \mathcal{F}(T_1) = \mathcal{F}(T_2) = \{1\}$;
(ii) W is a zero-demiclosed quasi-nonexpansive mapping.

Proof. It is easy to verify that $1 \in \mathcal{F}(W)$. On the other hand, let $p \in \mathcal{F}(W)$. Then we have

$$\begin{aligned} |p-1| &\le \frac{1}{2}|T_2U_1p - 1| + \frac{1}{2}|p-1| \\ &\le \frac{1}{2}|U_1p - 1| + \frac{1}{2}|p-1| \\ &= \frac{1}{2}\left|\frac{1}{2}T_1p + \frac{1}{2}p - 1\right| + \frac{1}{2}|p-1| \\ &\le \frac{1}{4}|T_1p-1| + \frac{1}{4}|p-1| + \frac{1}{2}|p-1| \\ &\le |p-1|, \end{aligned}$$

which implies the following conclusions hold:

(1) $\frac{1}{2}|U_1p - 1| + \frac{1}{2}|p-1| = |p-1|$;
(2) $\frac{1}{2}|T_2U_1p - 1| + \frac{1}{2}|p-1| = |p-1|$.

From (1), we have $U_1p = p$ which implies $T_1p = p$. By (2), we have $T_2p = p$. So $p \in \mathcal{F}(T_1) = \mathcal{F}(T_2) = \{1\}$ and hence $p = 1$. Thus $\mathcal{F}(W) = \{1\}$ and the conclusion (i) holds.

To see (ii), it is not hard to verify that W is quasi-nonexpansive, so it suffices to prove that W is zero-demiclosed. Let $\{x_n\} \subset C$ be a sequence satisfying $x_n - Wx_n \to 0$ and $x_n \to z$ as $n \to \infty$. From $x_n - Wx_n \to 0$, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \neq 1/5$ for all $l \in \mathbb{N}$. Indeed, let $\Lambda := \{n \in \mathbb{N} : x_n \neq 1/5\}$. If $\sharp(\Lambda)$, the cardinal number of Λ, is finite, then $x_n = 1/5$, $T_1x_n = 7/8$ and $U_1x_n = 43/80 \neq 1/5$ for all $n \in \mathbb{N} \setminus \Lambda$. So $Wx_n = 3/5$ for all $n \in \mathbb{N} \setminus \Lambda$ which implies $\lim_{n\to\infty}(x_n - Wx_n) \neq 0$, a contraction.

Now, we claim $z = 1$. For n_l, since $T_1x_{n_l} = 1$, $U_1x_{n_l} = 1/2 + \frac{1}{2}x_{n_l} \neq 1/5$ and $T_2U_1x_{n_l} = 1$, we have

$$Wx_{n_l} = 1/2 + \frac{1}{2}x_{n_l}$$

and

$$Wx_{n_l} - x_{n_l} = 1/2(1 - x_{n_l}).$$

Since $x_n - Wx_n \to 0$, we get $x_{n_l} \to 1$ which implies $z = 1$. Hence, we show that W is a zero-demiclosed mapping. $\square$

Lemma 2.13 (see [9]). *Let H be a real Hilbert space. Then for any $x_1, x_2, \dots, x_k \in H$ and $a_1, a_2, \dots, a_k \in [0,1]$ with $\sum_{i=1}^k a_i = 1$, $k \in \mathbb{N}$, we have*

$$\left\|\sum_{i=1}^k a_i x_i\right\|^2 = \sum_{i=1}^k a_i \|x_i\|^2 - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} a_i a_j \|x_i - x_j\|^2.$$

In particular, we have

(1) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ *for all $x, y \in H$ and $\alpha \in [0,1]$;*

(2) *the map $f : H \to \mathbb{R}$ defined by $f(x) = \|x\|^2$ is convex.*

3. Main results

In this section, we will introduce some new iterative algorithms for the split common solution problems. First of all, we need the following result.

