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POSITIVE GROUND STATE SOLUTIONS OF ASYMPTOTICALLY LINEAR SCHRÖDINGER-POISSON SYSTEMS

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Abstract. : In this paper, we study the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi(x)u = q(x)f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \lambda u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $V(x)$ is a real function on $\mathbb{R}^3$ and the parameter $\lambda \in (0, +\infty)$, the nonlinearity $f(s)/s$ tends to 0 and $l \in (0, +\infty)$, respectively, as $s \longrightarrow 0^+$ and $s \longrightarrow +\infty$. Under appropriate assumptions on V, q and f, we give the existence of a positive ground state solution resolved by variational methods, which depends on the parameter λ.

1. Introduction and the main results

This paper has been motivated by the problem:

$$\begin{cases} iV(x)\dfrac{\partial\psi}{\partial t} - \Delta\psi + K(x)\phi(x)\psi = q(x)\tilde{f}(\psi), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)|\psi|^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V(x), K(x)$ are real function on $\mathbb{R}^3$. We are interested in looking for a stationary solution, i.e., $\psi(x,t) = e^{-it}u(x)$ with $u > 0$ in $\mathbb{R}^3$. For this purpose, $\tilde{f}$ is a complex function and supposed to satisfy $\tilde{f}(e^{-it}u) = e^{-it}f(u)$, where f is an arbitrary real function on $(0, +\infty)$. Then it is not difficult to see that u must satisfy the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = q(x)f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

This coupling the nonlinear Schrödinger and the Poisson equations arises in an interesting physical model which describes the interaction of a charged particle with an electromagnet see [1, 5, 10] and the references therein.

Variational methods and critical point theory are powerful tools in studying nonlinear differential equations [16, 19, 26]. In recent years, system (1.2) has been studied extensively via modern variational methods under the various hypotheses. These researches mainly concern the multi-solutions [9, 11, 15, 17, 20]. The concentration of solutions, see [4, 14] and the positive solutions [1, 8, 11, 17, 21, 25, 27, 29, 30, 31].

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There has been a lot of research on the existence of the ground state solutions for system (1.2). In [2, 14, 27], the authors deal with the case when f is critical growth at infinity. In [3, 2, 8, 28], the authors deal with the case when f is super-linear at infinity. Specially, in [22], the authors deal with the case when f is asymptotically linear at infinity. Sun and Chen obtain result as follows:

Theorem A ([22])**.** *Suppose $V(x) \equiv 1$. Moreover, assume that the following conditions hold:*

(A_1) $f \in C(\mathbb{R}, \mathbb{R}^+), f(s) \equiv 0$ for all $s < 0$ and $\displaystyle\lim_{s \to 0^+} \frac{f(s)}{s} = 0$.

(A_2) *There exists $l \in (0, +\infty)$ such that* $\displaystyle\lim_{s \to +\infty} \frac{f(s)}{s} = l$.

(K_1) *$q(x)$ is a positive continuous function and there exists $T_0 > 0$ such that*

$$\sup\{f(s)/s : s > 0\} \leq \inf\{1/q(x) : |x| \geq T_0\}.$$

(K_2) *There exists a constant $\beta \in (0, 1)$ such that*

$$(1-\beta)l > \mu^* := \inf\left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx : u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} q(x)F(u)dx \geq \frac{l}{2}, \right.$$

$$\left. \text{and } \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2dx < 2\beta l \right\},$$

$$\text{where } F(u) = \int_0^u f(s)ds \text{ and } \phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)}{|x-y|}u^2(y)dy.$$

(K_3) $K \in L^2(\mathbb{R}^3)\backslash\{0\},\ K(x) \geq 0$ for all $x \in \mathbb{R}^3$.

Then system (1.2) *has a ground state solution in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.*

Motivated by the above fact, in this paper, our aim is to revisit system (1.2). We consider another case:

• when f is asymptotically linear at infinity, i.e., $\displaystyle\lim_{s \to +\infty} \frac{f(s)}{s} = l \in (0, +\infty)$, and $q \in L^2(\mathbb{R}^3)\backslash\{0\}$. We obtain the existence of a positive ground state solution via variational methods.

In the order to obtain our result, we have to overcome various difficulties. First, the competing effect of the non-local term with the nonlinear term in the functional I_λ gives rise to some difficulties, and I_λ is defined in Section 2. Second, it is not difficult to find that every (PS) sequence is bounded when $3 < q < 5$ in [8] because a variant of global Ambrosetti-Rabinowitz condition is satisfied when $3 < q < 5$ (see [12]). However, for the asymptotically linear case, we have to find another method to verify the boundedness of (PS) sequence. Third, since the embedding $H^1(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3), q \in [2, 6)$, is not compact, in order to recover the compactness, we establish a compactness lemma different from the one in [8]. In fact, this difficulty can be avoided, when autonomous problems are considered, restricting I_λ to the subspace of $H^1(\mathbb{R}^3)$ consisting of radially symmetric functions, or, when one is looking for semi-classical states, by using perturbation methods or a reduction to a finite dimension by the projections method.

