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THE SPLIT COMMON NULL POINT PROBLEM AND HALPERN-TYPE STRONG CONVERGENCE THEOREM IN HILBERT SPACES

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Abstract. Based on recent works by Byrne-Censor-Gibali-Reich [C. Byrne, Y. Censor, A. Gibali and S. Reich, The split common null point problem, J. Nonlinear Convex Anal. 13 (2012), 759–775] and third author [W. Takahashi, Strong convergence theorems for maximal and inverse-strongly monotone mappings in Hilbert spaces and applications, J. Optim. Theory Appl. 157 (2013), 781–802], we obtain a Halpern-type strong convergence theorem for finding a solution of the split common null point problem for three maximal monotone mappings which is related to the split feasibility problem by Censor and Elfving [Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994), 221–239]. The solution of the split common null point problem is characterized as a unique solution of the variational inequality of a nonlinear operator. As applications, we get two new strong convergence theorems which are connected with the split common null point problem and an equilibrium problem.

1. Introduction

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be two real Hilbert spaces. Given set-valued mappings $A_i : H_1 \to 2^{H_1},\ 1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2},\ 1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2,\ 1 \leq j \leq n$, the *split common null point problem* [6] is to find a point $z \in H_1$ such that

$$z \in \big(\cap_{i=1}^m A_i^{-1}0\big) \cap \big(\cap_{j=1}^n T_j^{-1}(B_j^{-1}0)\big),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Let C and Q be non-empty, closed and convex subsets of H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [7] is to find $z \in H_1$ such that $z \in C \cap T^{-1}Q$. Putting $A_i = \partial i_C$ for all i, $B_j = \partial i_Q$ for all j and

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∂i_C and ∂i_Q are the subdifferentials of the indicator functions i_C of C and i_Q of Q, respectively. Defining $U = T^*(I - P_Q)T$ in the split feasibility peoblem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator, where T^* is the adjoint operator of T and P_Q is the metric projection of H_2 onto Q. Furthermore, if $C \cap T^{-1}Q$ is non-empty, then $z \in C \cap T^{-1}Q$ is equivalent to $z = P_C(I - \lambda U)z$, where $\lambda > 0$ and P_C is the metric projection of H_1 onto C.

In this paper, motivated by these definitions and results, we establish a Haplern-type strong convergence theorem for finding a solution of the split common null point problem for three maximal monotone mappings which is characterized as a unique solution of the variational inequality of a nonlinear operator. As applications, we get two new strong convergence theorems which are connected with the split common null point problem and an equilibrium problem.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [20] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we have that for $x, y, u, v \in H$

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a non-empty, closed and convex subset of a Hilbert space H and let $T: C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points for T. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is called firmly nonexpansive if $\|Tx-Ty\|^2 \leq \langle Tx-Ty, x-y\rangle$ for all $x, y \in C$. If a mapping T is firmly nonexpansive, then it is nonexpansive. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [20]. For a non-empty, closed and convex subset C of H, the nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_Cx\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive; $\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y\rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_Cx, y - P_Cx\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [18].

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}: H \to \mathrm{dom}(B)$, which is called the resolvent of B for

$r > 0$. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [19] that

$$A_r x \in B J_r x, \quad \forall x \in H,\ r > 0. \tag{2.4}$$

Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \le \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.5}$$

Furthermore, we have that for $s, r \in \mathbb{R}$ with $s \ge r > 0$ and $x \in H$

$$\|x - J_s x\| \ge \|x - J_r x\|. \tag{2.6}$$

See [1] for a simpler proof of (2.6); see also [22] for a more general result. We also know the following lemma from [17].

Lemma 2.1. *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \ge \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.1, we have that

$$\|J_\lambda x - J_\mu x\| \le (|\lambda - \mu| / \lambda)\, \|x - J_\lambda x\|$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [9,18]. Let B be a maximal monotone mapping on H such that $B^{-1}0$ is non-empty. Let J_λ be the resolvent of B for $\lambda > 0$. Then

$$\langle x - J_\lambda x, J_\lambda x - y\rangle \ge 0 \tag{2.7}$$

for all $x \in H$ and $y \in B^{-1}0$. In fact, since J_λ is firmly nonexpansive and $J_\lambda y = y$ for all $y \in B^{-1}0$, we have that for all $x \in H$ and $y \in B^{-1}0$

$$\begin{aligned}
&\langle x - J_\lambda x, J_\lambda x - y\rangle \\
&\quad = \langle x - y + y - J_\lambda x, J_\lambda x - y\rangle \\
&\quad = \langle x - y, J_\lambda x - y\rangle + \langle y - J_\lambda x, J_\lambda x - y\rangle \\
&\quad \ge \|J_\lambda x - y\|^2 - \|J_\lambda x - y\|^2 \\
&\quad = 0.
\end{aligned}$$

We use this result for proving Lemma 3.1 in Section 3. Let C be a non-empty, closed and convex subset of H. If a mapping $T : C \to H$ is firmly nonexpansive, then $I - T : C \to H$ is firmly nonexpansive. In fact, put $S = I - T$. Since T is firmly nonexpansive, we have that

$$\|(I - S)x - (I - S)y\|^2 \le \langle x - y, (I - S)x - (I - S)y\rangle$$

for all $x, y \in C$. This implies that

$$\|x - y\|^2 - 2\langle x - y, Sx - Sy\rangle + \|Sx - Sy\|^2 \le \|x - y\|^2 - \langle x - y, Sx - Sy\rangle$$

and hence $\|Sx - Sy\|^2 \le \langle x - y, Sx - Sy\rangle$.

