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FINDING THE MINIMUM NORM COMMON ELEMENT OF MAXIMAL MONOTONE OPERATORS AND NONEXPANSIVE MAPPINGS WITHOUT INVOLVING PROJECTION

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Abstract. The purpose of this paper is to construct two simple algorithms without involving projection for finding the minimum norm common solution of maximal monotone operators and nonexpansive mappings in Hilbert spaces. Some applications are also included.

1. Introduction

In many problems, it is needed to find a solution with minimum norm. A typical example is the least-squares solution to the constrained linear inverse problem [25]

$$\begin{cases} Ax = b, \\ x \in C, \end{cases}$$

where A is a bounded linear operator from H to another real Hilbert space H_1 and b is a given point in H_1. Some related works on the minimum norm solution problems (or least squares problem), please refer to [8, 9, 12, 13, 15, 19–21, 23, 31, 32, 34–37]. We note that we may formulate such problems in an abstract way as finding a point $x^\dagger \in \Omega$ with the property

$$\|x^\dagger\| = \min_{x\in\Omega} \|x\|.$$

In another word, $x^\dagger$ is the (nearest point or metric) projection of the origin onto Ω,

$$x^\dagger = P_\Omega(0),$$

where P_Ω is the metric (or nearest point) projection from H onto Ω. This indicates that we can use projection to find the minimum norm solution. In this respect, very recently, some authors use projection algorithms that employ projections onto the set C, in order to iteratively reach the minimum norm solution of some nonlinear operators, see., e.g., [8, 21, 31, 32, 34–37].

Projection methods are used extensively in a variety of methods in optimization theory. Apart from theoretical interest, the main advantage of projection methods, which makes them successful in real-word applications, is computational. The field

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of projection methods is vast and we mention here only a few recent works that can give the reader some good starting points. Such a list includes, among many others, the works of [2, 5, 11, 16–18], the connection with variational inequalities, see, e.g., Solodov and Svaiter [26], Censor, Gibali and Reich [6], Noor [22], Yamada [33] which is motivated by real-word problems of signal processing, and the many contributions of Bauschke and Combettes, see, e.g., Bauschke, Combettes and Kruk [3]. We observe that in each iteration of the projection algorithm, in order to get the next iterative x_{n+1}, projection onto C is calculated, according to the iterative step. If the set C is simple enough, so that the projection onto it is easily executed, then this method is particularly useful; but, if C is a general closed and convex set, then a minimal distance problem has to be solved in order to obtain the next iterative. This might seriously affect the efficiency of the method. It remains however a challenge how to implement the projection algorithm in the case where the projection P_C fails to have closed-form expressions. Hence, it is an very interesting work of finding the minimum norm solution without involving projection.

The purpose of this paper is to construct two algorithms without using projection for finding the minimum norm common solution of maximal monotone operators and nonexpansive mappings in Hilbert spaces. Our work is mainly based on a recent work of Takahashi, Takahashi and Toyoda [28]. They proved the following convergence result

Theorem 1.1. *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself, such that $F(S) \cap (A + B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\alpha_n x + (1 - \alpha_n)J_{\lambda_n}^B(x_n - \lambda_n Ax_n)) \tag{1.1}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, 2\alpha)$,$\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < 2\alpha, \qquad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0,\ \lim_{n\to\infty}\alpha_n = 0 \text{ and } \sum_n \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point of $F(S) \cap (A + B)^{-1}0$.

Remark 1.2. We notice that the above method (1.1) does find the minimum-norm element in $F(S) \cap (A + B)^{-1}0$ if $0 \in C$. However, if $0 \notin C$, then this algorithm (1.1) does not work to find the minimum-norm element. The reason is simple: If $0 \notin C$, we cannot take $x = 0$ since $(1 - \alpha_n)J_{\lambda_n}^B(x_n - \lambda_n Ax_n)$ may not belong to C and consequently, x_{n+1} may be undefined. A natural idea is we can choose the initial point x in the whole space. Then we have to employ projection such that $P_C[\alpha_n x + (1 - \alpha_n)J_{\lambda_n}^B(x_n - \lambda_n Ax_n)] \in C$. Thus, we can construct algorithm $x_{n+1} = \beta_n x_n + (1 - \beta_n)SP_C[\alpha_n x + (1 - \alpha_n)J_{\lambda_n}^B(x_n - \lambda_n Ax_n)]$ to find the minimum-norm element. This is an active topic. But this is not our main purpose in the present paper due to this algorithm involves the computation of the projection.

