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FIXED POINT THEOREMS FOR α-ψ-CONTRACTIVE TYPE MAPPINGS OF INTEGRAL TYPE WITH APPLICATIONS

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Abstract. One interesting technique for obtaining fixed point results is the technique of contractive conditions of integral type. In order to generalize this technique, we introduce a new concept of α-ψ-contractive type mappings of integral type. Also, we establish fixed point theorems and coupled fixed point theorems for such mappings in complete metric spaces. As consequences of our main results, we further obtain, fixed point theorems on metric spaces endowed with a partial order. The theorems presented here generalize, enrich and improve the previous related results. To illustrate our results and to distinguish them from the existing ones, we equip the paper with examples.

1. Introduction

Fixed point theory is an important tool in the study of nonlinear analysis as it is considered to be the key connection between pure and applied mathematics with wide applications in physical sciences, economics and in almost all engineering fields. The Banach contraction principle is a very popular tool which is used to solve existence problems in many branches of mathematical analysis and its applications. Many authors have extended, generalized and improved this fundamental theorem in different directions by either modifying the basic contractive condition or by changing the ambiental space. Recently, a new category of contractive type mappings known as α-ψ-contractive type mappings has been introduced by Samet *et al.* [21]. The results obtained by Samet *et al.* [21] generalize the existing fixed point results in the literature, in particular the Banach contraction principle.

In recent years, various researchers have shown a significant interest in proving fixed point theorems for mappings in metric spaces satisfying general contractive integral type inequalities. The study of contractive conditions of integral type was initiated in 2002 by Branciari [9] who gave an integral version of the Banach contraction principle. Further, Rhoades [18] in 2003 obtained such an extension to two of the most general contractive conditions which embraces the Branciari's result as well as a result of Ćirić [10]. Thereafter, many authors like Aliouche [2], Djoudi and Aliouche [11], Rhoades [18], Samet and Vetro [20], Türkoğlu and Altun [23], Vetro [24], Vijayaraju *et al.* [25], and others have proved some fixed point theorems involving more general contractive conditions of integral type.

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In the present work, we introduce a new contractive condition of integral type known as α-ψ-contractive type mappings of integral type. Also, we study the existence and uniqueness of fixed points for such mappings. Our results improve, extend and generalize the results derived by Samet *et al.* in [21] and various other related results in the literature. Moreover, from our fixed point theorems, we will derive various fixed point results on metric spaces endowed with a partial order.

2. Preliminaries

First we introduce some notations and definitions that will be used subsequently. Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing.

(ii) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.

These functions are known as (c)-comparison functions in the literature. It can be easily verified that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.

Recently, Samet *et al.* [21] introduced the following new concepts of α-ψ-contractive type mappings and α-admissible mappings:

Definition 2.1. Let (X,d) be a metric space and $T : X \to X$ be a given self mapping. T is said to be an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,+\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y))$$

for all $x, y \in X$.

Definition 2.2. Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. T is said to be α-admissible if

$$x, y \in X,\ \alpha(x,y) \ge 1 \Rightarrow \alpha(Tx,Ty) \ge 1.$$

The following fixed point theorems are the main results in [21]:

Theorem 2.3. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$;*

(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 2.4. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha(x_n,x) \ge 1$ for all n.*

Then, T has a fixed point.

Samet *et al.* [21] added the following condition to the hypotheses of Theorem 2.3 and Theorem 2.4 to assure the uniqueness of the fixed point:

(C): For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Further, Samet *et al.* [21] derived the following coupled fixed point theorems in complete metric spaces using the previous obtained results.