Let C be a non-empty, closed and convex subset of a Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. If $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y-\lambda(Ax-Ay)\|^2\\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + (\lambda)^2\|Ax-Ay\|^2\\ &\leq \|x-y\|^2 - 2\lambda\alpha\|Ax-Ay\|^2 + (\lambda)^2\|Ax-Ay\|^2\\ &= \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2\\ &\leq \|x-y\|^2.\end{aligned}$$

Thus $I - \lambda A : C \to H$ is nonexpansive. A mapping $g : C \to H$ is a contraction if there exists $k \in (0,1)$ such that$\|g(x) - g(y)\| \leq k\|x - y\|$ for all $x, y \in C$. We also call such a mapping g a k-contraction. A linear bounded self-adjoint operator $G : H \to H$ is called strongly positive if there exists $\overline{\gamma} > 0$ such that $\langle Gx, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. We know the following lemmas from [21]; see also [1].

Lemma 2.2. *Let H be a Hilbert space. Let g be a k-contraction of H into itself and let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Take $\gamma > 0$ with $\gamma < \frac{\overline{\gamma}}{k}$ and $t > 0$ with $t(\|G\| + \gamma\ k)^2 < 2(\overline{\gamma} - \gamma\ k)$ and $2t(\overline{\gamma} - \gamma\ k) < 1$. Then*

$$0 < 1 - t\{2(\overline{\gamma} - \gamma\ k) - t(\|G\| + \gamma\ k)^2\} < 1$$

and $I - t(G - \gamma g)$ is a contraction of H into itself.

Lemma 2.3. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let g be a k-contraction of H into itself and let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Take $\gamma > 0$ with $\gamma < \frac{\overline{\gamma}}{k}$ and $t > 0$ with $t(\|G\| + \gamma\ k)^2 < 2(\overline{\gamma} - \gamma\ k)$ and $2t(\overline{\gamma} - \gamma\ k) < 1$. Let $w \in C$. Then the following are equivalent:*

(1) $w = P_C(I - t(G - \gamma g))w$;
(2) $\langle (G - \gamma g)w, w - q\rangle \leq 0, \quad \forall q \in C$;
(3) $w = P_C(I - G + \gamma g)w$.

Such $w \in C$ exists always and is unique.

The following lemma was proved by Marino and Xu [12].

Lemma 2.4. *Let H be a Hilbert space and let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. If $0 < \gamma \leq \|G\|^{-1}$, then $\|I - \gamma G\| \leq 1 - \gamma\overline{\gamma}$.*

To prove our main result, we need the following lemma:

Lemma 2.5 ([3]; see also [25])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

3. Strong Convergence Theorem

In this section, we prove a Halpern-type strong convergence theorem [10] for finding a solution of the split common null point problem in Hilbert spaces; see also [24]. Before proving the theorem, we need the following lemmas which were obtained by [1].

Lemma 3.1. *Let H_1 and H_2 be Hilbert spaces and let A and B be maximal monotone mappings on H_1 and H_2 such that $A^{-1}0$ and $B^{-1}0$ are non-empty, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator such that $A^{-1}0 \cap T^{-1}(B^{-1}0)$ is non-empty and let T^* be the adjoint operator of T. Let J_λ and Q_μ be the resolvents of A and B for $\lambda > 0$ and $\mu > 0$, respectively. Let $\lambda, \mu, \nu, r > 0$ and $z \in H$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rT^*(I - Q_\mu)T)z$;
(ii) $0 \in T^*(I - Q_\nu)Tz + Az$;
(iii) $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Lemma 3.2. *Let H_1 and H_2 be Hilbert spaces and let B be a maximal monotone mapping on H_2. Let $T : H_1 \to H_2$ be a bounded linear operator such that $T \neq 0$. Let Q_μ be the resolvent of B for $\mu > 0$. Then a mapping $T^*(I - Q_\mu)T : H_1 \to H_1$ is $\frac{1}{\|TT^*\|}$-inverse strongly monotone.*

Theorem 3.3. *Let H_1 and H_2 be Hilbert spaces. Let A and F be maximal monotone mappings on H_1 and let B be a maximal monotone mapping on H_2 such that $A^{-1}0$, $F^{-1}0$ and $B^{-1}0$ are non-empty. Let $T : H_1 \to H_2$ be a bounded linear operator such that $A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$ is non-empty. Let T^* be the adjoint operator of T. Let J_λ and T_r be the resolvents of A for $\lambda > 0$ and of F for $r > 0$, respectively and let Q_μ be the resolvent of B for $\mu > 0$. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let G be a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_n}(I - \lambda_n T^*(I - Q_{\mu_n})T)T_{r_n}x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\}, \{\mu_n\}, \{r_n\} \subset (0, \infty)$ satisfy

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2}{\|TT^*\|}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\liminf_{n\to\infty} \mu_n > 0, \ \sum_{n=1}^{\infty} |\mu_n - \mu_{n+1}| < \infty, \ \liminf_{n\to\infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$, where z_0 is a unique fixed point of $P_{A^{-1}0\cap T^{-1}(B^{-1}0)\cap F^{-1}0}(I - G + \gamma g)$. This point z_0 is also a unique solution of the variational inequality