Remark 1.3. We also note that in Theorem 1.1, the authors added an additional assumption: the domain of B is included in C (The reader can refer to Lemma 4.3 in the last section for a possible example which satisfies this assumption). This assumption is indeed not restrict in order to guarantee $J^B_{\lambda_n}(x_n-\lambda_n Ax_n) \in C$. Based on this fact, in the present paper we construct two simple algorithms with strong convergence to the minimum-norm element.

Remark 1.4. From the listed references, there exist a large number of problems which need to find the minimum norm solution. A useful path to circumvent this problem is to use projection. Bauschke and Browein [2] and Censor and Zenios [7] provide reviews of the field. The main difficult is in computation. We note that the algorithm (1.1) can not use to find the minimum norm element.

Motivated and inspired by the works in this field, we first suggest the following two algorithms without using projection:

$$x_t = SJ^B_\lambda\Big((1-t)x_t - \lambda Ax_t\Big), t \in (0,1)$$

and

$$x_{n+1} = \beta_n x_n + (1-\beta_n)SJ^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big), n \geq 0.$$

(Notice that these two algorithms are indeed well-defined (see the next section).) We will show the suggested algorithms converge strongly to a common point $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$ which is the minimum-norm element of $F(S)\cap(A+B)^{-1}0$. Some applications are also included.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. A mapping $A : C \to H$ is said to be α-inverse strongly-monotone iff $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$ for some $\alpha > 0$ and for all $x, y \in C$. It is known that if A is α-inverse strongly-monotone, then $\|Ax - Ay\| \leq 1/\alpha\|x - y\|$ for all $x, y \in C$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H iff $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in dom(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on H. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $J^B_\lambda = (I+\lambda B)^{-1} : H \to dom(B)$, which is called the resolvent of B for λ. It is known that the resolvent J^B_λ is firmly nonexpansive, i.e., $\|J^B_\lambda x - J^B_\lambda y\|^2 \leq \langle J^B_\lambda x - J^B_\lambda y, x - y\rangle$ for all $x, y \in C$ and $B^{-1}0 = F(J^B_\lambda)$ for all $\lambda > 0$.

The following resolvent identity is well-known: for $\lambda > 0$ and $\mu > 0$, there holds the identity

$$J_\lambda^B x = J_\mu^B\Big(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda^B x\Big), x \in H. \tag{2.1}$$

We use the following notation:

- $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x;
- $x_n \to x$ stands for the strong convergence of (x_n) to x.

We need the following lemmas for the next section.

Lemma 2.1 ([35]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $\lambda > 0$ be a constant. Then, we have*

$$\|(I-\lambda A)x - (I-\lambda A)y\|^2 \le \|x-y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2, \forall x, y \in C.$$

In particular, if $0 \le \lambda \le 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.2 ([21]). *Let C be a closed convex subset of a Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Then $F(S)$ is a closed convex subset of C and the mapping $I - S$ is demiclosed at 0, i.e. whenever $\{x_n\} \subset C$ is such that $x_n \rightharpoonup x$ and $(I-S)x_n \to 0$, then $(I-S)x = 0$.*

Lemma 2.3 ([36]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4 ([14]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^\infty \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^\infty |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we will prove our main results.

Theorem 3.1. *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself, such that $F(S) \cap (A+B)^{-1}0 \ne \emptyset$. Let λ be a constant satisfying $a \le \lambda \le b$ where $[a,b] \subset (0, 2\alpha)$. For $t \in (0, 1-\frac{\lambda}{2\alpha})$, let $\{x_t\} \subset C$ be a net generated by*

$$x_t = SJ_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big). \tag{3.1}$$

Then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$ which is the minimum norm element in $F(S) \cap (A+B)^{-1}0$.