We state our main result.

Theorem 1.1. *Suppose that* $K(x) \equiv \lambda \in (0, +\infty)$ *is a parameter and* $(A_1), (A_2)$ *hold. Moreover, assume that the following conditions hold:*

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R}) \bigcap L^\infty(\mathbb{R}^3, \mathbb{R})$ and $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^3$;
(A_3) $q \in L^2(\mathbb{R}^3)\backslash\{0\}$, $q(x) \geq 0$ for all $x \in \mathbb{R}^3$;
(A_4)

$$l > \Lambda := \inf \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx : u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} q(x)u^2 dx = 1 \right\}.$$

Then there exists $\lambda_0 > 0$ *such that system* (1.2) *has a positive ground state solution for any* $\lambda \in (0, \lambda_0)$.

Remark 1.2. Indeed, there are many functions V and q satisfying the conditions of Theorem 1.1.

Remark 1.3. Theorem 1.1 is different from Theorem A (see [22]). In fact, in our paper $q(x)$ can be unbounded in $\mathbb{R}^3$. However, in [22], from the conditions (A_2) and (K_1), we obtain $q(x)$ must be bounded in $\mathbb{R}^3$.

The remainder of this paper is organized as follow. In Section 2, some preliminary results are presented. In Section 3, we give several important lemmas and the proof of Theorem 1.1.

2. Preliminaries

In this section, we give the variant version of the Mountain Pass Theorem, which allows us to find a so-called Cerami-type (PS) sequence, the properties of this kind of (PS) sequence are very helpful in showing the boundedness of the sequence in the asymptotically linear case.

Lemma 2.1 ([13, Mountain Pass Theorem])**.** *Let* E *be a real Banach space with its dual space* E^**, and suppose that* $I \in C^1(E, \mathbb{R})$ *satisfies*

$$\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta, \rho > 0$ *and* $e \in E$ *with* $\|e\| > \rho$. *Let* $c \geq \eta$ *be characterized by*

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ *is the set of continuous paths joining* 0 *and* e*; then there exists a sequence* $\{u_n\} \subset E$ *such that*

$$I(u_n) \longrightarrow c \geq \eta \text{ and } (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

This kind of sequence is usually called a Cerami sequence.
Hereafter we use the following notations:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the scalar product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) dx; \ \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx$$

under (V_1). Specially, $\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + V(x)u^2)dx$, where $\Omega \subset \mathbb{R}^3$.

• $D^{1,2}(\mathbb{R}^3) = \left\{u \in L^6(\mathbb{R}^3) : \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^3),\ i = 1,2,3\right\}$. And the norm of $D^{1,2}(\mathbb{R}^3)$ is defined by

$$\|u\|^2_D = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

• $L^q(\Omega)(1 \le q \le \infty, \Omega \subset \mathbb{R}^3)$, denotes a Lebesgue space, the norm in $L^q(\Omega)$ is denoted by $\|u\|_{L^q(\Omega)}$.

• For any $\rho > 0$ and $z \in H^1(\mathbb{R}^3)$, $B_\rho(z)$ denotes the ball of radius ρ centered at z and $B^c_\rho(z) := \mathbb{R}^3 \setminus B_\rho(z)$.

• The measure of a set $E \subset \mathbb{R}^3$ is denoted by $|E|$.

• H^* denotes the dual space of $H^1(\mathbb{R}^3)$.

• C_i denotes various positive constants which can change from line to line.

• For any $u \in H^1(\mathbb{R}^3)$, the linear operator $T_u : D^{1,2}(\mathbb{R}^3) \longrightarrow \mathbb{R}$ defined as

$$T_u(\nu) = \int_{\mathbb{R}^3} \lambda u^2 \nu dx$$

is continuous. The Hölder inequality and the Sobolev inequality imply

$$|T_u(\nu)| \le \lambda \|u^2\|_{L^{6/5}} \|\nu\|_{L^6} \le \tilde{C}\lambda \|u\|^2 \|\nu\|_D, \text{ where } \tilde{C} > 0. \tag{2.1}$$

Then by the Lax-Milgram theorem there exists $\Phi[u] = \phi_u \in D^{1,2}(\mathbb{R}^3)$ such that for any $\nu \in D^{1,2}(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla\nu dx = \int_{\mathbb{R}^3} \lambda u^2 \nu dx. \tag{2.2}$$

Therefore, $-\Delta\phi_u = \lambda u^2$ in a weak sense. We can write an integral expression for ϕ_u in the form

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\lambda u^2(y)}{|x-y|} dy.$$

In addition, by (2.1) and (2.2), we easily obtain that $\|\phi_u\|_D \le \tilde{C}\lambda\|u\|^2$. Hence, we have