Lemma 3.1. *Let $I = \{1, 2, \dots, k\}$ be a finite index set. For each $i \in I$, let f_i be bi-functions from $K \times K$ into $\mathbb{R}$ satisfying the conditions (A1)-(A4) and for each $r > 0$, let $T_r^i \colon H \to K$ be defined as (2.1). Let $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\{x_n\} \subset H$ be given. Then the following statements hold.*

(1) *For each $(i,n) \in I \times \mathbb{N}$, $T_{r_n}^i$ is a firmly non-expansive single-valued mapping and $\mathcal{F}(T_{r_n}^i) = EP(f_i)$ is closed and convex.*

(2) *For each $(i,n) \in I \times \mathbb{N}$, let $u_n^i = T_{r_n}^i x_n$ and $z_n = \frac{u_n^1 + u_n^2 + \cdots + u_n^k}{k}$. Then we have*
   (i) $\|z_n - v\|^2 \le \|x_n - v\|^2 - \frac{1}{k}\sum_{i=1}^k \|u_n^i - x_n\|^2$ *for any $v \in \bigcap_{i=1}^k EP(f_i)$.*
   (ii) *If $\|u_n^i - x_n\| \to 0$ and $u_n^i \rightharpoonup z$ as $n \to \infty$, then $z \in \bigcap_{i=1}^k EP(f_i)$.*

Proof. The conclusion (1) follows from Lemma 2.9 immediately. To see (2), we first prove that (i) holds. For any $v \in \bigcap_{i=1}^k EP(f_i)$, by Lemma 2.9 and Lemma 2.7, we obtain

$$\begin{aligned} \|u_n^i - v\|^2 &= \|T_{r_n}^i x_n - T_{r_n}^i v\|^2 \le \langle T_{r_n}^i x_n - T_{r_n}^i v, x_n - v\rangle \\ &= \frac{1}{2}\left\{\|u_n^i - v\|^2 + \|x_n - v\|^2 - \|u_n^i - x_n\|^2\right\}, \end{aligned}$$

which yields that

$$\|u_n^i - v\|^2 \le \|x_n - v\|^2 - \|u_n^i - x_n\|^2. \tag{3.1}$$

By Lemma 2.13, we get

$$\|z_n - v\|^2 \le \frac{1}{k}\sum_{i=1}^k \|u_n^i - v\|^2. \tag{3.2}$$

Hence, it follows from (3.1) and (3.2) that

$$\|z_n - v\|^2 \le \|x_n - v\|^2 - \frac{1}{k}\sum_{i=1}^k \|u_n^i - x_n\|^2$$

and (i) is proved. Finally, we show (ii). For each $i \in I$, since

$$f_i(u_n^i, y) + \frac{1}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \ \forall\, y \in K,$$

it follows from (A2) that

$$\frac{1}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq f_i(y, u_n^i) + f_i(u_n^i, y) + \frac{1}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq f_i(y, u_n^i),$$

and hence

$$\langle y - u_n^i, \frac{u_n^i - x_n}{r_n}\rangle \geq f_i(y, u_n^i), \ \ \forall\, y \in K. \tag{3.3}$$

Applying (A4) and (3.3), we obtain $f_i(y, z) \leq 0, \ \ \forall\, y \in K$. Let $y \in K$ be given. Put

$$y_t = ty + (1-t)z, \ t \in (0,1).$$

Then $y_t \in K$ and $f_i(y_t, z) \leq 0$ for all $i \in I$. For each $i \in I$, by (A1) and (A4), we get

$$0 = f_i(y_t, y_t) \leq t f_i(y_t, y) + (1-t) f_i(y_t, z) \leq t f_i(y_t, y).$$

So $f_i(y_t, y) \geq 0$ for all $i \in I$. For any $i \in I$, by (A3), we have

$$f_i(z, y) \geq \lim_{t \downarrow 0} f_i(ty + (1-t)z, y) = \lim_{t \downarrow 0} f_i(y_t, y) \geq 0,$$

which implies $z \in \bigcap_{i=1}^k EP(f_i)$. □

Theorem 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a nonempty closed convex subset of H_1 and K a nonempty closed convex subset of H_2. Let $I := \{1, 2, \ldots, k\}$ denote a finite index set. For any $i \in I$, let $G_i : C \to C$ be quasi-nonexpansive mappings and $f_i : C \times C \to \mathbb{R}$ be bi-functions. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $T : K \to K$ be a zero-demiclosed quasi-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Let $\beta \in (0,1)$, ρ be the spectral radius of the operator A^*A and $\lambda \in (0, \frac{1}{\rho\beta})$. Let W be the W-mapping generated by $G_1, G_2, \ldots, G_k$ and $\gamma_1, \gamma_2, \ldots, \gamma_k$, where $\gamma_i \in [0,1]$ for all $i \in I$ with $\sum_{i \in I} \gamma_i = 1$.*