Theorem 2.5. *Let (X, d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ and a function $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that*

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \frac{1}{2}\psi(d(x, u) + d(y, v)),$$

for all $(x, y), (u, v) \in X \times X$. Suppose also that

(i) *for all $(x, y), (u, v) \in X \times X$, we have*

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)) \geq 1;$$

(ii) *there exists $(x_0, y_0) \in X \times X$ such that $\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)) \geq 1$ and $\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1$.*

(iii) *F is continuous.*

Then, F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Theorem 2.6. *Let (X, d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ and a function $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that*

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \frac{1}{2}\psi(d(x, u) + d(y, v)),$$

for all $(x, y), (u, v) \in X \times X$. Suppose also that

(i) *for all $(x, y), (u, v) \in X \times X$, we have*

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)) \geq 1;$$

(ii) *there exists $(x_0, y_0) \in X \times X$ such that $\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)) \geq 1$ and $\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1$.*

(iii) *if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1$ and $\alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1$, $x_n \to x \in X$ and $y_n \to y \in X$ as $n \to +\infty$, then $\alpha((x_n, y_n), (x, y)) \geq 1$ and $\alpha((y, x), (y_n, x_n)) \geq 1$ for all $n \in \mathbb{N}$.*

Then, F has a coupled fixed point.

Samet *et al.* [21] added the following condition to the hypotheses of Theorem 2.5 and Theorem 2.6 to assure the uniqueness of the coupled fixed point:

(C'): For all $(x, y), (u, v) \in X \times X$, there exists $(z_1, z_2) \in X \times X$ such that

$$\alpha((x, y), (z_1, z_2)) \geq 1, \;\; \alpha((z_2, z_1), (y, x)) \geq 1$$

and

$$\alpha((u, v), (z_1, z_2)) \geq 1, \;\; \alpha((z_2, z_1), (v, u)) \geq 1.$$

Recently, Berzig and Rus [7] introduced the notion of α-contractive mapping of Meir-Keeler type in complete metric spaces and proved the related theorems for this type of contraction. Berzig and Rus [7] introduced the following definitions:

Definition 2.7 ([7])**.** Let $N \in \mathbb{N}$. We say that α is N-transitive (on X) if

$$x_0, x_1, ..., x_{N+1} \in X : \alpha(x_i, x_{i+1}) \geq 1$$

for all $i \in \{0, 1, ..., N\} \Rightarrow \alpha(x_0, x_{N+1}) \geq 1$.
In particular, we say that α is transitive if it is 1-transitive, i.e.,

$$x, y, z \in X : \alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1.$$

As consequences of the Definition 2.7, we obtain the following remarks:

Remark 2.8 ([7])**.** Any function $\alpha : X \times X \to [0, +\infty)$ is 0-transitive.

Remark 2.9 ([7])**.** If α is N transitive, then it is kN-transitive for all $k \in \mathbb{N}$

Remark 2.10 ([7])**.** If α is transitive, then it is N-transitive for all $N \in \mathbb{N}$

Remark 2.11 ([7])**.** If α is N-transitive, then it is not necessarily transitive for all $N \in \mathbb{N}$

3. Main results

In this section, we present our main results.
Define $\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}\}$ such that φ is nonnegative, Lebesgue integrable, and satisfy

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ foreach } \epsilon > 0. \tag{3.1}$$

We introduce here a new concept of α-ψ-contractive type mappings of integral type as follows:

Definition 3.1. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-contractive mapping of integral type if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \varphi(t)dt \leq \psi\left(\int_0^{d(x,y)} \varphi(t)dt\right) \tag{3.2}$$

where $\varphi \in \Phi$.

Remark 3.2. If $T : X \to X$ is an α-ψ-contractive mapping, then T is an α-ψ-contractive mapping of integral type, where $\varphi(t) = 1$ for each $t \geq 0$.

We now present our main results.

Theorem 3.3. *Let (X, d) be a complete metric space and $\alpha : X \times X \to [0, +\infty)$ be a transitive mapping. Suppose that $T : X \to X$ be an α-ψ-contractive mapping of integral type and satisfies the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $z \in X$ such that $Tz = z$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then $x^* = x_n$ is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \tag{3.3}$$

By induction, we get $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \geq 0$. On applying the inequality (3.2) with $x = x_{n-1}$ and $y = x_n$ and using (3.3), we obtain

$$\begin{aligned} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)dt &\leq \alpha(x_{n-1}, x_n) \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt\right). \end{aligned}$$

Therefore, by induction, we get for all $n \in \mathbb{N}$

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)dt\right) = \psi^n(d) \tag{3.4}$$

where $d = \int_0^{d(x_0,x_1)} \varphi(t)dt$.