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \le \|u\|^2_{L^{12/5}(\mathbb{R}^3)} \|\phi_u\|_{L^6(\mathbb{R}^3)} \le \tilde{C}_1 \lambda \|u\|^4, \text{ where } \tilde{C}_1 > 0.$$

Lemma 2.2 ([10])**.** *For any $u \in H^1(\mathbb{R}^3)$, we have*

(i) $\phi_u \ge 0$;

(ii) $\phi_{tu} = t^2\phi_u$, *for any* $t > 0$;

(iii) *if* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$ *and*

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \le \liminf_{n\longrightarrow\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx.$$

To find a weak solution $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, it is sufficient to seek a solution of the first equation of system (1.2) with $\phi = \phi_u$. We define a functional $I_\lambda : H^1(\mathbb{R}^3) \longrightarrow \mathbb{R}$ by, for all $u \in H^1(\mathbb{R}^3)$

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\lambda \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \int_{\mathbb{R}^3} q(x)F(u)dx, \tag{2.3}$$

where $F(u) = \int_0^u f(s)ds$. It is easy to show that $I_\lambda \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$, and for all $u, \varphi \in H^1(\mathbb{R}^3)$

$$\begin{aligned} &\langle I_\lambda'(u), \varphi\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \varphi + V(x)u\varphi)dx + \lambda \int_{\mathbb{R}^3} \phi_u(x)u\varphi dx - \int_{\mathbb{R}^3} q(x)f(u)\varphi dx. \end{aligned} \tag{2.4}$$

Hence if $u \in H^1(\mathbb{R}^3)$ is a critical point of I_λ, then the pair (u, ϕ) is a solution of system (1.2).

3. Proof of Theorem

In what follows, we ensure that the functional I_λ has what is called the mountain pass geometry.

Lemma 3.1. *Suppose that (V_1) and $(A_1)-(A_3)$ hold, then there exist $\rho > 0$ and $\eta > 0$ such that*

$$\inf\{I_\lambda(u) : u \in H^1(\mathbb{R}^3) \text{ with } \|u\| = \rho\} > \eta.$$

Proof. For any $\varepsilon > 0$, it follows from (A_1) and (A_2) that there exists $C_\varepsilon > 0$ such that, for all $s \in \mathbb{R}$

$$|f(s)| \le \varepsilon|s| + C_\varepsilon s^2, \tag{3.1}$$

and, for all $s \in \mathbb{R}$ then

$$|F(s)| \le \frac{\varepsilon}{2}s^2 + \frac{C_\varepsilon}{3}|s|^3.$$

From (A_3), the Hölder inequality and the Sobolev inequality, we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}^3} q(x)F(u)dx\right| &\le \int_{\mathbb{R}^3} \frac{\varepsilon}{2} q(x)u^2dx + \int_{\mathbb{R}^3} \frac{C_\varepsilon}{3} q(x)|u|^3dx \\ &\le \frac{C_1\varepsilon}{2}\|u\|^2 + C_\varepsilon\|u\|^3 \end{aligned}$$

for all $u \in H^1(\mathbb{R}^3)$.

By Lemma 2.2, this yields

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\lambda \int_{\mathbb{R}^3} \phi_u(x)u^2dx - \int_{\mathbb{R}^3} q(x)F(u)dx \\ &\ge \frac{1-C_1\varepsilon}{2}\|u\|^2 - C_\varepsilon\|u\|^3. \end{aligned} \tag{3.2}$$

So, by fixing $\varepsilon \in (0, \frac{1}{C_1})$, letting $\|u\| = \rho > 0$, small enough, it is easy to see that there is $\eta > 0$ such that this lemma holds. □

Lemma 3.2. *Suppose that (V_1) and $(A_1)-(A_4)$ hold , then there exist $u_0 \in H^1(\mathbb{R}^3)$ with $\|u_0\| > \rho$ and $\lambda_0 > 0$ such that $I_\lambda(u_0) < 0$ for $\lambda \in (0, \lambda_0)$, where ρ is given by Lemma 3.1.*

Proof. By (A_4), there is $\xi \in H^1(\mathbb{R}^3)$ such that $\xi \geq 0$, $\int_{\mathbb{R}^3} q(x)\xi^2 dx = 1$ and $\Lambda \leq \|\xi\|^2 < l$. Then from $(A_1), (A_2)$ and the Dominated Convergence Theorem, we obtain

$$\begin{aligned}
\lim_{t \longrightarrow +\infty} \frac{I_0(t\xi)}{t^2} &= \frac{1}{2}\|\xi\|^2 - \lim_{t \longrightarrow +\infty} \int_{\mathbb{R}^3} q(x) \frac{F(t\xi)}{t^2} dx \\
&= \frac{1}{2}\|\xi\|^2 - \lim_{t \longrightarrow +\infty} \int_{\{x \in \mathbb{R}^3 : \xi(x) \neq 0\}} q(x) \frac{F(t\xi)}{(t\xi)^2} \xi^2 dx \\
&= \frac{1}{2}\|\xi\|^2 - \frac{l}{2} \int_{\{x \in \mathbb{R}^3 : \xi(x) \neq 0\}} q(x)\xi^2 dx \\
&= \frac{1}{2}\|\xi\|^2 - \frac{l}{2} \int_{\mathbb{R}^3} q(x)\xi^2 dx \\
&= \frac{1}{2}(\|\xi\|^2 - l) < 0.
\end{aligned}$$