Proof. First, we show the net $\{x_t\}$ is well-defined. For any $t \in (0, 1-\frac{\lambda}{2\alpha})$, we define a mapping $T := SJ_\lambda^B\Big((1-t)I - \lambda A\Big)$. Note that S, J_λ^B and $I - \frac{\lambda}{1-t}A$ (see Lemma 2.1) are nonexpansive. For any $x, y \in C$, we have

$$\begin{aligned} \|Tx - Ty\| &= \Big\|SJ_\lambda^B\Big((1-t)x - \lambda Ax\Big) - SJ_\lambda^B\Big((1-t)y - \lambda Ay\Big)\Big\| \\ &\le \Big\|(1-t)(x - \frac{\lambda}{1-t}Ax) - (1-t)(y - \frac{\lambda}{1-t}Ay)\Big\| \\ &\le (1-t)\|x-y\|, \end{aligned}$$

which implies the mapping T is a contraction on C. We use x_t to denote the unique fixed point of T in C. Therefore, $\{x_t\}$ is well-defined.

Take any $z \in F(S) \cap (A+B)^{-1}0$. It is obvious that $z = J_\lambda^B(z - \lambda Az)$ for all $\lambda > 0$. So, we have $z = J_\lambda^B(z - \lambda Az) = J_\lambda^B\Big(tz + (1-t)(z - \lambda Az/(1-t))\Big)$ for all $t \in (0,1)$. Since J_λ^B is nonexpansive for all $\lambda > 0$, we have

$$\begin{aligned} &\Big\|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\ &= \Big\|J_\lambda^B\Big((1-t)(x_t - \lambda Ax_t/(1-t))\Big) - J_\lambda^B\Big(tz + (1-t)(z - \lambda Az/(1-t))\Big)\Big\|^2 \\ &\le \Big\|\Big((1-t)(x_t - \lambda Ax_t/(1-t))\Big) - \Big(tz + (1-t)(z - \lambda Az/(1-t))\Big)\Big\|^2 \\ &= \Big\|(1-t)\Big((x_t - \lambda Ax_t/(1-t)) - (z - \lambda Az/(1-t))\Big) + t(-z)\Big\|^2. \end{aligned} \tag{3.2}$$

By using the convexity of $\|\cdot\|$ and the α-inverse strong monotonicity of A, we derive

$$\begin{aligned} &\Big\|(1-t)\Big((x_t - \lambda Ax_t/(1-t)) - (z - \lambda Az/(1-t))\Big) + t(-z)\Big\|^2 \\ &\le (1-t)\|(x_t - \lambda Ax_t/(1-t)) - (z - \lambda Az/(1-t))\|^2 + t\|z\|^2 \\ &= (1-t)\|(x_t - z) - \lambda(Ax_t - Az)/(1-t)\|^2 + t\|z\|^2 \\ &= (1-t)\Big(\|x_t - z\|^2 - \frac{2\lambda}{1-t}\langle Ax_t - Az, x_t - z\rangle + \frac{\lambda^2}{(1-t)^2}\|Ax_t - Az\|^2\Big) \\ &\quad + t\|z\|^2 \\ &\le (1-t)\Big(\|x_t - z\|^2 - \frac{2\alpha\lambda}{1-t}\|Ax_t - Az\|^2 + \frac{\lambda^2}{(1-t)^2}\|Ax_t - Az\|^2\Big) + t\|z\|^2 \\ &= (1-t)\Big(\|x_t - z\|^2 + \frac{\lambda}{(1-t)^2}(\lambda - 2(1-t)\alpha)\|Ax_t - Az\|^2\Big) + t\|z\|^2. \end{aligned} \tag{3.3}$$

By the assumption, we have $\lambda - 2(1-t)\alpha \le 0$ for all $t \in (0, 1-\frac{\lambda}{2\alpha})$. Then, from (3.2) and (3.3), we obtain

$$\Big\|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2$$

$$\leq (1-t)\Big(\|x_t - z\|^2 + \frac{\lambda}{(1-t)^2}(\lambda - 2(1-t)\alpha)\|Ax_t - Az\|^2\Big) + t\|z\|^2 \tag{3.4}$$
$$\leq (1-t)\|x_t - z\|^2 + t\|z\|^2.$$

It follows from (3.1) and (3.4) that

$$\begin{aligned}\|x_t - z\|^2 &\leq \|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\|^2 \\ &\leq (1-t)\|x_t - z\|^2 + t\|z\|^2.\end{aligned} \tag{3.5}$$

It follows that

$$\|x_t - z\| \leq \|z\|.$$

Therefore, $\{x_t\}$ is bounded. Since A is α-inverse strongly monotone, it is $\frac{1}{\alpha}$-Lipschitz continuous. We deduce immediately that $\{Ax_t\}$ is also bounded.