Letting $n \to +\infty$, we obtain from the property of ψ that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0$$

which, from (3.1), implies that

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty. \tag{3.5}$$

We shall now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequences $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \epsilon, d(x_{m(p)}, x_{n(p)-1}) < \epsilon. \tag{3.6}$$

From (3.5), (3.6) and the triangular inequality, we have

$$\begin{aligned} \epsilon \leq d(x_{m(p)}, x_{n(p)}) &\leq d(x_{m(p)}, x_{n(p)-1}) + d(x_{n(p)-1}, x_{n(p)}) \\ &< \epsilon + d(x_{n(p)-1}, x_{n(p)}). \end{aligned}$$

Taking the limit as $p \to \infty$, we get

$$\lim_p d(x_{m(p)}, x_{n(p)}) = \epsilon.$$

Due to the fact that α is transitive, we infer from (3.3) that

$$\alpha(x_{m(p)-1}, x_{n(p)-1}) \geq 1.$$

Now, from (3.2) and the above inequality, we obtain

$$\int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)dt = \int_0^{d(Tx_{m(p)-1},Tx_{n(p)-1})} \varphi(t)dt$$

$$\begin{aligned} &\leq \alpha(x_{m(p)-1}, x_{n(p)-1}) \int_0^{d(Tx_{m(p)-1}, Tx_{n(p)-1})} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \varphi(t)dt\right). \end{aligned}$$

Letting $p \to \infty$ in the above equation, we get

$$\int_0^{\epsilon} \varphi(t)dt \leq \psi\left(\int_0^{\epsilon} \varphi(t)dt\right) < \int_0^{\epsilon} \varphi(t)dt,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to +\infty$. We infer from the continuity of T that $Tx_n \to Tz$ as $n \to +\infty$, that is, $x_{n+1} \to Tz$ as $n \to +\infty$. By the uniqueness of the limit, we obtain $z = Tz$. Therefore, z is a fixed point of T. □

In the next theorem, we exclude the continuity hypothesis of T.

Theorem 3.4. *Let (X, d) be a complete metric space and $\alpha : X \times X \to [0, +\infty)$ be a transitive mapping. Suppose that $T : X \to X$ be an α-ψ-contractive mapping of integral type and satisfies the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then T has a fixed point, that is, there exists $z \in X$ such that $Tz = z$.

Proof. From the proof of Theorem 3.3, we infer that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ converges to $z \in X$. We obtain, from hypothesis (iii) and (3.3), that there exists a subsequence $\{x_{n(k)}\}$ of x_n such that $\alpha(x_{n(k)}, z) \geq 1$ for all k. Now applying the inequality (3.2) we obtain for all k,

$$\begin{aligned} \int_0^{d(x_{n(k)+1}, Tz)} \varphi(t)dt = \int_0^{d(Tx_{n(k)}, Tz)} \varphi(t)dt &\leq \alpha(x_{n(k)}, z) \int_0^{d(Tx_{n(k)}, Tz)} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{n(k)}, z)} \varphi(t)dt\right). \end{aligned}$$

Suppose that $d(z, Tz) > 0$. Letting $k \to +\infty$, since ψ is continuous at $t = 0$, we obtain

$$\int_0^{d(z,Tz)} \varphi(t)dt = 0,$$

which, from (3.1), implies that $d(z, Tz) = 0$, or $Tz = z$. □

The following example shows that hypotheses of Theorem 3.3 and Theorem 3.4 do not guarantee the uniqueness of the fixed point.

Example 3.5. Let $X = \{(1,0),(0,1)\} \subset \mathbb{R}^2$ endowed with the Euclidean distance $d((x,y),(u,v)) = |x-u| + |y-v|$ for all $(x,y),(u,v) \in X$. Clearly, (X,d) is a complete metric space. Let us define the mapping $\alpha : X \times X \to [0,+\infty)$ by

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & \text{if } (x,y) = (u,v), \\ 0 & \text{if } (x,y) \neq (u,v). \end{cases}$$

Clearly, α is transitive.