So, if $I_0(t\xi) \longrightarrow -\infty$ as $t \longrightarrow +\infty$, then there exists $u_0 \in H^1(\mathbb{R}^3)$ with $\|u_0\| > \rho$ such that $I_0(u_0) < 0$. Since $I_\lambda(u_0) \longrightarrow I_0(u_0)$ as $\lambda \longrightarrow 0^+$, there exists $\lambda_0 > 0$ such that $I_\lambda(u_0) < 0$, for all $\lambda \in (0, \lambda_0)$. □

By Lemmas 3.1, 3.2 and Lemma 2.1, there is a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$I_\lambda(u_n) \longrightarrow c > 0 \text{ and } (1 + \|u_n\|)\|I_\lambda'(u_n)\|_{H^*} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.3}$$

Lemma 3.3. *Suppose that* (V_1) *and* $(A_1)-(A_4)$ *hold, then the sequence* $\{u_n\}$ *defined in* (3.3) *is bounded in* $H^1(\mathbb{R}^3)$.

Proof. By (A_1) and (A_2), it is sufficient to define $L_0 = \sup_{s>0} \frac{f(s)}{s} \in (0, +\infty)$ such that

$$0 \leq \frac{f(s)}{s} \leq L_0 \tag{3.4}$$

for all $s \in \mathbb{R}$. From (3.3), we obtain $\langle I_\lambda'(u_n), u_n \rangle \leq \|u_n\|$, i.e.

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 + \lambda \phi_{u_n} u_n^2 - q(x)f(u_n)u_n) dx \leq \|u_n\|,$$

which implies that

$$\begin{aligned}
\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 + \lambda \phi_{u_n} u_n^2) dx &\leq \|u_n\| + \int_{\mathbb{R}^3} q(x)f(u_n)u_n dx \\
&\leq \|u_n\| + L_0 \int_{\mathbb{R}^3} q(x)u_n^2 dx \qquad (3.5)
\end{aligned}$$

by (A_3) and (3.4).

Since $-\Delta \phi_{u_n} = \lambda u_n^2$ and (2.2), by the Young inequality, we have

$$\begin{aligned}
\lambda \int_{\mathbb{R}^3} |u_n|^3 dx &= \int_{\mathbb{R}^3} \nabla \phi_{u_n} \cdot \nabla |u_n| dx \\
&\leq \int_{\mathbb{R}^3} |\nabla \phi_{u_n}||\nabla u_n| dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2+|\nabla\phi_{u_n}|^2)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2+\lambda\phi_{u_n}u_n^2)dx. \qquad (3.6)
\end{aligned}
$$

From (3.5) and (3.6), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx \leq \|u_n\| - \int_{\mathbb{R}^3} g(x,u_n)dx,$$

i.e.

$$\frac{1}{2}\|u_n\|^2 \leq \|u_n\| - \int_{\mathbb{R}^3} g(x,u_n)dx, \tag{3.7}$$

where $g(x,u_n)=\lambda|u_n|^3-L_0q(x)u_n^2,\ x\in\mathbb{R}^3$. From $q\in L^2(\mathbb{R}^3)$, we can conclude that for every $\varepsilon>0$, there exist $R_\varepsilon>0, M_\varepsilon>0$ such that

$$\int_{|x|\geq R_\varepsilon} q^2(x)dx<\varepsilon \text{ and } \int_{\{x\in\mathbb{R}^3:|x|\leq R_\varepsilon,q(x)\geq M_\varepsilon\}} q^2(x)dx<\varepsilon.$$