Let $\{x_n\}$ and $\{u_n^i\}$ be sequences generated in the following manner:

$$\begin{cases} x_1 \in C, \\ u_n^i = T_{r_n}^i x_n, \quad \forall\, i \in I, \\ v_n = \frac{u_n^1 + \cdots + u_n^k}{k}, \\ x_{n+1} = (1-\alpha_n) y_n + \alpha_n W y_n, \\ y_n = P_C(v_n + \lambda\beta A^*(T - I)A v_n), \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{3.4}$$

where P_C is a projection operator from H_1 into C and the control coefficient sequences $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0, +\infty)$ satisfy the following restrictions:

(D1) *there exists $\xi \in (0,1)$ such that $\alpha_n \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$;*

(D2) $\liminf_{n \to \infty} r_n > 0$.

If W is zero-demiclosed, $\Omega = \left(\bigcap_{i=1}^k EP(f_i)\right) \bigcap \left(\bigcap_{i=1}^k \mathcal{F}(G_i)\right) \neq \emptyset$ and $\Gamma = \{p \in \Omega : Ap \in \mathcal{F}(T)\} \neq \emptyset$, then the sequences $\{x_n\}$ and $\{u_n^i\}$, converge weakly to an element $q \in \Gamma$.

Proof. Let $p \in \Gamma$. So $Ap \in \mathcal{F}(T)$. For each $n \in \mathbb{N}$, applying (2) of Lemma 3.1, we have

$$\|v_n - p\|^2 \leq \|x_n - p\|^2 - \frac{1}{k}\sum_{i=1}^{k}\|u_n^i - x_n\|^2 \tag{3.5}$$

and

$$\|u_n^i - p\| = \|T_{r_n}^i x_n - p\| \leq \|x_n - p\|. \tag{3.6}$$

Since T is quasi-nonexpansive,

$$\|TAv_n - Ap\| \leq \|Av_n - Ap\| \quad \text{for each } n \in \mathbb{N}. \tag{3.7}$$

For each $n \in \mathbb{N}$, by (b) of Lemma 2.7 and (3.7), we have

$$\begin{aligned} 2\lambda\beta\langle v_n - p, A^*(T-I)Av_n\rangle =& 2\lambda\beta\langle A(v_n - p) + (T-I)Av_n \\ & - (T-I)Av_n, (T-I)Av_n\rangle \\ =& \ 2\lambda\beta(\langle TAv_n - Ap, (T-I)Av_n\rangle - \|(T-I)Av_n\|^2) \\ =& \ 2\lambda\beta\Big(\frac{1}{2}\|TAv_n - Ap\|^2 + \frac{1}{2}\|(T-I)Av_n\|^2 \\ & - \frac{1}{2}\|Av_n - Ap\|^2 - \|(T-I)Av_n\|^2\Big) \\ \leq& \ 2\lambda\beta\Big(\frac{1}{2}\|(T-I)Av_n\|^2\|(T-I)Av_n\|^2\Big) \\ =& \ -\lambda\beta\|(T-I)Av_n\|^2. \end{aligned} \tag{3.8}$$

Since ρ is the spectral radius of A^*A, we have

$$\begin{aligned} \lambda^2\beta^2\langle (T-I)Av_n, AA^*(T-I)Av_n\rangle &\leq \rho\lambda^2\beta^2\langle (T-I)Av_n, (T-I)Av_n\rangle \\ &= \rho\lambda^2\beta^2\|(T-I)Av_n\|^2 \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.9}$$