The mapping $T(x,y) = (x,y)$ is trivially a continuous mapping and satisfies for any $\psi \in \Psi$ and $\varphi \in \Phi$

$$\alpha((x,y),(u,v)) \int_0^{d(T(x,y),T(u,v))} \varphi(t)dt \le \psi\left(\int_0^{d((x,y),(u,v))} \varphi(t)dt\right)$$

for all $(x,y),(u,v) \in X$. Thus, T is an α-ψ-contractive mapping of integral type.

Now, we have to prove that T is α-admissible. For this, let $\alpha((x,y),(u,v)) \ge 1$ for all $(x,y),(u,v) \in X$. It implies from the definition of α that $(x,y) = (u,v)$ which further implies that $T(x,y) = T(u,v)$. Again from the definition of α we obtain that $\alpha((T(x,y),T(u,v))) = 1$. Thus, T is α-admissible.

Also, for all $(x,y) \in X$, we have $\alpha((x,y),T(x,y)) \ge 1$. Thus, all the hypotheses of Theorem 3.3 are satisfied.

Indeed, if $\{(x_n,y_n)\}$ is a sequence in X that converges to some point $(x,y) \in X$ with $\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \ge 1$ for all n, then we have from the definition of α that $(x_n,y_n) = (x_{n+1},y_{n+1})$ for all n. This implies that $(x_n,y_n) = (x,y)$ for all n implying thereby $\alpha((x_n,y_n),(x,y)) = 1$ for all n.

Now, all the hypotheses of Theorem 3.4 are also satisfied. Consequently, T has a fixed point. In this example, T has two fixed points.

To guarantee the uniqueness of the fixed point, we will consider the following hypothesis:

(H): For all $a,b \in Fix(T)$, there exists $c \in X$ such that $\alpha(a,c) \ge 1$ and $\alpha(b,c) \ge 1$. Here, Fix(T) denotes the set of fixed points of T.

Theorem 3.6. *Adding condition (H) to the hypotheses of Theorem 3.3(resp. Theorem 3.4), we obtain uniqueness of the fixed point of T.*

Proof. Suppose that a and b are two fixed points of T. From (H), there exists $c \in X$ such that

$$\alpha(a,c) \ge 1 \text{ and } \alpha(b,c) \ge 1. \tag{3.7}$$

Due to the fact that T is α-admissible, we obtain from (3.7) that

$$\alpha(a,T^n(c)) \ge 1 \text{ and } \alpha(b,T^n(c)) \ge 1 \text{ forall } n \in \mathbb{N}. \tag{3.8}$$

Using the inequalities (3.2) and (3.8), we get

$$\begin{aligned} \int_0^{d(a,T^n(c))} \varphi(t)dt &= \int_0^{d(Ta,T(T^{n-1}(c)))} \varphi(t)dt \\ &\le \alpha(a,T^{n-1}(c)) \int_0^{d(T(a),T(T^{n-1}(c)))} \varphi(t)dt \end{aligned}$$

$$\leq \quad \psi\left(\int_0^{d(a,T^{n-1}(c))} \varphi(t)dt\right).$$

By induction, we obtain for all $n \in \mathbb{N}$

$$\int_0^{d(a,T^n(c))} \varphi(t)dt \leq \psi^n\left(\int_0^{d(a,c)} \varphi(t)dt\right).$$

Letting $n \to \infty$ in the above inequality, the property (ii) of the function ψ implies

$$\int_0^{d(a,T^n(c))} \varphi(t)dt = 0,$$

which, from (3.1), implies that

$$T^n c \to a \text{ as } n \to \infty. \tag{3.9}$$

Similarly, from (3.2) and (3.8), one can show that

$$T^n c \to b \text{ as } n \to \infty. \tag{3.10}$$

From (3.9) and (3.10), we find $a = b$ due to the uniqueness of the limit. This proves that a is the unique fixed point of T. □

The following example shows that our Theorem 3.6 is generalization of Theorem 2.3 of [21].

Example 3.7. Suppose that $X = \left\{\frac{1}{n} | n \in \mathbb{N}\right\} \cup \{0\}$ with the usual metric induced by $\mathbb{R}$. Since X is a closed subset of $\mathbb{R}$, it is a complete metric space. We consider a mapping $f : X \to X$ defined by

$$f(x) = \begin{cases} \dfrac{1}{n+1} & \text{if } x = \frac{1}{n}, \\ 0 & \text{if } x = 0. \end{cases}$$

Define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & y = 0, \\ 0 & otherwise. \end{cases}$$

Clearly, α is transitive.