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \ge 0, \quad \forall q \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0.$$

Proof. Define $A_n = T^*(I - Q_{\mu_n})T$ for all $n \in \mathbb{N}$. Put $u_n = T_{r_n}x_n$ and $y_n = J_{\lambda_n}(I - \lambda_n A_n)T_{r_n}x_n$ for all $n \in \mathbb{N}$. Let $z \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$. Then we have $z = T_{r_n}z$, $z = J_{\lambda_n}z$, $(I - Q_{\mu_n})Tz = 0$ and $z = J_{\lambda_n}(I - \lambda_n A_n)z$. Since $I - Q_{\mu_n}$ is 1-inverse strongly monotone, we have from $0 < \limsup_{n\to\infty} \lambda_n < \frac{2}{\|TT^*\|}$ that

$$\begin{aligned}
\|y_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A_n)u_n - J_{\lambda_n}(I - \lambda_n A_n)z\|^2 \\
&\le \|(I - \lambda_n A_n)u_n - (I - \lambda_n A_n)z\|^2 \\
&= \|u_n - z - \lambda_n A_n u_n\|^2 \\
&= \|u_n - z\|^2 - 2\lambda_n\langle u_n - z, A_n u_n\rangle + (\lambda_n)^2\|A_n u_n\|^2 \qquad (3.1)\\
&= \|u_n - z\|^2 - 2\lambda_n\langle Tu_n - Tz, (I - Q_{\mu_n})Tu_n\rangle + (\lambda_n)^2\langle A_n u_n, A_n u_n\rangle \\
&\le \|u_n - z\|^2 - 2\lambda_n\|(I - Q_{\mu_n})Tu_n\|^2 + (\lambda_n)^2\|TT^*\|\|(I - Q_{\mu_n})Tu_n\|^2 \\
&= \|u_n - z\|^2 + \lambda_n(\lambda_n\|TT^*\| - 2)\|(I - Q_{\mu_n})Tu_n\|^2 \\
&\le \|u_n - z\|^2 \\
&\le \|x_n - z\|^2.
\end{aligned}$$

Since $x_{n+1} = \alpha_n\gamma g(x_n) + (I - \alpha_n G)y_n$ and $z = \alpha_n Gz + z - \alpha_n Gz$, we have that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(\gamma g(x_n) - Gz) + (I - \alpha_n G)(y_n - z)\| \\
&\le \alpha_n \|\gamma g(x_n) - Gz\| + \|I - \alpha_n G\| \|x_n - z\| \\
&\le \alpha_n\gamma\, k \|x_n - z\| + \alpha_n \|\gamma g(z) - Gz\| + (1 - \alpha_n\overline{\gamma}) \|x_n - z\| \\
&= \{1 - \alpha_n(\overline{\gamma} - \gamma\, k)\} \|x_n - z\| + \alpha_n \|\gamma g(z) - Gz\| \\
&= \{1 - \alpha_n(\overline{\gamma} - \gamma\, k)\} \|x_n - z\| + \alpha_n(\overline{\gamma} - \gamma\, k)\frac{\|\gamma g(z) - Gz\|}{\overline{\gamma} - \gamma\, k}.
\end{aligned}$$

Putting $K = \max\{\frac{\|\gamma g(z) - Gz\|}{\overline{\gamma} - \gamma\, k}, \|x_1 - z\|\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_m - z\| \le K$ for some $m \in \mathbb{N}$. Then we have that

$$\begin{aligned}
\|x_{m+1} - z\| &\le \{1 - \alpha_m(\overline{\gamma} - \gamma\, k)\} \|x_m - z\| + \alpha_m(\overline{\gamma} - \gamma\, k)\frac{\|\gamma g(z) - Gz\|}{\overline{\gamma} - \gamma\, k} \\
&\le \{1 - \alpha_m(\overline{\gamma} - \gamma\, k)\}K + \alpha_m(\overline{\gamma} - \gamma\, k)K \\
&= K.
\end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$ and $\{y_n\}$ are bounded. Since

$$\begin{aligned}
x_{n+2} - x_{n+1} &= \alpha_{n+1}\gamma g(x_{n+1}) + (I - \alpha_{n+1}G)y_{n+1} - (\alpha_n\gamma g(x_n) + (I - \alpha_n G)y_n) \\
&= \alpha_{n+1}\gamma g(x_{n+1}) - \alpha_{n+1}\gamma g(x_n) + \alpha_{n+1}\gamma g(x_n) - \alpha_n\gamma g(x_n) \\
&\quad + (I - \alpha_{n+1}G)y_{n+1} - (I - \alpha_{n+1}G)y_n \\
&\quad + (I - \alpha_{n+1}G)y_n - (I - \alpha_n G)y_n,
\end{aligned}$$

we have that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\le \alpha_{n+1}\gamma\, k\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\gamma\|g(x_n)\| \\
&\quad + (1 - \alpha_{n+1}\overline{\gamma})\|y_{n+1} - y_n\| + |\alpha_{n+1} - \alpha_n|\|Gy_n\|
\end{aligned}$$

$$\leq \alpha_{n+1}\gamma\ k\|x_{n+1}-x_n\| + (1-\alpha_{n+1}\overline{\gamma})\|y_{n+1}-y_n\| + |\alpha_{n+1}-\alpha_n|M_1,$$

where $M_1 = \sup\{\gamma\|g(x_n)\| + \|Gy_n\| : n \in \mathbb{N}\}$. Putting $z_n = (I-\lambda_n A_n)T_{r_n}x_n$, we have from Lemma 2.1 that

