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ADVANCES ON THE BANACH-MAZUR CONJECTURE FOR ROTATIONS

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This manuscript is dedicated to María Inmaculada Díaz Bellido

Abstract. The Banach-Mazur Conjecture for Rotations states that every transitive and separable Banach space must be a Hilbert space. The weak form of this conjecture states that every transitive and separable Banach space must be rotund. In this paper we show that every transitive Banach space in which all faces of the unit ball are *invariant faces* must be rotund. As a consequence, a transitive and separable Banach space in which "most of the faces" of the unit ball are *invariant faces* must be rotund.

1. Introduction

In 1932 (see [2]) Banach and Mazur conjectured that every transitive and separable Banach space is a Hilbert space. A transitive space is a Banach space in which the group of its surjective, linear isometries acts transitively on the unit sphere. We refer the reader to [3] for a wide perspective on this problem and for an extensive study on weakenings of transitivity. Among other results, in [3] it is proved that a transitive Banach whose norm is Frechet-differentiable must be uniformly convex. We also refer the reader to [1] for partial results. Among other results, in [1] it is proved that a transitive smooth Banach space with normal structure must be rotund. In this paper we show that every transitive Banach space in which all faces of the unit ball are *invariant faces* must be rotund. As a consequence, a transitive and separable Banach space in which "most of the faces" of the unit ball are *invariant faces* must be rotund.

1.1. **Notation.** To end this introduction, we introduce the proper notation used throughout this manuscript.

- Given a vector space X, the linear span or vector subspace generated by a subset A of X will be denoted by $\operatorname{span}(A)$.
- Given a vector space X and non-empty subset A of X, $\operatorname{bl}(A), \operatorname{co}(A)$, and $\operatorname{aco}(A)$ will denote the balanced hull, the convex hull, and the absolutely convex hull of A, respectively.
- If X is now a topological space and A is a subset of X, then $\operatorname{int}(A)$, $\operatorname{cl}(A)$, and $\operatorname{bd}(A)$ will denote the topological interior, the topological closure, and the topological boundary of A, respectively.
- Given a normed space X, the open unit ball of X will be denoted by U_X, the closed unit ball or simply the unit ball of X will be denoted by B_X, and the unit sphere of X is S_X.

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2. Tools

In this section we intend to collect the geometrical tools we will be making use of in the further sections in order to prove our advances on the Banach-Mazur Conjecture for Rotations.

2.1. **Inner structure.**

We will strongly rely on the concept of *inner point.* We refer the reader to [4] for a wider perspective on this relatively new geometrical concept.

Definition 2.1 (García-Pacheco, [4])**.** Let X be a real vector space and consider M a non-empty subset of X. Let $x \in X$.

(1) We will say that x is an inner point of M provided that the following happens: Let S be a bounded or unbounded maximal segment of M such that $x \in S$, then $x \in \operatorname{int}(S)$.
(2) We will say that x is an outer point of M provided that the following happens: There exists a bounded or unbounded maximal segment S of M such that $x \in \operatorname{ext}(S)$.

The set of inner and outer points of M will be denoted by $\operatorname{inn}(M)$ and $\operatorname{out}(M)$, respectively. The reader may observe that $\{\operatorname{inn}(M), \operatorname{out}(M) \cap M\}$ is always a partition of M. Even more, if M is convex, then

- $\operatorname{out}(M) = \operatorname{out}(X \setminus M)$, and
- $\{\operatorname{inn}(M), \operatorname{out}(M), \operatorname{inn}(X \setminus M)\}$ is a partition of X.

In case M is not convex, then the previous two items do not hold as shown in the next example.

Example 2.2. Let $X := \mathbb{R}^2$ and consider

$$M := (-\infty, 0) \times \{0\} \cup \left\{ \left(\frac{1}{n}, 0 \right) \in \mathbb{R}^2 : n \in \mathbb{N} \right\}.$$

It is pretty clear that $0 \in \operatorname{out}(M) \setminus M$. However, the only maximal segments of $X \setminus M$ that contain 0 are the straight lines $\{(x, mx) : x \in \mathbb{R}\}$ with $m \neq 0$. As a consequence, $0 \in \operatorname{inn}(X \setminus M)$.

In case X is endowed with a vector topology, then the relation between the inner and outer points and the topological interior and closure is the following for convex subsets (see [4, Chapter 1]):

- $\operatorname{int}(M) \subseteq \operatorname{inn}(M)$.
- If $\operatorname{int}(M) \neq \varnothing$, then $\operatorname{int}(M) = \operatorname{inn}(M)$.
- $\operatorname{out}(M) \subseteq \operatorname{bd}(M)$.