For each $n \in \mathbb{N}$, from (3.4)-(3.9) we have

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C(v_n + \lambda\beta A^*(T-I)Av_n) - P_C p\|^2 \\ &\leq \|v_n + \lambda\beta A^*(T-I)Av_n - p\|^2 \\ &= \|v_n - p\|^2 + \|\lambda\beta A^*(T-I)Av_n\|^2 \\ &\quad + 2\lambda\beta\langle v_n - p, A^*(T-I)Av_n\rangle \\ &= \|v_n - p\|^2 + \lambda^2\beta^2\langle (T-I)Av_n, AA^*(T-I)Av_n\rangle \\ &\quad + 2\lambda\beta\langle v_n - p, A^*(T-I)Av_n\rangle \\ &\leq \|v_n - p\|^2 + \rho\lambda^2\beta^2\|(T-I)Av_n\|^2 \\ &\quad - \lambda\beta\|(T-I)Av_n\|^2 \\ &= \|v_n - p\|^2 - \lambda\beta(1 - \rho\lambda\beta)\|(T-I)Av_n\|^2 \\ &\leq \|x_n - p\|^2 - \lambda\beta(1 - \rho\lambda\beta)\|(T-I)Av_n\|^2. \end{aligned} \tag{3.10}$$

Since $\lambda \in (0, \frac{1}{\rho\beta})$, $1 - \rho\lambda\beta > 0$, by (3.10), we have

$$\|y_n - p\|^2 \leq \|v_n - p\|^2$$

and

$$\|y_n - p\|^2 \le \|x_n - p\|^2 \quad \text{for each } n \in \mathbb{N}.$$

On the other hand, from (3.4) and (3.10), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(y_n - p) + \alpha_n(Wy_n - p)\|^2 \\
&= (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\|Wy_n - p\|^2 - (1-\alpha_n)\alpha_n\|y_n - Wy_n\|^2 \\
&\le (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\|y_n - p\|^2 - (1-\alpha_n)\alpha_n\|y_n - Wy_n\|^2 \qquad (3.11)\\
&= \|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|Wy_n - y_n\|^2 \\
&\le \|v_n - p\|^2 - \alpha_n(1-\alpha_n)\|Wy_n - y_n\|^2 - \lambda\beta(1-\rho\lambda\beta)\|(T-I)Av_n\|^2 \\
&\le \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|Wy_n - y_n\|^2 - \lambda\beta(1-\rho\lambda\beta)\|(T-I)Av_n\|^2.
\end{aligned}$$

Hence we know from (3.11) that the sequence $\{\|x_n - p\|\}$ is nonincreasing and $\lim_{n\to\infty} \|x_n - p\|$ exists. By above inequalities, we also obtain

$$\ell := \lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|v_n - p\| = \lim_{n\to\infty} \|y_n - p\|.$$

From (3.11) again and the condition (D1),

$$\lim_{n\to\infty} \|Wy_n - y_n\| = \lim_{n\to\infty} \|(T-I)Av_n\| = 0. \tag{3.12}$$

Again from (3.5), we have

$$\lim_{n\to\infty} \|u_n^i - x_n\| = 0, \quad \forall i \in I. \tag{3.13}$$

It follows from (3.13) and Lemma 2.5 that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.14}$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergence subsequence $\{x_{n_l}\}$. Let $x_{n_l} \rightharpoonup q$ for some $q \in C$. Then $u_{n_l}^i \rightharpoonup q$, $v_{n_l} \rightharpoonup q$ and $Av_{n_l} \rightharpoonup Aq \in K$ by (3.12)-(3.14). Since A is bounded and $\lim_{n\to\infty} \|(T-I)Av_n\| = 0$, we obtain

$$\begin{aligned}
\|y_n - v_n\| &= \|P_C(v_n + \lambda\beta A^*(T-I)Av_n) - P_C v_n\| \\
&\le \|(v_n + \lambda\beta A^*(T-I)Av_n) - v_n\| \\
&= \|\lambda\beta A^*(T-I)Av_n\| \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

which yields that $y_{n_l} \rightharpoonup q$ as $n_l \to \infty$. Since W is a zero-demiclosed quasi-nonexpansive mapping, by (3.12), we have $q \in \mathcal{F}(W)$. Notice that $u_{n_l}^i \rightharpoonup q$, so, from (2) of Lemma 3.1, (3.13) and (D2), we have $q \in \bigcap_{i=1}^k EP(f_i)$. So, $q \in \mathcal{F}(W) \bigcap (\bigcap_{i=1}^k EP(f_i)) = \Omega$. On the other hand, since T is also a zero-demiclosed mapping, it follows (3.12) that $Aq \in \mathcal{F}(T)$ which implies $q \in \Gamma$.