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|J_{\lambda_{n+1}}(I-\lambda_{n+1}A_{n+1})T_{r_{n+1}}x_{n+1} - J_{\lambda_n}(I-\lambda_n A_n)T_{r_n}x_n\| \\
&\leq \|J_{\lambda_{n+1}}(I-\lambda_{n+1}A_{n+1})T_{r_{n+1}}x_{n+1} - J_{\lambda_{n+1}}(I-\lambda_{n+1}A_{n+1})T_{r_n}x_n\| \\
&\quad + \|J_{\lambda_{n+1}}(I-\lambda_{n+1}A_{n+1})T_{r_n}x_n - J_{\lambda_{n+1}}(I-\lambda_n A_n)T_{r_n}x_n\| \\
&\quad + \|J_{\lambda_{n+1}}(I-\lambda_n A_n)T_{r_n}x_n - J_{\lambda_n}(I-\lambda_n A_n)T_{r_n}x_n\| \\
&\leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\| \\
&\quad + \|(I-\lambda_{n+1}A_{n+1})T_{r_n}x_n - (I-\lambda_n A_n)T_{r_n}x_n\| \\
&\quad + \|J_{\lambda_{n+1}}z_n - J_{\lambda_n}z_n\| \\
&\leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\| \\
&\quad + \|\lambda_{n+1}A_{n+1}T_{r_n}x_n - \lambda_n A_n T_{r_n}x_n\| + \|J_{\lambda_{n+1}}z_n - J_{\lambda_n}z_n\| \\
&\leq \|T_{r_{n+1}}x_{n+1} - T_{r_{n+1}}x_n\| + \|T_{r_{n+1}}x_n - T_{r_n}x_n\| \\
&\quad + \|\lambda_{n+1}A_{n+1}T_{r_n}x_n - \lambda_n A_{n+1}T_{r_n}x_n\| \\
&\quad + \|\lambda_n A_{n+1}T_{r_n}x_n - \lambda_n A_n T_{r_n}x_n\| + \|J_{\lambda_{n+1}}z_n - J_{\lambda_n}z_n\| \\
&\leq \|x_{n+1}-x_n\| + \|T_{r_{n+1}}x_n - T_{r_n}x_n\| + |\lambda_{n+1}-\lambda_n|\|A_{n+1}T_{r_n}x_n\| \\
&\quad + \lambda_n\|T\|\|(I-Q_{\mu_{n+1}})TT_{r_n}x_n - (I-Q_{\mu_n})TT_{r_n}x_n\| \\
&\quad + \|J_{\lambda_{n+1}}z_n - J_{\lambda_n}z_n\| \\
&\leq \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}x_n - x_n\| \\
&\quad + |\lambda_{n+1}-\lambda_n|\|A_{n+1}T_{r_n}x_n\| + \lambda_n\|T\|\|Q_{\mu_{n+1}}TT_{r_n}x_n - Q_{\mu_n}TT_{r_n}x_n\| \\
&\quad + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J_{\lambda_{n+1}}z_n - z_n\| \\
&\leq \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}x_n - x_n\| \\
&\quad + |\lambda_{n+1}-\lambda_n|\|A_{n+1}T_{r_n}x_n\| \\
&\quad + \lambda_n\|T\|\frac{|\mu_{n+1}-\mu_n|}{\mu_{n+1}}\|Q_{\mu_{n+1}}TT_{r_n}x_n - TT_{r_n}x_n\| \\
&\quad + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J_{\lambda_{n+1}}z_n - z_n\| \\
&\leq \|x_{n+1}-x_n\| + |r_{n+1}-r_n|M_2 + |\lambda_{n+1}-\lambda_n|M_2 \\
&\quad + |\mu_{n+1}-\mu_n|M_2 + |\lambda_{n+1}-\lambda_n|M_2,
\end{aligned}$$

where M_2 is the maximum value of $\sup_{n\in\mathbb{N}} \frac{\|T_{r_{n+1}}x_n - x_n\|}{r_{n+1}}$, $\sup_{n\in\mathbb{N}} \|A_{n+1}T_{r_n}x_n\|$, $\sup_{n\in\mathbb{N}} \frac{\lambda_n\|T\|\|Q_{\mu_{n+1}}TT_{r_n}x_n - TT_{r_n}x_n\|}{\mu_{n+1}}$ and $\sup_{n\in\mathbb{N}} \frac{\|J_{\lambda_{n+1}}z_n - z_n\|}{\lambda_{n+1}}$. Then we have that

$$\|x_{n+2}-x_{n+1}\| \leq \alpha_{n+1}\gamma\ k\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\ M_1$$

$$
\begin{aligned}
&\quad + (1-\alpha_{n+1}\overline{\gamma})\|y_{n+1}-y_n\| \\
&\le \alpha_{n+1}\gamma\ k\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\ M_1 \\
&\quad + (1-\alpha_{n+1}\overline{\gamma})\{\|x_{n+1}-x_n\| + |r_{n+1}-r_n|M_2 \\
&\quad + 2|\lambda_{n+1}-\lambda_n|M_2 + |\mu_{n+1}-\mu_n|M_2\} \\
&\le \{1-\alpha_{n+1}(\overline{\gamma}-\gamma\ k)\}\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\ M_3 \\
&\quad + |r_{n+1}-r_n|M_3 + |\lambda_{n+1}-\lambda_n|M_3 + |\mu_{n+1}-\mu_n|M_3,
\end{aligned}
$$