So, if we fix ε small enough, we can obtain

$$
\begin{aligned}
L_0\int_{|x|\geq R_\varepsilon} q(x)u_n^2dx &\leq L_0\Big(\int_{|x|\geq R_\varepsilon} q^2(x)dx\Big)^{\frac{1}{2}}\Big(\int_{|x|\geq R_\varepsilon} u_n^4dx\Big)^{\frac{1}{2}} \\
&\leq C_2\Big(\int_{|x|\geq R_\varepsilon} q^2(x)dx\Big)^{\frac{1}{2}}\|u_n\|^2 \\
&\leq \frac{1}{8}\|u_n\|^2, \qquad (3.8)
\end{aligned}
$$

and

$$L_0\int_{\{x\in\mathbb{R}^3:|x|\leq R_\varepsilon,q(x)\geq M_\varepsilon\}} q(x)u_n^2dx \leq \frac{1}{8}\|u_n\|^2,$$

which implies that

$$
\begin{aligned}
L_0\int_{|x|\leq R_\varepsilon} q(x)u_n^2dx &= L_0\int_{\{x\in\mathbb{R}^3:|x|\leq R_\varepsilon,q(x)\leq M_\varepsilon\}} q(x)u_n^2dx \\
&\quad +L_0\int_{\{x\in\mathbb{R}^3:|x|\leq R_\varepsilon,q(x)\geq M_\varepsilon\}} q(x)u_n^2dx \\
&\leq \int_{|x|\leq R_\varepsilon} L_0M_\varepsilon u_n^2dx+\frac{1}{8}\|u_n\|^2. \qquad (3.9)
\end{aligned}
$$

Denote $g_0(t)=\lambda|t|^3-L_0M_\varepsilon t^2$. Let $\mu=\inf_{t\in\mathbb{R}} g_0(t)$, then $\mu\in(-\infty,0)$. Now from (3.9) and (3.8), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3} g(x,u_n)dx &= \int_{|x|\leq R_\varepsilon} g(x,u_n)dx+\int_{|x|\geq R_\varepsilon} g(x,u_n)dx \\
&\geq \int_{|x|\leq R_\varepsilon} g_0(u_n)dx-\frac{1}{8}\|u_n\|^2+\int_{|x|\geq R_\varepsilon} g(x,u_n)dx
\end{aligned}
$$

$$\begin{aligned}
&\geq \mu|B_{R_\varepsilon}(0)| - \frac{1}{8}\|u_n\|^2 - \int_{|x|\geq R_\varepsilon} L_0 q(x) u_n^2 dx \\
(3.10)\qquad &\geq \mu|B_{R_\varepsilon}(0)| - \frac{1}{4}\|u_n\|^2.
\end{aligned}$$

Using (3.7) and (3.10), we have

$$\frac{1}{2}\|u_n\|^2 \leq \|u_n\| + \frac{1}{4}\|u_n\|^2 + |\mu||B_{R_\varepsilon}(0)|,$$

which yields that $\|u_n\|$ is bounded. □

To prove that the Cerami sequence $\{u_n\}$ in (3.3) converges to a nonzero critical point of I_λ, the following compactness lemma is useful.

Lemma 3.4. *Assume that* (V_1) *and* $(A_1)-(A_4)$ *hold, then for any* $\varepsilon > 0$ *and there exist* $R_\varepsilon > 0$ and $N_\varepsilon > 0$ *such that* $\int_{|x|\geq R} (|\nabla u_n|^2 + V(x)u_n^2)dx \leq \varepsilon$, *if* $R \geq R_\varepsilon, n \geq N_\varepsilon$.

Proof. Let $\xi_R : \mathbb{R}^3 \longrightarrow [0,1]$ be a smooth function such that

$$\xi_R(x) = \begin{cases} 0, & 0 \leq |x| \leq R, \\ 1, & |x| \geq 2R, \end{cases}$$

and, for some constant $C > 0$ (independent of R)

$$|\nabla \xi_R(x)| \leq \frac{C}{R}, \text{ for all } x \in \mathbb{R}^3. \tag{3.11}$$

Then for any $\varepsilon > 0$ ($\varepsilon < \frac{1}{2}$), there exists $R_\varepsilon' > 0$ such that

$$\frac{C^2}{R^2} \leq 4\varepsilon^2 V_0, \text{ for any } R \geq R_\varepsilon'. \tag{3.12}$$

From the Young inequality, (3.11), (3.12) and (V_1), we get, for all $n \in \mathbb{N}$ and $R \geq R_\varepsilon'$

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla u_n \cdot \nabla \xi_R u_n \xi_R| dx &= \int_{\mathbb{R}^3} \Big|(\sqrt{2\varepsilon}\xi_R \nabla u_n)\cdot\Big(\frac{1}{\sqrt{2\varepsilon}} u_n \nabla \xi_R\Big)\Big| dx \\
&\leq \varepsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R^2 dx + \frac{1}{4\varepsilon}\int_{\mathbb{R}^3} u_n^2 |\nabla \xi_R|^2 dx \\
&\leq \varepsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4\varepsilon}\int_{|x|\leq 2R} u_n^2 \frac{C^2}{R^2} dx \\
&\leq \varepsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \varepsilon \int_{|x|\leq 2R} V_0 u_n^2 dx \\
&\leq \varepsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \varepsilon \int_{\mathbb{R}^3} V(x) u_n^2 dx \\
(3.13)\qquad &= \varepsilon \|u_n\|^2,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^3} |\nabla(u_n\xi_R)|^2 dx = \int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R^2 dx + \int_{\mathbb{R}^3} u_n^2 |\nabla \xi_R|^2 dx$$