By (3.4) and (3.5), we obtain

$$\|x_t - z\|^2 \leq (1-t)\|x_t - z\|^2 + \frac{\lambda}{(1-t)}(\lambda - 2(1-t)\alpha)\|Ax_t - Az\|^2 + t\|z\|^2.$$

So,

$$\frac{\lambda}{(1-t)}(2(1-t)\alpha - \lambda)\|Ax_t - Az\|^2 \leq t\|z\|^2 - t\|x_t - z\|^2 \to 0.$$

This implies that

$$\lim_{t\to 0+}\|Ax_t - Az\| = 0. \tag{3.6}$$

Next, we show $\|x_t - Sx_t\| \to 0$. By using the firm nonexpansivity of J_λ^B, we have

$$\begin{aligned} &\Big\|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\ = &\Big\|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - J_\lambda^B\Big(z - \lambda Az\Big)\Big\|^2 \\ \leq &\Big\langle (1-t)x_t - \lambda Ax_t - (z - \lambda Az), J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\rangle \\ = &\frac{1}{2}\Big(\|(1-t)x_t - \lambda Ax_t - (z - \lambda Az)\|^2 + \Big\|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\ &-\Big\|(1-t)x_t - \lambda(Ax_t - \lambda Az) - J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big)\Big\|^2\Big). \end{aligned}$$

By the nonexpansivity of $I - \lambda A/(1-t)$, we have

$$\begin{aligned} &\|(1-t)x_t - \lambda Ax_t - (z - \lambda Az)\|^2 \\ = &\|(1-t)((x_t - \lambda Ax_t/(1-t) - (z - \lambda Az/(1-t))) + t(-z)\|^2 \\ \leq &(1-t)\|(x_t - \lambda Ax_t/(1-t) - (z - \lambda Az/(1-t))\|^2 + t\|z\|^2 \\ \leq &(1-t)\|x_t - z\|^2 + t\|z\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} &\Big\|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\ \leq &\frac{1}{2}\Big((1-t)\|x_t - z\|^2 + t\|z\|^2 + \Big\|J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \end{aligned}$$

$$-\Big\|(1-t)x_t - J_\lambda^B\Big((1-t)x_t - \lambda Ax_t\Big) - \lambda(Ax_t - Az)\Big\|^2\Big).$$

That is,

$$\begin{aligned}
&\Big\|J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)-z\Big\|^2\\
\leq\ & (1-t)\|x_t-z\|^2+t\|z\|^2\\
&-\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)-\lambda(Ax_t-Az)\Big\|^2\Big)\\
=\ & (1-t)\|x_t-z\|^2+t\|z\|^2-\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|^2\\
&+2\lambda\Big\langle(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big),Ax_t-Az\Big\rangle-\lambda^2\|Ax_t-Az\|^2\\
\leq\ & (1-t)\|x_t-z\|^2+t\|z\|^2-\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|^2\\
&+2\lambda\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|\|Ax_t-Az\|.
\end{aligned}$$

This together with (3.5) imply that

$$\begin{aligned}
\|x_t-z\|^2 \leq\ & \Big\|J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)-z\Big\|^2\\
\leq\ & (1-t)\|x_t-z\|^2+t\|z\|^2-\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|^2\\
&+2\lambda\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|\|Ax_t-Az\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|^2\\
\leq\ & t\|z\|^2+2\lambda\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|\|Ax_t-Az\|.
\end{aligned}$$

Since $\|Ax_t - Az\| \to 0$, we deduce

$$\lim_{t\to 0+}\Big\|(1-t)x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|=0.$$