where $M_3 = M_1 + 2M_2$. Using Lemma 2.5, we obtain that

$$\|x_{n+2}-x_{n+1}\| \to 0. \tag{3.2}$$

We also have from $x_{n+1} = \alpha_n\gamma g(x_n) + (I-\alpha_n G)y_n$ that

$$
\begin{aligned}
\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1}-y_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma g(x_n) - Gy_n\|.
\end{aligned}
$$

From $\alpha_n \to 0$ and $\|x_{n+1}-x_n\| \to 0$, we get

$$y_n - x_n \to 0. \tag{3.3}$$

For $z \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$, we have from (2.5) that

$$
\begin{aligned}
2\|u_n - z\|^2 &= 2\|T_{r_n}x_n - T_{r_n}z\|^2 \\
&\le 2\langle x_n - z, u_n - z\rangle \\
&= \|x_n - z\|^2 + \|u_n - z\|^2 - \|u_n - x_n\|^2
\end{aligned}
$$

and hence

$$\|u_n - z\|^2 \le \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.4}$$

Then we have from (2.1), (3.1) and (3.4) that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &= \|(I-\alpha_n G)(y_n - z) + \alpha_n(\gamma g(x_n) - Gz)\|^2 \\
&\le (1-\alpha_n\overline{\gamma})^2\ \|y_n - z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Gz, x_{n+1}-z\rangle \\
&\le (1-\alpha_n\overline{\gamma})^2(\|u_n - z\|^2 + \lambda_n(\lambda_n\|TT^*\| - 2)\|(I-Q_{\mu_n})Tu_n\|^2) \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Gz, x_{n+1}-z\rangle \\
&\le (1-\alpha_n\overline{\gamma})^2(\|x_n - z\|^2 - \|x_n - u_n\|^2) \\
&\quad + (1-\alpha_n\overline{\gamma})^2\lambda_n(\lambda_n\|TT^*\| - 2)\|(I-Q_{\mu_n})Tu_n\|^2 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Gz, x_{n+1}-z\rangle \\
&\le (1-\alpha_n\overline{\gamma})^2(\|x_n - z\|^2 - \|x_n - u_n\|^2) \\
&\quad + (1-\alpha_n\overline{\gamma})^2\lambda_n(\lambda_n\|TT^*\| - 2)\|(I-Q_{\mu_n})Tu_n\|^2 \\
&\quad + 2\alpha_n\gamma\ k\|x_n - z\|\|x_{n+1}-z\| + 2\alpha_n\|\gamma g(z) - Gz\|\|x_{n+1}-z\| \\
&\le \|x_n - z\|^2 - (1-\alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 \\
&\quad + (1-\alpha_n\overline{\gamma})^2\lambda_n(\lambda_n\|TT^*\| - 2)\|(I-Q_{\mu_n})Tu_n\|^2 \\
&\quad + 2\alpha_n\gamma\ k\|x_n - z\|\|x_{n+1}-z\| + 2\alpha_n\|\gamma g(z) - Gz\|\|x_{n+1}-z\|
\end{aligned}
$$

and hence

$$\begin{aligned}(1-\alpha_n\overline{\gamma})^2\lambda_n(2-\lambda_n\|TT^*\|)\|(I-Q_{\mu_n})Tu_n\|^2+(1-\alpha_n\overline{\gamma})^2\|x_n-u_n\|^2\\ \le \|x_n-z\|^2-\|x_{n+1}-z\|^2\\ +2\alpha_n\gamma\ k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\|\gamma g(z)-Gz\|\|x_{n+1}-z\|.\end{aligned}$$

Then we have that

$$\begin{aligned}(1-\alpha_n\overline{\gamma})^2\lambda_n(2-\lambda_n\|TT^*\|)\|(I-Q_{\mu_n})Tu_n\|^2\\ \le \|x_n-z\|^2-\|x_{n+1}-z\|^2\\ +2\alpha_n\gamma\ k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\|\gamma g(z)-Gz\|\|x_{n+1}-z\|\end{aligned}$$

and

$$\begin{aligned}(1-\alpha_n\overline{\gamma})^2\|x_n-u_n\|^2 \le \|x_n-z\|^2-\|x_{n+1}-z\|^2\\ +2\alpha_n\gamma\ k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\|\gamma g(z)-Gz\|\|x_{n+1}-z\|.\end{aligned}$$

From $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and $0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<\frac{2}{\|TT^*\|}$, we have that

$$\|(I-Q_{\mu_n})Tu_n\|\to 0 \quad \text{and} \quad \|x_n-u_n\|\to 0. \tag{3.5}$$