$$
\begin{aligned}
&+2\int_{\mathbb{R}^3}|\nabla u_n\cdot\nabla\xi_R u_n\xi_R|dx\\
\leq\ & \int_{\mathbb{R}^3}|\nabla u_n|^2dx+\int_{\mathbb{R}^3}\frac{C^2}{R^2}u_n^2dx+2\varepsilon\|u_n\|^2\\
\leq\ & \int_{\mathbb{R}^3}|\nabla u_n|^2dx+\int_{\mathbb{R}^3}V_0u_n^2dx+2\varepsilon\|u_n\|^2\\
\leq\ & \|u_n\|^2+2\varepsilon\|u_n\|^2\\
\leq\ & 2\|u_n\|^2.
\end{aligned}
$$

Hence we have

$$\|u_n\xi_R\|\leq\sqrt{3}\|u_n\| \tag{3.14}$$

for all $n\in\mathbb{N}$ and $R\geq R_\varepsilon^{'}$. By (3.3), one has $\|I_\lambda^{'}(u_n)\|_{H^*}\|u_n\|\longrightarrow 0,$ as $n\longrightarrow\infty$. Then for $\varepsilon>0$ above, there exists $N_\varepsilon>0$ such that

$$\|I_\lambda^{'}(u_n)\|_{H^*}\|u_n\|\leq\frac{\varepsilon}{\sqrt{3}},\ \text{for all } n\geq N_\varepsilon. \tag{3.15}$$

It follows from (3.14) and (3.15) that, for all $n\geq N_\varepsilon$ and $R\geq R_\varepsilon^{'}$

$$|\langle I_\lambda^{'}(u_n),u_n\xi_R\rangle|\leq\|I_\lambda^{'}(u_n)\|_{H^*}\|u_n\xi_R\|\leq\varepsilon. \tag{3.16}$$

Similar to (3.13), we get, for all $n\in\mathbb{N}$ and $R\geq R_\varepsilon^{'}$

$$\int_{\mathbb{R}^3}|\nabla u_n\cdot\nabla\xi_R u_n|dx\leq\varepsilon\|u_n\|^2.$$

By Lemma 2.2, (A_3) and (3.4), for all $n\in\mathbb{N}$ and $R\geq R_\varepsilon^{'}$, we have

$$
\begin{aligned}
\langle I_\lambda^{'}(u_n),u_n\xi_R\rangle\ \geq\ & \int_{\mathbb{R}^3}|\nabla u_n|^2\xi_Rdx+\int_{\mathbb{R}^3}V(x)u_n^2\xi_Rdx\\
&+\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\xi_R u_ndx-\int_{\mathbb{R}^3}q(x)f(u_n)u_n\xi_Rdx\\
\geq\ & \int_{\mathbb{R}^3}|\nabla u_n|^2\xi_Rdx+\int_{\mathbb{R}^3}V(x)u_n^2\xi_Rdx\\
&-\varepsilon\|u_n\|^2-\int_{\mathbb{R}^3}L_0q(x)\xi_Ru_n^2dx. \qquad (3.17)
\end{aligned}
$$

From the Hölder inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3}L_0q(x)\xi_Ru_n^2dx\ \leq\ & L_0\int_{|x|\geq R}q(x)u_n^2dx\\
\leq\ & L_0\Big(\int_{|x|\geq R}q^2(x)dx\Big)^{\frac{1}{2}}\Big(\int_{|x|\geq R}u_n^4dx\Big)^{\frac{1}{2}}\\
\leq\ & C_3\Big(\int_{|x|\geq R}q^2(x)dx\Big)^{\frac{1}{2}}\|u_n\|^2.
\end{aligned}
$$

So, for $\varepsilon>0$ above, there exists $R_\varepsilon^{''}>0$ such that

$$\int_{\mathbb{R}^3}L_0q(x)\xi_Ru_n^2dx\leq\varepsilon\|u_n\|^2,\ \text{for any } R\geq R_\varepsilon^{''}. \tag{3.18}$$

Choose $R_\varepsilon = \max\{R_\varepsilon^{'}, R_\varepsilon^{''}\}$, by (3.16) – (3.18) and Lemma 3.3, there exists $C_0 > 0$ such that, for all $n \geq N_\varepsilon$ and $R \geq R_\varepsilon$

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R dx + \int_{\mathbb{R}^3} V(x) u_n^2 \xi_R dx \leq C_0 \varepsilon,$$

which complete the proof. □

Lemma 3.5. *Let* (V_1) *and* (A_1)–(A_4) *hold. Then* $u_n \longrightarrow u$ in $H^1(\mathbb{R}^3)$, as $n \longrightarrow \infty$, *for some* $u \in H^1(\mathbb{R}^3)\backslash\{0\}$.