Therefore,

$$\lim_{t\to 0+}\Big\|x_t-J_\lambda^B\Big((1-t)x_t-\lambda Ax_t\Big)\Big\|=0. \tag{3.7}$$

From (3.1) and (3.2), we have

$$\begin{aligned}
\|x_t-z\|^2 \leq\ & \Big\|(1-t)\Big((x_t-\frac{\lambda}{1-t}Ax_t)-(z-\frac{\lambda}{1-t}Az)\Big)-tz\Big\|^2\\
=\ & (1-t)^2\Big\|(x_t-\frac{\lambda}{1-t}Ax_t)-(z-\frac{\lambda}{1-t}Az)\Big\|^2\\
&-2t(1-t)\Big\langle z,(x_t-\frac{\lambda}{1-t}Ax_t)-(z-\frac{\lambda}{1-t}Az)\Big\rangle+t^2\|z\|^2\\
\leq\ & (1-t)^2\|x_t-z\|^2-2t(1-t)\Big\langle z,x_t-\frac{\lambda}{1-t}(Ax_t-Az)-z\Big\rangle+t^2\|z\|^2
\end{aligned}$$

$$= (1-2t)\|x_t - z\|^2 + 2t\Big\{ -(1-t)\Big\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\rangle + t^2(\|z\|^2 + \|x_t - z\|^2)\Big\}.$$

It follows that

$$\begin{aligned} \|x_t - z\|^2 &\le -\Big\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\rangle + \frac{t}{2}(\|z\|^2 + \|x_t - z\|^2) \\ &\quad + t\|z\|\Big\|x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\| \\ &\le -\Big\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\rangle + tM, \qquad (3.8) \end{aligned}$$

where M is some constant such that

$$\sup\Big\{\|z\|^2 + \|x_t - z\|^2 + \|z\|\|x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\|, t \in (0, 1 - \frac{\lambda}{2\alpha})\Big\} \le M.$$

Next we show that $\{x_t\}$ is relatively norm-compact as $t \to 0+$. Assume $\{t_n\} \subset (0, 1 - \frac{\lambda}{2\alpha})$ is such that $t_n \to 0+$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.8), we have

$$\begin{aligned} \|x_n - z\|^2 &\le -\Big\langle z, x_n - \frac{\lambda}{1-t_n}(Ax_n - Az) - z\Big\rangle \\ &\quad + t_n M, \ z \in F(S) \cap (A+B)^{-1}0. \qquad (3.9) \end{aligned}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_n \rightharpoonup \tilde{x} \in C$. Hence, $x_n - \frac{\lambda}{1-t_n}(Ax_n - Az) \rightharpoonup \tilde{x}$ because of $\|Ax_n - Az\| \to 0$. From (3.7), we have $\lim_{t\to 0+} \|x_t - Sx_t\| = 0$. Thus,

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \qquad (3.10)$$

We can use Lemma 2.2 to (3.10) to deduce $\tilde{x} \in F(S)$. Further, we show that $\tilde{x}$ is also in $(A+B)^{-1}0$. Let $v \in Bu$. Set $z_n = J_\lambda^B((1-t_n)x_n - \lambda Ax_n)$ for all n. Then, we have

$$(1-t_n)x_n - \lambda Ax_n \in (I + \lambda B)z_n \Rightarrow \frac{1-t_n}{\lambda}x_n - Ax_n - \frac{z_n}{\lambda} \in Bz_n.$$

Since B is monotone, we have, for $(u, v) \in B$,

$$\begin{aligned} & \Big\langle \frac{1-t_n}{\lambda}x_n - Ax_n - \frac{z_n}{\lambda} - v, z_n - u\Big\rangle \ge 0 \\ \Rightarrow\ & \langle (1-t_n)x_n - \lambda Ax_n - z_n - \lambda v, z_n - u\rangle \ge 0 \\ \Rightarrow\ & \langle Ax_n + v, z_n - u\rangle \le \frac{1}{\lambda}\langle x_n - z_n, z_n - u\rangle - \frac{t_n}{\lambda}\langle x_n, z_n - u\rangle \\ \Rightarrow\ & \langle A\tilde{x} + v, z_n - u\rangle \le \frac{1}{\lambda}\langle x_n - z_n, z_n - u\rangle - \frac{t_n}{\lambda}\langle x_n, z_n - u\rangle + \langle A\tilde{x} - Ax_n, z_n - u\rangle \\ \Rightarrow\ & \langle A\tilde{x} + v, z_n - u\rangle \le \frac{1}{\lambda}\|x_n - z_n\|\|z_n - u\| + \frac{t_n}{\lambda}\|x_n\|\|z_n - u\| \\ & \qquad + \|A\tilde{x} - Ax_n\|\|z_n - u\|. \end{aligned}$$