Define $\varphi \in \Phi$ by $\varphi(t) = t^{1/t-2}[1 - logt]$ for $t > 0$ and $\varphi(0) = 0$. Then, for any $a > 0$, we have

$$\int_0^a \varphi(t)dt = a^{1/a}.$$

Let us define $\psi \in \Psi$ by $\psi(t) = \frac{t}{2}$ for all $t \geq 0$.

We observe that this case is not applicable to α-ψ-contractive mappings since we have

$$d(f(\tfrac{1}{2}), f(0)) = \tfrac{1}{3} > \tfrac{1}{4} = \tfrac{1}{2}(d(\tfrac{1}{2}, 0)) = \psi(d(x,y)).$$

Using ([9], Example 3.6), we have for $x = \frac{1}{n}$ and $y = 0$;

$$\int_0^{d(fx,f0)} \varphi(t)dt \quad = \quad d(fx, f0)^{1/d(fx,f0)}$$

$$\begin{aligned} &= \left|\frac{1}{n+1} - 0\right|^{1/|(1/n+1)-0|} \\ &\leq \frac{1}{2}\left|\frac{1}{n}\right|^{1/|(1/n)|} \\ &= \frac{1}{2}d(x,y)^{1/d(x,y)} = \frac{1}{2}\int_0^{d(x,y)} \varphi(t)dt. \end{aligned}$$

Therefore, f is an α-ψ-contractive mapping of integral type with α as defined above and $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. In fact, for all $x, y \in X$, we have

$$\alpha(x,y)\int_0^{d(fx,fy)} \varphi(t)dt \leq \psi\left(\int_0^{d(x,y)} \varphi(t)dt\right).$$

There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. In fact, for $x_0 = 0$, we obtain

$$\alpha(0, f0) = \alpha(0,0) = 1$$

Now, let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies that $y = 0$ and by the definition of f and α, we have $fy = 0$, that is, $\alpha(fx, fy) = 1$. Thus, f is α-admissible.

Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$ for some $x \in X$. From the definition of α, for all n, we have $x_n = 0$ for all n. Consequently, $x = 0$ and $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}$.

Let $(x,y) \in X \times X$. It is easy to show that, for $z = 0$, we have $\alpha(x,z) = \alpha(y,z) = 1$. So, condition (H) is satisfied. Now, all the hypotheses of Theorem 3.6 are satisfied; thus T has a unique fixed point $u \in X$. In this case, we have $u = 0$.

4. Coupled fixed point theorems

In this section, we derive coupled fixed point theorems in complete metric spaces from our previous obtained results. For this, we recollect the following definition:

Definition 4.1. (Bhaskar and Lakshmikantham [8]) Let $F : X \times X \to X$ be a given mapping. We say that $(x,y) \in X \times X$ is a coupled fixed point of F if

$$F(x,y) = x \text{ and } F(y,x) = y.$$

Samet *et al.* [21] gave the following useful lemma which states that a coupled fixed point is a fixed point:

Lemma 4.2. *Let $F : X \times X \to X$ be a given mapping. Define the mapping $T : X \times X \to X \times X$ by*

$$T(x,y) = (F(x,y), F(y,x)), \text{ forall } (x,y) \in X \times X. \tag{4.1}$$

Then, (x,y) is a coupled fixed point of F if and only if (x,y) is a fixed point of T.