Then we have from (3.3) and (3.5) that

$$\|y_n-u_n\|\le\|y_n-x_n\|+\|x_n-u_n\|\to 0. \tag{3.6}$$

From $\sum_{n=1}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty$, we have that $\{\lambda_n\}$ is a Cauchy sequence. Then we have $\lambda_n\to\lambda_0\in(0,\frac{2}{\|TT^*\|})$. Put $A_\mu=T^*(I-Q_\mu)T$, where $0<\mu<\liminf_{n\to\infty}\mu_n$. For $u_n=T_{r_n}x_n$, $z_n=(I-\lambda_nA_n)T_{r_n}x_n$ and $y_n=J_{\lambda_n}(I-\lambda_nA_n)T_{r_n}x_n$, we have from Lemma 2.1 and (2.6) that

$$\begin{aligned}&\|J_{\lambda_0}(I-\lambda_0A_\mu)u_n-y_n\|\\ &\le\|J_{\lambda_0}(I-\lambda_0A_\mu)u_n-J_{\lambda_0}(I-\lambda_nA_n)u_n\|+\|J_{\lambda_0}(I-\lambda_nA_n)u_n-y_n\|\\ &\le\|(I-\lambda_0A_\mu)u_n-(I-\lambda_nA_n)u_n\|+\|J_{\lambda_0}z_n-J_{\lambda_n}z_n\|\\ &=\|\lambda_0A_\mu u_n-\lambda_nA_nu_n\|+\|J_{\lambda_0}z_n-J_{\lambda_n}z_n\|\\ &=\|\lambda_0A_\mu u_n-\lambda_0A_nu_n+\lambda_0A_nu_n-\lambda_nA_nu_n\|+\|J_{\lambda_0}z_n-J_{\lambda_n}z_n\|\\ &\le\lambda_0\|T\|\|(I-Q_\mu)Tu_n-(I-Q_{\mu_n})Tu_n\|\\ &\quad+\|\lambda_0A_nu_n-\lambda_nA_nu_n\|+\|J_{\lambda_0}z_n-J_{\lambda_n}z_n\|\\ &\le\lambda_0\|T\|(\|(I-Q_\mu)Tu_n\|+\|(I-Q_{\mu_n})Tu_n\|)\\ &\quad+\|\lambda_0A_nu_n-\lambda_nA_nu_n\|+\|J_{\lambda_0}z_n-J_{\lambda_n}z_n\|\\ &\le2\lambda_0\|T\|\|(I-Q_{\mu_n})Tu_n\|+\|\lambda_0A_nu_n-\lambda_nA_nu_n\|+\|J_{\lambda_0}z_n-J_{\lambda_n}z_n\|\\ &\le2\lambda_0\|T\|\|(I-Q_{\mu_n})Tu_n\|+|\lambda_n-\lambda_0|\|A_nu_n\|+\frac{|\lambda_n-\lambda_0|}{\lambda_0}\|J_{\lambda_0}z_n-z_n\|.\end{aligned} \tag{3.7}$$

We also have from (3.6) and (3.7) that

$$\|u_n-J_{\lambda_0}(I-\lambda_0A_\mu)u_n\|\le\|u_n-y_n\|+\|y_n-J_{\lambda_0}(I-\lambda_0A_\mu)u_n\|. \tag{3.8}$$

We will use (3.7) and (3.8) later. From Lemma 2.3, we can take a unique solution $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$ of the variational inequality

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0.$$

We show that $\limsup_{n\to\infty} \langle (G - \gamma g)z_0, x_n - z_0 \rangle \geq 0$. Put

$$l = \limsup_{n\to\infty} \langle (G - \gamma g)z_0, x_n - z_0 \rangle .$$

Without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $l = \lim_{i\to\infty} \langle (G - \gamma g)z_0, x_{n_i} - z_0 \rangle$ and $\{x_{n_i}\}$ converges weakly to some point $w \in H_1$. From $\|x_n - u_n\| \to 0$, we also have that $\{u_{n_i}\}$ converges weakly to $w \in H_1$. On the other hand, from $\lambda_n \to \lambda_0 \in (0, \frac{2}{\|TT^*\|})$, we have $\lambda_{n_i} \to \lambda_0 \in (0, \frac{2}{\|TT^*\|})$. Using (3.7), we have that

$$\|J_{\lambda_0}(I - \lambda_0 A_\mu)u_{n_i} - y_{n_i}\| \to 0.$$

Furthermore, using (3.8), we have that

$$\|u_{n_i} - J_{\lambda_0}(I - \lambda_0 A_\mu)u_{n_i}\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A_\mu)$ is nonexpansive, we have from [20, p. 114] that $w = J_{\lambda_0}(I - \lambda_0 A_\mu)w$. From Lemma 3.1 we have that $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. We show $w \in F^{-1}0$. Since F is a maximal monotone operator, we have from (2.4) that $A_{r_{n_i}} x_{n_i} \in FT_{r_{n_i}} x_{n_i}$. Furthermore, we have that for any $(u, v) \in F$

$$\langle u - u_{n_i}, v - \frac{x_{n_i} - u_{n_i}}{r_{n_i}} \rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_i} \rightharpoonup w$ and $x_{n_i} - u_{n_i} \to 0$, we have

$$\langle u - w, v \rangle \geq 0.$$

Since F is a maximal monotone operator, we have $0 \in Fw$ and hence $w \in F^{-1}0$. Thus we have $w \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$. Then we have

$$l = \lim_{i\to\infty} \langle (G - \gamma g)z_0, x_{n_i} - z_0 \rangle = \langle (G - \gamma g)z_0, w - z_0 \rangle \geq 0.$$