Proof. By Lemma 3.3, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Subject to a subsequence, we can assume that, there exists $u \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^3);\ u_n \longrightarrow u \text{ a.e. in } \mathbb{R}^3; \\ &u_n \longrightarrow u \text{ in } L^s(B), \text{ with } B \subset \mathbb{R}^3 \text{ is bounded and } s = 2, 3. \end{aligned}$$

Note that,

$$\begin{aligned} \langle I_\lambda^{'}(u_n), u_n \rangle &= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x) u_n^2) dx \\ &\quad + \lambda \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} q(x) f(u_n) u_n dx \\ &= o(1), \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \langle I_\lambda^{'}(u_n), u \rangle &= \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla u + V(x) u_n u) dx \\ &\quad + \lambda \int_{\mathbb{R}^3} \phi_{u_n} u_n u dx - \int_{\mathbb{R}^3} q(x) f(u_n) u dx \\ &= o(1). \end{aligned} \tag{3.20}$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we can see

$$\int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla u + V(x) u_n u) dx = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2) dx + o(1). \tag{3.21}$$

By Lemma 3.4, for any $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ and $R_\varepsilon > 0$ such that for any $n \in \mathbb{N}$ with $n \geq N_\varepsilon$ and $R \geq R_\varepsilon$, one has $\|u_n\|_{H^1(B_R^c(0))} \leq \varepsilon$, and $\|u_n - u\|_{L^s(B_R(0))} \leq \varepsilon$ for $s = 2, 3$. It follows that

$$\begin{aligned} \|u_n - u\|_{L^s(\mathbb{R}^3)} &\leq \|u_n - u\|_{L^s(B_R(0))} + \|u_n - u\|_{L^s(B_R^c(0))} \\ &\leq \|u_n - u\|_{L^s(B_R(0))} + \|u_n\|_{L^s(B_R^c(0))} + \|u\|_{L^s(B_R^c(0))} \\ &\leq \varepsilon + C_4(\|u_n\|_{H^1(B_R^c(0))} + \|u\|_{H^1(B_R^c(0))}) \\ &\leq (1 + 2C_4)\varepsilon, \end{aligned}$$

for any $n \in \mathbb{N}$ with $n \geq N_\varepsilon$ and $s = 2, 3$. Therefore, for $s = 2, 3$, we have

$$u_n \longrightarrow u \text{ in } L^s(\mathbb{R}^3),\ \text{as } n \longrightarrow \infty. \tag{3.22}$$

By (3.4), (A_3), the Hölder inequality and (3.22), one has

$$\left| \int_{\mathbb{R}^3} q(x) f(u_n)(u_n - u) dx \right| \quad \leq \quad \int_{\mathbb{R}^3} |q(x) f(u_n)| |u_n - u| dx$$

$$\begin{aligned} &\le \int_{\mathbb{R}^3} L_0|q(x)u_n||u_n-u|dx \\ &\le L_0\|q\|_{L^2(\mathbb{R}^3)}\|u_n\|_{L^6(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)} \\ &\longrightarrow 0, \text{ as } n\longrightarrow\infty. \end{aligned} \tag{3.23}$$

From the Hölder inequality and (3.22), we get

$$\begin{aligned} \left|\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u)dx\right| &\le \int_{\mathbb{R}^3}|\phi_{u_n}u_n||u_n-u|dx \\ &\le \|\phi_{u_n}\|_{L^6(\mathbb{R}^3)}\|u_n\|_{L^3(\mathbb{R}^3)}\|u_n-u\|_{L^2(\mathbb{R}^3)} \\ &\longrightarrow 0, \text{ as } n\longrightarrow\infty. \end{aligned} \tag{3.24}$$

By $(3.19), (3.20), (3.23)$ and (3.24), we have

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx=\int_{\mathbb{R}^3}(\nabla u_n\cdot\nabla u+V(x)u_nu)dx+o(1),$$

which implies that

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx=\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)u^2)dx+o(1)$$

by (3.21). i.e., $\|u_n\|\longrightarrow\|u\|$, as $n\longrightarrow\infty$. This together with $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^3)$, shows that $u_n\longrightarrow u$ in $H^1(\mathbb{R}^3)$, as $n\longrightarrow\infty$.

From $u_n\longrightarrow u$ in $H^1(\mathbb{R}^3)$ and (3.3), we obtain $I_\lambda(u)=c>0$. So $u\in H^1(\mathbb{R}^3)\backslash\{0\}$. □

Now we give the proof of the main result.