We now prove the following coupled fixed point theorems:

Theorem 4.3. *Let (X,d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ and a function $\alpha : X^2 \times X^2 \to [0, +\infty)$*

such that for all $(x,y),(u,v) \in X \times X$, we have

$$\alpha((x,y),(u,v)) \int_0^{d(F(x,y),F(u,v))} \varphi(t)dt \le \frac{1}{2}\psi\left(\int_0^{d(x,u)+d(y,v)} \varphi(t)dt\right), \tag{4.2}$$

where $\varphi \in \Phi$. Suppose also that

(i) *for all $(x,y),(u,v) \in X \times X$, we have*

$$\alpha((x,y),(u,v)) \ge 1 \Rightarrow \alpha((F(x,y),F(y,x)),(F(u,v),F(v,u)))) \ge 1$$

(ii) *there exists $(x_0,y_0) \in X \times X$ such that*

$$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \ge 1 \text{ and } \alpha((F(y_0,x_0),F(x_0,y_0)),(y_0,x_0)) \ge 1.$$

(iii) *F is continuous.*

(iv) $\int_0^{a+b} \varphi(t)dt \le \int_0^{a} \varphi(t)dt + \int_0^{b} \varphi(t)dt.$

Then, F has a coupled fixed point, that is, there exists $(u,v) \in X \times X$ such that $u = F(u,v)$ and $v = F(v,u)$.

Proof. We convert this problem to the complete metric space (Y,δ), where $Y = X \times X$ and $\delta((x,y),(u,v)) = d(x,u) + d(y,v)$ for all $(x,y),(u,v) \in X \times X$. From (4.2), we obtain

$$\alpha((x,y),(u,v)) \int_0^{d(F(x,y),F(u,v))} \varphi(t)dt \le \frac{1}{2}\psi\left(\int_0^{\delta((x,y),(u,v))} \varphi(t)dt\right), \tag{4.3}$$

and

$$\alpha((v,u),(y,x)) \int_0^{d(F(v,u),F(y,x))} \varphi(t)dt \le \frac{1}{2}\psi\left(\int_0^{\delta((x,y),(u,v))} \varphi(t)dt\right). \tag{4.4}$$

Adding equations (4.3) and (4.4), we have

$$\begin{aligned}\beta((x,y),(u,v)) &\left(\int_0^{d(F(x,y),F(u,v))} \varphi(t)dt + \int_0^{d(F(v,u),F(y,x))} \varphi(t)dt\right) \\ &\le \psi\left(\int_0^{\delta((x,y),(u,v))} \varphi(t)dt\right),\end{aligned} \tag{4.5}$$

where $\beta : Y \times Y \to [0,+\infty)$ is the function defined by

$$\beta((x,y),(u,v)) = \min\{\alpha((x,y),(u,v)), \alpha((v,u),(y,x))\},$$

which further implies from hypothesis (iv) that

$$\beta((x,y),(u,v)) \left(\int_0^{d(F(x,y),F(u,v))+d(F(v,u),F(y,x))} \varphi(t)dt\right) \le \psi\left(\int_0^{\delta((x,y),(u,v))} \varphi(t)dt\right). \tag{4.6}$$

Henceforth, for all $(x,y),(u,v) \in X \times X$, we have

$$\beta((x,y),(u,v)) \int_0^{\delta(T(x,y),T(u,v))} \varphi(t)dt \le \psi\left(\int_0^{\delta((x,y),(u,v))} \varphi(t)dt\right), \tag{4.7}$$

where $T : Y \to Y$ is given by (4.1).

Therefore, for all $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in X \times X$, we get

$$\beta(\mu, \nu) \int_0^{\delta(T(\mu),T(\nu))} \varphi(t)dt \le \psi \left(\int_0^{\delta(\mu,\nu)} \varphi(t)dt \right). \tag{4.8}$$

Consequently, from (4.8) and condition(iii) of the hypotheses, we obtain that T is continuous and β-ψ-contractive mapping of integral type.

Let $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in Y$ such that $\beta(\mu, \nu) \ge 1$. Using condition (i), we obtain immediately that $\beta(T\mu, T\nu) \ge 1$. Then T is β-admissible. From hypotheses (ii), we infer that there exists $(x_0, y_0) \in Y$ such that $\beta((x_0, y_0), T(x_0, y_0)) \ge 1$. All the hypotheses of Theorem 3.3 are satisfied and so we deduce the existence of a fixed point of T. Thus, we obtain the existence of a coupled fixed point of F from Lemma 4.2. □

In the next theorem, we omit the continuity hypotheses of F.