It follows that

$$\langle A\tilde{x} + v, \tilde{x} - u\rangle \le \frac{1}{\lambda}\|x_n - z_n\|\|z_n - u\| + \frac{t_n}{\lambda}\|x_n\|\|z_n - u\|$$

$$+\|A\tilde{x} - Ax_n\|\|z_n - u\| + \langle A\tilde{x} + v, \tilde{x} - z_n\rangle. \tag{3.11}$$

Since

$$\langle x_n - \tilde{x}, Ax_n - A\tilde{x}\rangle \geq \alpha\|Ax_n - A\tilde{x}\|^2,$$

$Ax_n \to Az$ and $x_n \rightharpoonup \tilde{x}$, we have $Ax_n \to A\tilde{x}$. We also observe that $t_n \to 0$, $\|x_n - z_n\| \to 0$ (by (3.7)) and $z_n \rightharpoonup \tilde{x}$. Then, from (3.11), we derive

$$\langle -A\tilde{x} - v, \tilde{x} - u\rangle \geq 0.$$

Since B is maximal monotone, we have $-A\tilde{x} \in B\tilde{x}$. This shows that $0 \in (A+B)\tilde{x}$. Hence, we have $\tilde{x} \in F(S) \cap (A+B)^{-1}0$. Therefore we can substitute $\tilde{x}$ for z in (3.9) to get

$$\|x_n - \tilde{x}\|^2 \leq -\Big\langle \tilde{x}, x_n - \frac{\lambda}{1-t_n}(Ax_n - A\tilde{x}) - \tilde{x}\Big\rangle + t_n M.$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t \to 0+$.

Now we return to (3.9) and take the limit as $n \to \infty$ to get

$$\|\tilde{x} - z\|^2 \leq -\langle z, \tilde{x} - z\rangle,\ z \in F(S) \cap (A+B)^{-1}0.$$

Equivalently,

$$\|\tilde{x}\|^2 \leq \langle \tilde{x}, z\rangle, \quad z \in F(S) \cap (A+B)^{-1}0.$$

This clearly implies that

$$\|\tilde{x}\| \leq \|z\|, \quad z \in F(S) \cap (A+B)^{-1}0.$$

Therefore, $\tilde{x}$ is the minimum-norm element in $F(S) \cap (A+B)^{-1}0$. This completes the proof. □

Theorem 3.2. *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself, such that $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S J_{\lambda_n}^B\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) \tag{3.12}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, 2\alpha)$,$\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$ *where* $[a,b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.12) converges strongly to a point $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$ which is the minimum norm element in $F(S) \cap (A+B)^{-1}0$.

Proof. Pick up $z \in F(S) \cap (A+B)^{-1}0$. It is obvious that $z = J_{\lambda_n}^B(z - \lambda_n Az) = J_{\lambda_n}^B\Big(\alpha_n z + (1-\alpha_n)(z - \lambda_n Az/(1-\alpha_n))\Big)$ for all $n \geq 0$. Since J_λ^B is nonexpansive for all $\lambda > 0$, we have

$$\Big\|J_{\lambda_n}^B\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big) - z\Big\|^2$$

$$
\begin{aligned}
&= \Big\| J_{\lambda_n}^B \Big((1-\alpha_n)(x_n - \lambda_n A x_n/(1-\alpha_n)) \Big) \\
&\qquad - J_{\lambda_n}^B \Big(\alpha_n z + (1-\alpha_n)(z - \lambda_n A z/(1-\alpha_n)) \Big) \Big\|^2 \\
&\le \Big\| \Big((1-\alpha_n)(x_n - \lambda_n A x_n/(1-\alpha_n)) \Big) notag \\
&\qquad - \Big(\alpha_n z + (1-\alpha_n)(z - \lambda_n A z/(1-\alpha_n)) \Big) \Big\|^2 \\
&= \Big\| (1-\alpha_n) \Big((x_n - \lambda_n A x_n/(1-\alpha_n)) - (z - \lambda_n A z/(1-\alpha_n)) \Big) + \alpha_n(-z) \Big\|^2. \qquad (3.13)
\end{aligned}
$$