Finally, we prove $\{x_n\}$ converges weakly to $q \in \Gamma$. Otherwise, if there exists other subsequence of $\{x_n\}$ which is denoted by $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup z \in \Gamma$ with $z \neq q$. Then, by Opial's condition, we get

$$\liminf_{j\to\infty} \|x_{n_j} - z\| < \liminf_{j\to\infty} \|x_{n_j} - q\| < \liminf_{j\to\infty} \|x_{n_j} - z\|,$$

a contradiction. Hence $\{x_n\}$ and $\{u_n^i\}$ converge weakly to an element in Γ, respectively and we obtain the desired result. The proof is completed. □

Remark 3.3. We know that any nonexapansive mapping is quasi-nonexpansive mappings, so Theorem 3.2 also holds when G_i (or T) is nonexpansive for all $i \in I$.

Here, we give a simple example illustrating Theorem 3.2.

Example 3.4. Let $H_1 = H_2 = H = \mathbb{R}$, $C := [0, 1]$ and $K := [-1, 0]$. Let T_1, T_2, W be the same as Examples 2.6 and 2.12. Let $Ax = -x$ for all $x \in \mathbb{R}$. Then A is a bounded linear operator from C into K and A^*(the adjoint of A) $= A$. Let $f_1(x, y) = x - y$ and $f_2(x, y) = 2(x - y)$ for all $x, y \in C$. Then f_1 and f_2 satisfy the condition (A1)-(A4) and $EP(f_1) = EP(f_2) = \{1\}$.

Let T be defined by

$$Tx = \begin{cases} -1 & x \neq -1/5, \\ -7/9 & x = -1/5 \end{cases} \quad \text{for all } x \in K.$$

Then $\Gamma := \left\{p \in \left(\bigcap_{i=1}^{2} EP(f_i)\right) \bigcap \left(\bigcap_{i=1}^{2} F(T_i)\right) : Ap \in F(T)\right\} = \{1\}$. Moreover, following a similar argument as the proof of Example 2.6 or Example 2.12, one can see that T is a zero-demiclosed quasi-nonexpansive mapping.

Let $\{x_n\}$ and $\{u_n^i\}$, $i = 1, 2$, be sequences generated by

$$\begin{cases} f_i(u_n^i, y) + \frac{1}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \ y \in C, i = 1, 2, \\ v_n = \frac{u_n^1 + u_n^2}{2}, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n W y_n, \\ y_n = P_C(v_n + \lambda\beta A^*(T - I)Av_n), \ \forall\, n \in \mathbb{N}, \end{cases}$$

where P_C is a projection operator from H into C, $\lambda, \beta \in (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and the control coefficient sequence $\{r_n\} \subset (0, +\infty)$ satisfies $r_n \geq 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{u_n^i\}$ converge strongly to an element $q \in \Gamma$ for $i = 1, 2$. Moreover, $x_{n+1} = u_n^i = 1$ for all $n \in \mathbb{N}$.

Proof. For $i \in \{1, 2\}$, let

$$\varphi_i(y, z, w, r) = i(z - y) + \frac{1}{r}\langle y - z, z - w\rangle = (z - y)(i + \frac{z - w}{r}), \forall\, y, z, w \in C, \forall\, r \geq 1.$$

It is easy to verify that there exists a unique element $z = 1 \in C$ such that for each $i \in \{1, 2\}$,

$$\varphi_i(y, z, w, r) = i(z - y) + \frac{1}{r}\langle y - z, z - w\rangle \geq 0, \forall\, y, w \in C, \forall\, r \geq 1.$$

So, from $f_i(u_n^i, y) + \frac{1}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0$, $i \in \{1, 2\}$, and $r_n \geq 1$, we have $u_n^1 = u_n^2 = 1$ for all $n \in \mathbb{N}$. Further, $v_n = 1$ for all $n \in \mathbb{N}$. From the definition of T and A, we obtain $(T - I)Av_n = 0$ and $A^*(T - I)Av_n = 0$. Hence $y_n = P_C(v_n + \lambda\beta A^*(T - I)Av_n) = 1$ and $x_{n+1} = u_n^i = 1$ for all $n \in \mathbb{N}$. □

In Theorem 3.2, if the index set I is a singleton, we have the following Corollary 3.5.