Theorem 4.4. *Let (X, d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ and a function $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that for all $(x, y), (u, v) \in X \times X$, we have*

$$\alpha((x, y), (u, v)) \int_0^{d(F(x,y),F(u,v))} \varphi(t)dt \le \frac{1}{2}\psi \left(\int_0^{d(x,u)+d(y,v)} \varphi(t)dt \right), \tag{4.9}$$

where $\varphi \in \Phi$. Suppose also that

(i) *for all $(x, y), (u, v) \in X \times X$, we have*

$$\alpha((x, y), (u, v)) \ge 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)))) \ge 1$$

(ii) *there exists $(x_0, y_0) \in X \times X$ such that*

$$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \ge 1 \text{ and } \alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \ge 1.$$

(iii) *if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \ge 1$ and $\alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \ge 1$ for all $n \in \mathbb{N}$. $x_n \to x \in X$ and $y_n \to y \in X$ as $n \to +\infty$, then $\alpha((x_n, y_n), (x, y)) \ge 1$ and $\alpha((y, x), (y_n, x_n)) \ge 1$ for all $n \in \mathbb{N}$.*

Then, F has a coupled fixed point.

Proof. Using the same notations of the proof of previous theorem, let $\{(x_n, y_n)\}$ be a sequence in Y such that $\beta((x_n, y_n), (x_{n+1}, y_{n+1})) \ge 1$ and $(x_n, y_n) \to (x, y)$ as $n \to +\infty$. From hypotheses (iii), we obtain that $\beta((x_n, y_n), (x, y)) \ge 1$. Consequently, all the hypotheses of Theorem 3.4 are satisfied. Thus, we deduce the existence of a fixed point of T that gives us from Lemma 4.2 the existence of a coupled fixed point of F. □

For the uniqueness of the coupled fixed point, we will consider the following hypothesis.

(H$'$): For all $(a, b), (c, d) \in X \times X$, there exists $(u_1, u_2) \in X \times X$ such that

$$\alpha((a, b), (u_1, u_2)) \ge 1, \alpha((u_2, u_1), (b, a)) \ge 1,$$

and

$$\alpha((c,d),(u_1,u_2)) \geq 1, \alpha((u_2,u_1),(d,c)) \geq 1.$$

Theorem 4.5. *Adding condition ($H^{'}$) to the hypotheses of Theorem 4.3(resp. Theorem 4.4) we obtain the uniqueness of the coupled fixed point of F.*

Proof. Under hypotheses ($H^{'}$) it is easy to show that T and β satisfy the hypotheses (H). Therefore, our result follows from Theorem 3.6 and Lemma 4.2. □

5. Consequences

In this section, we will show that many existing results in the literature can be deduced easily from our Theorem 3.6

5.1. **Standard fixed point theorems.** By taking $\alpha(x,y) = 1$ for all $x,y \in X$ and $\psi(t) = kt$ for $k \in [0,1)$ in Theorem 3.6, we obtain the following corollary:

Corollary 5.1 ([9, Branciari])**.** *Let (X,d) be a complete metric space, $k \in [0,1)$, and let $T : X \to X$ be a mapping such that for each $x,y \in X$,*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq k \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi \in \Phi$. Then T has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to+\infty} T^n x = a$.

By taking $y = Tx$ in Corollary 5.1, we obtain the following corollary:

Corollary 5.2 ([19, Rhoades and Abbas])**.** *Let T be a self map of a complete metric space (X,d) satisfying*

$$\int_0^{d(Tx,T^2x)} \varphi(t)dt \leq k \int_0^{d(x,Tx)} \varphi(t)dt,$$

for all $x \in X$ and $k \in [0,1)$, where $\varphi \in \Phi$. Then T has a unique fixed point $a \in X$.

By taking $\varphi(t) = 1$ for all $t \geq 0$ in Theorem 3.3, we obtain the following fixed point theorem:

Corollary 5.3 ([21, Samet et al.])**.** . *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

By taking $\alpha(x,y) = 1$ for all $x,y \in X$ and $\varphi(t) = 1$ for all $t \geq 0$ in Theorem 3.6, we obtain the following:

Corollary 5.4 ([4, Berinde])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi(d(x,y)),$$

for all $x,y \in X$. Then, T has a unique fixed point.