Since $x_{n+1} - z_0 = \alpha_n(\gamma g(x_n) - Gz_0) + (I - \alpha_n G)(y_n - z_0)$, we have from (2.1) that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq (1 - \alpha_n \overline{\gamma})^2 \|y_n - z_0\|^2 + 2\langle \alpha_n(\gamma g(x_n) - Gz_0), x_{n+1} - z_0 \rangle \\
&\leq (1 - \alpha_n \overline{\gamma})^2 \|x_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Gz_0, x_{n+1} - z_0 \rangle \\
&\leq (1 - \alpha_n \overline{\gamma})^2 \|x_n - z_0\|^2 + 2\alpha_n \gamma\, k \|x_n - z_0\| \|x_{n+1} - z_0\| \\
&\qquad + 2\alpha_n \langle \gamma g(z_0) - Gz_0, x_{n+1} - z_0 \rangle \\
&\leq (1 - \alpha_n \overline{\gamma})^2 \|x_n - z_0\|^2 + \alpha_n \gamma\, k(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\qquad + 2\alpha_n \langle \gamma g(z_0) - Gz_0, x_{n+1} - z_0 \rangle \\
&= \{(1 - \alpha_n \overline{\gamma})^2 + \alpha_n \gamma\, k\} \|x_n - z_0\|^2 \\
&\quad + \alpha_n \gamma\, k \|x_{n+1} - z_0\|^2 + 2\alpha_n \langle \gamma g(z_0) - Gz_0, x_{n+1} - z_0 \rangle
\end{aligned}$$

and hence

$$\|x_{n+1} - z_0\|^2 \leq \frac{1 - 2\alpha_n \overline{\gamma} + (\alpha_n \overline{\gamma})^2 + \alpha_n \gamma\, k}{1 - \alpha_n \gamma\, k} \|x_n - z_0\|^2$$

$$
\begin{aligned}
&+\frac{2\alpha_n}{1-\alpha_n\gamma\ k}\langle\gamma g(z_0)-Gz_0,x_{n+1}-z_0\rangle\\
&=\left(1-\frac{2(\overline{\gamma}-\gamma\ k)\alpha_n}{1-\alpha_n\gamma\ k}\right)\|x_n-z_0\|^2+\frac{(\alpha_n\overline{\gamma})^2}{1-\alpha_n\gamma\ k}\|x_n-z_0\|^2\\
&+\frac{2\alpha_n}{1-\alpha_n\gamma\ k}\langle\gamma g(z_0)-Gz_0,x_{n+1}-z_0\rangle\\
&=\left(1-\frac{2(\overline{\gamma}-\gamma\ k)\alpha_n}{1-\alpha_n\gamma\ k}\right)\|x_n-z_0\|^2+\frac{\alpha_n\cdot\alpha_n\overline{\gamma}^2}{1-\alpha_n\gamma\ k}\|x_n-z_0\|^2\\
&+\frac{2\alpha_n}{1-\alpha_n\gamma\ k}\langle\gamma g(z_0)-Gz_0,x_{n+1}-z_0\rangle\\
&=(1-\beta_n)\|x_n-z_0\|^2\\
&+\beta_n\left(\frac{\alpha_n\overline{\gamma}^2\|x_n-z_0\|^2}{2(\overline{\gamma}-\gamma\ k)}+\frac{1}{\overline{\gamma}-\gamma\ k}\langle\gamma g(z_0)-Gz_0,x_{n+1}-z_0\rangle\right),
\end{aligned}
\tag{3.9}
$$

where $\beta_n=\frac{2(\overline{\gamma}-\gamma\ k)\alpha_n}{1-\alpha_n\gamma\ k}$. Since $\sum_{n=1}^{\infty}\beta_n=\infty$, we have from Lemma 2.5 and (3.9) we have that $x_n\to z_0$, where $z_0=P_{A^{-1}0\cap T^{-1}(B^{-1}0)\cap F^{-1}0}(I-G+\gamma g)z_0$. This completes the proof. □

4. Applications

In this section, using Theorem 3.3, we can obtain well-known and new strong convergence theorems which are related to the split common null point problem and an equilibrium problem in Hilbert spaces. Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty,\infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x)=\{z\in H: f(x)+\langle z,y-x\rangle\le f(y),\quad \forall y\in H\}$$

for all $x\in H$. From Rockafellar [15], we know that ∂f is a maximal monotone operator. Let C be a non-empty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x)=\begin{cases}0, & x\in C,\\ \infty, & x\notin C.\end{cases}$$

Then i_C is a proper, lower semicontinuous and convex function on H and then the subdifferential ∂i_C of i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda>0$, i.e.,

$$J_\lambda x=(I+\lambda\partial i_C)^{-1}x$$

for all $x\in H$. We have that for any $x\in H$ and $u\in C$

$$
\begin{aligned}
u=&J_\lambda x\Longleftrightarrow x\in u+\lambda\partial i_C u\Longleftrightarrow x\in u+\lambda N_C u\\
&\Longleftrightarrow x-u\in\lambda N_C u\\
&\Longleftrightarrow \frac{1}{\lambda}\langle x-u,v-u\rangle\le 0,\quad \forall v\in C\\
&\Longleftrightarrow \langle x-u,v-u\rangle\le 0,\quad \forall v\in C
\end{aligned}
$$