Corollary 3.5. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a closed convex subset of H_1 and K a closed convex subset of H_2. Let $S : C \to C$ be a zero-demiclosed quasi-nonexpansive mappings and f be a bi-functions from $C \times C$ into $\mathbb{R}$ with $\Omega = EP(f) \bigcap F(S) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator with* *its adjoint A^* and $T : K \to K$ be a zero-demiclosed quasi-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Let $\beta \in (0,1)$, ρ be the spectral radius of the operator A^*A and $\lambda \in (0, \frac{1}{\rho\beta})$.*

Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:

$$\begin{cases} x_1 \in C, \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ y \in C, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n S y_n, \\ y_n = P_C(u_n + \lambda\beta A^*(T-I)Au_n), \ \forall\ n \in \mathbb{N}, \end{cases}$$

where P_C is a projection operator from H_1 into C and the control coefficient sequences $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfy the following restrictions:

(D1) *there exists $\xi \in (0,1)$ such that $\alpha_n \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$;*

(D2) $\liminf_{n\to\infty} r_n > 0$.

If $\Gamma = \{p \in \Omega : Ap \in \mathcal{F}(T)\} \neq \emptyset$, then the sequences $\{x_n\}$ and $\{u_n\}$, converge weakly to an element $q \in \Gamma$.

If $S = I$ in Corollary 3.5, we have the following Corollary 3.6.

Corollary 3.6. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a closed convex subset of H_1 and K a closed convex subset of H_2. Let f be a bi-function from $C \times C$ into $\mathbb{R}$ with $\Omega = EP(f) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $T : K \to K$ be a zero-demiclosed quasi-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Let $\beta \in (0,1)$, ρ be the spectral radius of the operator A^*A and $\lambda \in (0, \frac{1}{\rho\beta})$.*

Let $x_1 \in C$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:

$$\begin{cases} x_1 \in C, \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ y \in C, \\ x_{n+1} = P_C(u_n + \lambda\beta A^*(T-I)Au_n), \ \forall\ n \in \mathbb{N}, \end{cases}$$

where P_C is a projection operator from H_1 into C and the control coefficient sequences $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfy the following restrictions:

(D1) *there exists $\xi \in (0,1)$ such that $\alpha_n \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$;*

(D2) $\liminf_{n\to\infty} r_n > 0$.

If $\Gamma = \{p \in EP(f) : Ap \in \mathcal{F}(T)\} \neq \emptyset$, then the sequences $\{x_n\}$ and $\{u_n\}$, converge weakly to an element $q \in \Gamma$.

In Corollary 3.5, if $f(x, y) = 0$ for all $x, y \in C$, we have the following Corollary 3.7.

Corollary 3.7. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a closed convex subset of H_1 and K a closed convex subset of H_2. Let $S : C \to C$ be a zero-demiclosed quasi-nonexpansive mapping with $\mathcal{F}(S) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $T : K \to K$ be a zero-demiclosed quasi-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Let $\beta \in (0,1)$, ρ be the spectral radius of the operator A^*A and $\lambda \in (0, \frac{1}{\rho\beta})$.*

Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n S y_n, \\ y_n = P_C(x_n + \lambda\beta A^*(T-I)Ax_n), \ \forall\ n \in \mathbb{N}, \end{cases}$$

where P_C is a projection operator from H_1 into C and there exists $\xi \in (0,1)$ such that $\alpha_n \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$. If $\Gamma = \{p \in EP(f) : Ap \in \mathcal{F}(T)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element $q \in \mathcal{F}(T)$.

Remark 3.8. The class of problems considered in Corollary 3.7 is so-called the split common fixed-point problem which is a generalization of the spilt feasibility problem and the convex feasibility problem; for more detail, see [3, 13].

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