By using the convexity of $\|\cdot\|$ and the α-inverse strong monotonicity of A, we derive

$$
\begin{aligned}
&\Big\| (1-\alpha_n) \Big((x_n - \lambda_n A x_n/(1-\alpha_n)) - (z - \lambda_n A z/(1-\alpha_n)) \Big) + \alpha_n(-z) \Big\|^2 \\
&\le (1-\alpha_n) \| (x_n - \lambda_n A x_n/(1-\alpha_n)) - (z - \lambda_n A z/(1-\alpha_n)) \|^2 + \alpha_n \|z\|^2 \\
&= (1-\alpha_n) \| (x_n - z) - \lambda_n (A x_n - A z)/(1-\alpha_n) \|^2 + \alpha_n \|z\|^2 \\
&= (1-\alpha_n) \Big(\|x_n - z\|^2 - \frac{2\lambda_n}{1-\alpha_n} \langle A x_n - A z, x_n - z \rangle + \frac{\lambda_n^2}{(1-\alpha_n)^2} \|A x_n - A z\|^2 \Big) \\
&\quad + \alpha_n \|z\|^2 \\
&\le (1-\alpha_n) \Big(\|x_n - z\|^2 - \frac{2\alpha\lambda_n}{1-\alpha_n} \|A x_n - A z\|^2 + \frac{\lambda_n^2}{(1-\alpha_n)^2} \|A x_n - A z\|^2 \Big) \\
&\quad + \alpha_n \|z\|^2 \\
&= (1-\alpha_n) \Big(\|x_n - z\|^2 + \frac{\lambda_n}{(1-\alpha_n)^2} (\lambda_n - 2(1-\alpha_n)\alpha) \|A x_n - A z\|^2 \Big) \\
&\quad + \alpha_n \|z\|^2. \qquad (3.14)
\end{aligned}
$$

By condition (iii), we get $\lambda_n - 2(1-\alpha_n)\alpha \le 0$ for all $n \ge 0$. Then, from (3.13) and (3.14), we obtain

$$
\begin{aligned}
&\Big\| J_{\lambda_n}^B \Big((1-\alpha_n) x_n - \lambda_n A x_n \Big) - z \Big\|^2 \\
&\le (1-\alpha_n) \Big(\|x_n - z\|^2 + \frac{\lambda_n}{(1-\alpha_n)^2} (\lambda_n - 2(1-\alpha_n)\alpha) \|A x_n - A z\|^2 \Big) \\
&\quad + \alpha_n \|z\|^2 \qquad (3.15) \\
&\le (1-\alpha_n) \|x_n - z\|^2 + \alpha_n \|z\|^2.
\end{aligned}
$$

It follows from (3.12) and (3.15) that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \Big\| \beta_n (x_n - z) + (1-\beta_n) \Big(S J_{\lambda_n}^B \Big((1-\alpha_n) x_n - \lambda_n A x_n \Big) - z \Big) \Big\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n) \Big\| S J_{\lambda_n}^B \Big((1-\alpha_n) x_n - \lambda_n A x_n \Big) - z \Big\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n) \Big\| J_{\lambda_n}^B \Big((1-\alpha_n) x_n - \lambda_n A x_n \Big) - z \Big\|^2 \qquad (3.16) \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n) ((1-\alpha_n) \|x_n - z\|^2 + \alpha_n \|z\|^2)
\end{aligned}
$$

$$
\begin{aligned}
&= [1-(1-\beta_n)\alpha_n]\|x_n - z\|^2 + (1-\beta_n)\alpha_n\|z\|^2 \\
&\leq \max\{\|x_n - z\|^2, \|z\|^2\}.
\end{aligned}
$$