Finally, it is worth mentioning that all the extreme points of a convex set are trivially outer points, however the converse to the previous assertion is not true. For this it suffices to consider the unit ball B_X of any non-rotund Banach space. Indeed, all points in the unit sphere S_X of X will be outer points of B_X, however not all of these points will be extreme points of B_X due to the fact that X is not rotund.

2.2. **Extremal structure.** The extremal structure of convex sets will also be crucial towards accomplishing the main results in this paper. In this subsection we collect all we need.

Definition 2.3. Let X be a real vector space. Consider a convex subset M of X. We say that a convex subset C of M is a face of M exactly when the extremal condition is verified, that is, if $m, n \in M$ and there exists $t \in (0,1)$ such that $tm + (1-t)\,n \in C$, then $m, n \in C$.

The next two lemmas are the tools we will be making use of in order to prove the main result in this manuscript.

Lemma 2.4. *Let X be a real vector space. Let M be a convex subset of X. If D is a face of M and C is a convex subset of M such that $D \cap \operatorname{inn}(C) \neq \varnothing$, then $C \subseteq D$.*

Proof. Let $c \in C$ and $d \in D \cap \operatorname{inn}(C)$. By definition of inner point, there must exit $e \in C$ such that $d \in (c, e) \subset C$. Since D is a face of M, by the extremal condition we deduce that $c, e \in D$. $\square$

Lemma 2.5. *Let X be a real Banach space. Let M be a bounded closed convex subset of X. If C is a face of M, then $C \setminus \operatorname{inn}(C) \neq \varnothing$.*

Proof. We may assume C is not a singleton since otherwise $\operatorname{inn}(C) = \varnothing$. Let S be any non-trivial maximal segment of C. Since M is bounded and closed, there are $m \neq n \in M$ such that $[m, n]$ is a maximal segment of M containing S. Since C is a face of M we deduce that $S = [m, n]$. Finally observe that $m, n \in C \setminus \operatorname{inn}(C)$. $\square$

The next theorem will also be crucial toward our goals.

Theorem 2.6. *Let X be a real vector space. Let M be a non-singleton convex subset of X. If $x \in \operatorname{out}(M) \cap M$, then there exists a proper face of M containing x.*

Proof. Notice that we can assume without any loss of generality that $x = 0$. Consider the set

$$C = \bigcup \{S \subset M : S \text{ is a maximal segment of } M \text{ whose interior contains } 0\}.$$

Notice that if $C = \varnothing$, then 0 is an extreme point of M. On the other hand, since $0 \in M \setminus \operatorname{inn}(M)$ there exists a maximal segment in M one of its endpoints is 0, thus $C \neq M$. To see that C is convex, we take $x, y \in C$ and $t \in [0,1]$. There exists $\alpha < 0$ such that $\alpha x, \alpha y \in M$. Then,

$$\alpha\,(tx + (1-t)\,y) = t\,(\alpha x) + (1-t)\,(\alpha y) \in M.$$

This means that $tx + (1-t)\,y \in C$. Finally, to see that C is a face of M we take $x, y \in M$ and $t \in (0,1)$ such that $tx + (1-t)\,y \in C$. There exists $\alpha < 0$ such that $\alpha\,(tx + (1-t)\,y) \in M$. Then,

$$\frac{\alpha t}{1-\alpha\,(1-t)}x = \frac{-\alpha\,(1-t)}{1-\alpha\,(1-t)}y + \frac{1}{1-\alpha\,(1-t)}\alpha\,(tx + (1-t)\,y) \in M$$

and

$$\frac{\alpha\,(1-t)}{1-\alpha t}y = \frac{-\alpha t}{1-\alpha t}x + \frac{1}{1-\alpha t}\alpha\,(tx + (1-t)\,y) \in M.$$

This proves that $x, y \in C$. $\square$

3. The main result

The idea is to look for *geometrical invariants* in order to reach a contradiction. In this case, our geometrical invariants will be the *invariant faces.*

Definition 3.1. Let X be a real Banach space. Let C be a proper face of B_X. We say that C is an invariant face of B_X exactly when for every $T \in \mathcal{G}_X$ such that $C \subseteq T(C)$ we have that $T(C) = C$.

We would like to point out that by $\mathcal{G}_X$ we mean the set of all surjective linear isometries on X. On the other hand, trivial examples of invariant faces are maximal faces and extreme points.

Theorem 3.2. *Let X be a transitive real Banach space. If C is an invariant face of B_X, then* $\operatorname{inn}(C) = \varnothing$.