$$\iff u = P_C x,$$

where $N_C u$ is the normal cone to C at u, i.e.,

$$N_C u = \{z \in H : \langle z, v - u\rangle \le 0,\ \forall v \in C\}.$$

Theorem 4.1. *Let H_1 and H_2 be Hilbert spaces. Let C and D be non-empty, closed and convex subsets of H_1 and let Q be a non-empty, closed and convex subset of H_2. Let $T : H_1 \to H_2$ be a bounded linear operator such that $C \cap T^{-1}Q \cap D$ is non-empty. Let T^* be the adjoint operator of T. Let P_C and P_D be the metric projections of H_1 onto C and D, respectively and let P_Q be the metric projection of H_2 onto Q. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let G be a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n G)P_C(I - \lambda_n T^*(I - P_Q)T)P_D x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2}{\|TT^*\|}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to $z_0 \in C \cap T^{-1}Q \cap D$, where z_0 is a unique fixed point of $P_{C\cap T^{-1}Q\cap D}(I - G + \gamma g)$. This point z_0 is also a unique solution of the variational inequality

$$\langle (G - \gamma g)z_0, q - z_0\rangle \ge 0, \quad \forall q \in C \cap T^{-1}Q \cap D.$$

Proof. Put $A = \partial i_C$, $F = \partial i_D$ and $B = \partial i_Q$ in Theorem 3.3. Then we have that for $\lambda_n > 0$, $r_n > 0$ and $\mu_n > 0$, $J_{\lambda_n} = P_C$, $T_{r_n} = P_D$ and $Q_{\mu_n} = P_Q$. Furthermore, we have $(\partial i_C)^{-1}0 = C$, $(\partial i_D)^{-1}0 = D$ and $(\partial i_Q)^{-1}0 = Q$. Taking $\mu_n = r_n = 1$, we obtain the desired result by Theorem 3.3. □

Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Then an equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \ge 0, \quad \forall y \in C. \tag{4.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \ge 0,\ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \le f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

We know the following lemma which appears implicitly in Blum and Oettli [5].

Lemma 4.2 (Blum and Oettli)**.** *Let C be a nonempty, closed and convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [8].

Lemma 4.3. *Assume that $f : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}.$$

Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 4.2 and 4.3, Takahashi, Takahashi and Toyoda [17] obtained the following lemma. See [2] for a more general result.

Lemma 4.4. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy $(A1)-(A4)$. Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $\mathrm{dom}(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,

$$T_r x = (I + rA_f)^{-1} x.$$

Using Theorem 3.3, we can also prove a strong convergence theorem for finding solutions of equilibrium problems in Hilbert spaces.

Theorem 4.5. *Let H_1 and H_2 be Hilbert spaces. Let C and D be non-empty, closed and convex subsets of H_1 and let Q be a non-empty, closed and convex subset of H_2. Let f_1 and f_2 be bifunctions of $C \times C$ into $\mathbb{R}$ and $D \times D$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let f_3 be a bifunction of $Q \times Q$ into $\mathbb{R}$ satisfying $(A1)-(A4)$ such that $EP(f_1)$, $EP(f_2)$ and $EP(f_3)$ are non-empty. Let $T : H_1 \to H_2$ be a bounded linear operator such that $EP(f_1) \cap T^{-1}EP(f_3) \cap EP(f_2)$ is non-empty. Let T^* be the adjoint operator of T. Let J_λ and T_r be the resolvents of f_1 for $\lambda > 0$ and of f_2 for $r > 0$, respectively and let Q_μ be the resolvent of f_3 for $\mu > 0$. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let G be a strongly positive bounded linear* *self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n T^*(I - Q_{\mu_n})T) T_{r_n} x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\}, \{\mu_n\}, \{r_n\} \subset (0, \infty)$ satisfy

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|TT^*\|}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\liminf_{n\to\infty} \mu_n > 0, \ \sum_{n=1}^{\infty} |\mu_n - \mu_{n+1}| < \infty, \ \liminf_{n\to\infty} r_n > 0 \ \textit{and} \ \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to $z_0 \in EP(f_1) \cap T^{-1}EP(f_3) \cap EP(f_2)$, where z_0 is a unique fixed point of $P_{EP(f_1)\cap T^{-1}EP(f_3)\cap EP(f_2)}(I - G + \gamma g)$. This point z_0 is also a unique solution of the variational inequality

$$\langle (G - \gamma g) z_0, q - z_0 \rangle \geq 0, \quad \forall q \in EP(f_1) \cap T^{-1}EP(f_3) \cap EP(f_2).$$

Proof. For the bifunctions $f_1 : C \times C \to \mathbb{R}$, $f_2 : D \times D \to \mathbb{R}$ and $f_3 : Q \times Q \to \mathbb{R}$, we can define A_{f_1}, A_{f_2} and A_{f_3} in Lemma 4.4. Putting $A = A_{f_1}$, $F = A_{f_2}$ and $B = A_{f_3}$ in Theorem 3.3, we obtain from Lemma 4.4 that $J_{\lambda_n} = (I + \lambda_n A_{f_1})^{-1}$, $T_{r_n} = (I + r_n A_{f_2})^{-1}$ and $Q_{\mu_n} = (I + \mu_n A_{f_3})^{-1}$ for all $\lambda_n > 0$, $r_n > 0$ and $\mu_n > 0$, respectively. Thus we obtain the desired result by Theorem 3.3. □

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