Clearly, we also obtain the following corollary:

Corollary 5.5 ([3, Banach]). *Let (X,d) be a complete metric space and $T: X \to X$ be a given mapping satisfying*

$$d(Tx,Ty) \leq kd(x,y) \text{ for all } x,y \in X.$$

where $k \in [0,1)$. Then T has a unique fixed point.

5.2. **Fixed point theorems on ordered metric spaces.** Recently, authors have initiated a new trend in fixed point theory by considering the existence and uniqueness of a fixed point in partially ordered sets. The first result in this direction was given by Turinici [22], where he extended the Banach contraction principle in partially ordered sets. Some applications of Turinici's theorem to matrix equations were presented by Ran and Reurings [17]. After this fascinating paper, many useful results have been obtained in this direction(see, for example, [1, 6, 5, 8, 13, 14, 15, 16] and the references cited therein). In this section, we will derive various fixed point results on a metric space endowed with a partial order. For this, we require the following concepts:

Definition 5.6. Let $(X,\preceq)$ be a partially ordered set and $T: X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x,y \in X, x \preceq y \Rightarrow Tx \preceq Ty.$$

Definition 5.7. Let $(X,\preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 5.8 ([12]). Let $(X,\preceq)$ be a partially ordered set and d be a metric on X. We say that $(X,\preceq,d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

Now, we have the following result.

Corollary 5.9. *Let $(X,\preceq)$ be a partially ordered set and d be a metric on X such that (X,d) is complete. Let $T: X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that for all $x,y \in X$ with $x \preceq y$, we have*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \psi\left(\int_0^{d(x,y)} \varphi(t)dt\right),$$

where $\varphi \in \Phi$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X,\preceq,d)$ is regular.*

Then, T has a fixed point. Moreover, if for all $x,y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Proof. Let us define the mapping $\alpha: X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \textit{otherwise}. \end{cases}$$

Clearly, α is transitive. Also, from the definition of α, we obtain that T is an α-ψ-contractive mapping of integral type, that is,

$$\alpha(x,y)\int_0^{d(Tx,Ty)}\varphi(t)dt \le \psi\left(\int_0^{d(x,y)}\varphi(t)dt\right),$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, Tx_0) \ge 1$. Now, we have to prove that T is α-admissible. For this, let $\alpha(x,y) \ge 1$ for all $x, y \in X$. Moreover, from the monotone property of T, we have for all $x, y \in X$

$$\alpha(x,y) \ge 1 \Rightarrow x \preceq y \Rightarrow Tx \preceq Ty \Rightarrow \alpha(Tx,Ty) \ge 1.$$

Thus, T is α-admissible. Now, if T is continuous, the existence of a fixed point follows from Theorem 3.3. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Due to the fact that the space $(X, \preceq, d)$ is regular, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. Now, from the definition of α, we obtain that $\alpha(x_{n(k)}, x) \ge 1$ for all k. In this case, we obtain the existence of a fixed point from Theorem 3.4. Now, we have to prove the uniqueness of the fixed point. For this, let $x, y \in X$. By hypothesis, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$ which implies from the definition of α that $\alpha(x,z) \ge 1$ and $\alpha(y,z) \ge 1$. Therefore, we deduce the uniqueness of the fixed point from Theorem 3.6. □

We can now easily derive the following results from Corollary 5.9.

Corollary 5.10 ([12, Karapinar and Samet])**.** *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi(d(x,y)),$$

for all $x, y \in X$ with $x \preceq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then, T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Proof. By taking $\varphi(t) = 1$ for all $t \ge 0$ in Corollary 5.9, we get the proof of this corollary. □

Corollary 5.11. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that for all $x, y \in X$ with $x \preceq y$, we have*

$$\int_0^{d(Tx,Ty)}\varphi(t)dt \le k\int_0^{d(x,y)}\varphi(t)dt,$$

where $\varphi \in \Phi$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then, T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Proof. By taking $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$ in Corollary 5.9, we get the proof of this corollary. □

Corollary 5.12 ([17, Ran and Reurings], [16, Nieto and Lopez]). *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$ with $x \preceq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then, T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Proof. Taking $\varphi(t) = 1$ for all $t \geq 0$ in Corollary 5.11, we get the proof of this corollary. □

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