Proof. Assume that there exists $c \in \operatorname{inn}(C)$. By Lemma 2.5, we can find $d \in C \setminus \operatorname{inn}(C)$. Let $T \in \mathcal{G}_X$ such that $T(d) = c$. Note that $T(C) \cap \operatorname{inn}(C) \neq \varnothing$, so by Lemma 2.4 we have that $C \subseteq T(C)$. Since C is an invariant face, we deduce that $C = T(C)$. Finally,

$$\begin{aligned} c &= T(d) \\ &\in T(C \setminus \operatorname{inn}(C)) \\ &= T(C) \setminus T(\operatorname{inn}(C)) \\ &= T(C) \setminus \operatorname{inn}(T(C)) \\ &= C \setminus \operatorname{inn}(C), \end{aligned}$$

which is impossible. Thus, $\operatorname{inn}(C) = \varnothing$. □

Corollary 3.3. *Let X be a transitive real Banach space. If all proper faces of B_X are invariant, then X is rotund.*

Proof. Assume that X is not rotund and take $x \in \mathsf{S}_X \setminus \operatorname{ext}(\mathsf{B}_X)$. By the separation theorem, there is a closed hyperplane supporting B_X at x. In particular, x is contained in a proper face of B_X. Consider the minimal face of B_X containing x, that is,

$$\mathsf{C}_x := \bigcap \{C \subset \mathsf{B}_X : C \text{ is a proper face of } \mathsf{B}_X \text{ and } x \in C\}.$$

We will show now that $x \in \operatorname{inn}(\mathsf{C}_x)$, which is already a contradiction in virtue of Theorem 3.2. So, suppose that $x \in \operatorname{out}(\mathsf{C}_x) \cap \mathsf{C}_x$. Since x is not an extreme point of B_X we are entitled to apply Theorem 2.6 to deduce the existence of a proper face D of C_x containing x. Notice that D is also a face of B_X which contains x and is strictly contained in C_x. This contradicts the fact that C_x is the minimal face of B_X containing x. As a consequence, $x \in \operatorname{inn}(\mathsf{C}_x)$. □

4. Complementary results

Let us first introduce a bit of notation. Let X be a real Banach space. We define the following sets for $n \in \mathbb{N}$:

(1) If $n = 1$ then

$$\mathcal{C}_1 := \{C \subset \mathsf{S}_X : C \text{ is a maximal face of } \mathsf{B}_X\}.$$

(2) If $n > 1$ then

$$\mathcal{C}_n := \{C \subset \mathsf{S}_X : C \text{ is a maximal face of some element of } \mathcal{C}_{n-1}\}.$$

As mentioned before, all elements in $\mathcal{C}_1$ are invariant faces. The next theorem shows that under certain conditions all elements in $\mathcal{C}_2$ are also invariant faces.

Theorem 4.1. *Let X be a real Banach space. If all elements in $\mathcal{C}_1$ are pairwise disjoint, then all elements in $\mathcal{C}_2$ are invariant faces.*

Proof. Assume that $C \in \mathcal{C}_2$ and let $T \in \mathcal{G}_X$ such that $C \subseteq T(C)$. There exists $D \in \mathcal{C}_1$ such that C is a maximal face of D. Look at the following chain of inclusions:

$$C \subseteq T(C) \subseteq T(D).$$

This implies that $T(D) \cap D \neq \varnothing$. Since $T(D) \in \mathcal{C}_1$, by hypothesis we deduce that $T(D) = D$. Since C is maximal in D, we deduce that $C = T(C)$. $\square$

Examples of real Banach spaces in which all elements in $\mathcal{C}_1$ are pairwise disjoint include all rotund spaces and smooth spaces. Notice that the proof of Theorem 4.2 can be adapted to prove the following more general result:

Theorem 4.2. *Let X be a real Banach space. If all elements in $\mathcal{C}_n$ are pairwise disjoint, then all elements in $\mathcal{C}_{n+1}$ are invariant faces.*

Since every transitive and separable Banach space must be smooth in virtue of a result of Mazur (see [5]), we have the following partial positive solution to the weak form of the Banach-Mazur Conjecture for Rotations.

Corollary 4.3. *Let X be a transitive and separable real Banach space. Then all faces in $\mathcal{C}_1 \cup \mathcal{C}_2$ are invariant. Hence if all other faces in the unit ball are invariant, then X is rotund.*

References

[1] A. Aizpuru and F. J. García-Pacheco, *Rotundity in transitive and separable Banach spaces*, Quaest. Math. **30** (2007), 85–96.
[2] S. Banach, *Théorie des Opérations Linéaires*, Monografie Matematyczne **1**, Warszawa, 1932.
[3] J. Becerra-Guerrero and A. Rodríguez-Palacios, *Transitivity of the norm on Banach spaces*, Extracta Math. **17** (2002), 1–58.
[4] F. J. García-Pacheco, *Four Non-Linear Problems on Normed Spaces. Volumen I*, VDM Verlag, Berlin, 2008.
[5] S. Mazur, *Über konvexe mengen in linearem normieten Räumen*, Studia Math. **4** (1933), 70–84.

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