Proof of Theorem 1.1. Set the Nehari manifold

$$\mathcal{N}:=\{u\in H^1(\mathbb{R}^3)\backslash\{0\}:\langle I_\lambda'(u),u\rangle=0\}.$$

From Lemma 3.5, $\mathcal{N}$ is nonempty. For any $u\in\mathcal{N}$, by Lemma 2.2, we have

$$\begin{aligned} 0 &= \langle I_\lambda'(u),u\rangle \\ &= \|u\|^2+\lambda\int_{\mathbb{R}^3}\phi_u u^2dx-\int_{\mathbb{R}^3}q(x)f(u)udx \\ &\ge \|u\|^2-\int_{\mathbb{R}^3}q(x)f(u)udx. \end{aligned} \tag{3.25}$$

Now, choose $\varepsilon\in(0,\frac{1}{C_1})$ in the proof of Lemma 3.1 and use (3.1) to get

$$\begin{aligned} \left|\int_{\mathbb{R}^3}q(x)f(u)udx\right| &\le \int_{\mathbb{R}^3}(\varepsilon q(x)u^2+C_\varepsilon q(x)|u|^3)dx \\ &\le C_1\varepsilon\|u\|^2+C_\varepsilon\int_{\mathbb{R}^3}q(x)|u|^3dx \\ &\le C_1\varepsilon\|u\|^2+C_\varepsilon\|u\|^3. \end{aligned} \tag{3.26}$$

Therefore, by (3.25) and (3.26), for every $u\in\mathcal{N}$, we have

$$0\ge\|u\|^2-C_1\varepsilon\|u\|^2-C_\varepsilon\|u\|^3,$$

which implies that

$$\|u\| \geq \frac{1-C_1\varepsilon}{C_\varepsilon} > 0, \text{ for any } u \in \mathcal{N}.$$

Hence any limit point of a sequence in the Nehari manifold is different from zero.

We claim that I_λ is bounded from below on $\mathcal{N}$, i.e., there exists $M_1 > 0$ such that $I_\lambda(u) \geq -M_1$, for any $u \in \mathcal{N}$. Otherwise, there exists $\{u_n\} \subset \mathcal{N}$ such that

$$I_\lambda(u_n) < -n, \text{ for any } n \in \mathbb{N}. \tag{3.27}$$

From (3.2), we obtain

$$I_\lambda(u_n) \geq \frac{1-C_1\varepsilon}{2}\|u_n\|^2 - C_\varepsilon\|u_n\|^3.$$

This and (3.27) imply that $\|u_n\| \longrightarrow +\infty$, as $n \longrightarrow \infty$. Because $\{u_n\} \subset \mathcal{N}$, as in the proof of Lemma 3.3, we obtain that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, so $\|u_n\| \longrightarrow +\infty$ is impossible. Then, I_λ is bounded from below on $\mathcal{N}$. So we may define

$$\bar{c} = \inf\{I_\lambda(u) : \ u \in \mathcal{N}\}, \text{ and } \bar{c} \geq -M_1, \text{ where } M_1 > 0.$$

Let $\{\bar{u}_n\} \subset \mathcal{N}$, such that $I_\lambda(\bar{u}_n) \longrightarrow \bar{c}$ as $n \longrightarrow \infty$. Following the same procedures as the proof of Lemmas 3.3, 3.4 and Lemma 3.5, we can show that $\{\bar{u}_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and it has a convergent subsequence, strongly to $\bar{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$. Thus $I_\lambda(\bar{u}) = \bar{c}$ and $I_\lambda'(\bar{u}) = 0$. Therefore $(\bar{u}, \phi_{\bar{u}}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a ground state solution of system (1.2).

If we denote $\bar{u}^\pm = \max\{\pm\bar{u}, 0\}$ the positive (negative) part of $\bar{u}$ and by (A_1), we have

$$\begin{aligned} 0 = \langle I_\lambda'(\bar{u}), \bar{u}^-\rangle &= -\|\bar{u}^-\|^2 - \lambda\int_{\mathbb{R}^3}\phi_{\bar{u}^-}(\bar{u}^-)^2dx - \int_{\mathbb{R}^3} q(x)f(\bar{u})\bar{u}^-dx \\ &= -\|\bar{u}^-\|^2 - \lambda\int_{\mathbb{R}^3}\phi_{\bar{u}^-}(\bar{u}^-)^2dx, \end{aligned}$$

i.e.

$$\|\bar{u}^-\|^2 + \lambda\int_{\mathbb{R}^3}\phi_{\bar{u}^-}(\bar{u}^-)^2dx = 0$$

and therefore, $\bar{u} \geq 0$ in $\mathbb{R}^3$. By the standard arguments, see [6, 23], we have $\bar{u} \in L^\infty(\mathbb{R}^3)$, and $\bar{u} \in C^{1,\alpha}_{loc}(\mathbb{R}^3)$ with $0 < \alpha < 1$. Moreover, by the Harnack's inequality, see [24], $\bar{u}(x) > 0$ for all $x \in \mathbb{R}^3$.

So $(\bar{u}, \phi_{\bar{u}}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a positive ground state solution of system (1.2). □

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