By induction, we have

$$\|x_{n+1} - z\|^2 \leq \max\{\|x_0 - z\|^2, \|z\|^2\}.$$

Therefore, $\{x_n\}$ is bounded. Since A is α-inverse strongly monotone, it is $\frac{1}{\alpha}$-Lipschitz continuous. We deduce immediately that $\{Ax_n\}$ is also bounded. Set $u_n = (1-\alpha_n)x_n - \lambda_n Ax_n$ and $y_n = SJ^B_{\lambda_n}u_n$ for all $n \geq 0$. Noticing that S and $J^B_{\lambda_n}$ are nonexpansive, we can check easily that $\{u_n\}$, $\{J^B_{\lambda_n}u_n\}$ and $\{y_n\}$ are bounded.

We can rewrite (3.12) as $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$ for all $n \geq 0$. Next, we estimate $\|x_{n+1} - x_n\|$. In fact, we have

$$
\begin{aligned}
& \|y_{n+1} - y_n\| \\
= & \Big\|SJ^B_{\lambda_{n+1}}\Big((1-\alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1}\Big) - SJ^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| \\
\leq & \Big\|J^B_{\lambda_{n+1}}\Big((1-\alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1}\Big) - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| \\
\leq & \Big\|J^B_{\lambda_{n+1}}\Big((1-\alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1}\Big) - J^B_{\lambda_{n+1}}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| \\
& + \Big\|J^B_{\lambda_{n+1}}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big) - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| \\
\leq & \Big\|\Big((1-\alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1}\Big) - \Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| \\
& + \Big\|J^B_{\lambda_{n+1}}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big) - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| \\
= & \|(I-\lambda_{n+1}A)x_{n+1} - (I-\lambda_{n+1}A)x_n + (\lambda_n - \lambda_{n+1})Ax_n + \alpha_n x_n - \alpha_{n+1}x_{n+1}\| \\
& + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \\
\leq & \|(I-\lambda_{n+1}A)x_{n+1} - (I-\lambda_{n+1}A)x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\
& + \alpha_n\|x_n\| + \alpha_{n+1}\|x_{n+1}\| + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\|.
\end{aligned}
$$

Since $I - \lambda_{n+1}A$ is nonexpansive for $\lambda_{n+1} \in (0, 2\alpha)$, we have $\|(I-\lambda_{n+1}A)x_{n+1} - (I-\lambda_{n+1}A)x_n\| \leq \|x_{n+1} - x_n\|$. By the resolvent identity (2.1), we have

$$J^B_{\lambda_{n+1}}u_n = J^B_{\lambda_n}\Big(\frac{\lambda_n}{\lambda_{n+1}}u_n + (1 - \frac{\lambda_n}{\lambda_{n+1}})J^B_{\lambda_{n+1}}u_n\Big).$$

It follows that

$$
\begin{aligned}
\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| &= \Big\|J^B_{\lambda_n}\Big(\frac{\lambda_n}{\lambda_{n+1}}u_n + (1 - \frac{\lambda_n}{\lambda_{n+1}})J^B_{\lambda_{n+1}}u_n\Big) - J^B_{\lambda_n}u_n\Big\| \\
&\leq \Big\|\Big(\frac{\lambda_n}{\lambda_{n+1}}u_n + (1 - \frac{\lambda_n}{\lambda_{n+1}})J^B_{\lambda_{n+1}}u_n\Big) - u_n\Big\| \\
&\leq \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\|u_n - J^B_{\lambda_{n+1}}u_n\|.
\end{aligned}
$$

So,

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \alpha_n\|x_n\| + \alpha_{n+1}\|x_{n+1}\|$$

$$+\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|u_n - J^B_{\lambda_{n+1}}u_n\|.$$

Then,

$$\begin{aligned}\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \quad &\le \quad |\lambda_{n+1}-\lambda_n|\|Ax_n\| + \alpha_n\|x_n\| + \alpha_{n+1}\|x_{n+1}\| \\ &\quad +\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|u_n - J^B_{\lambda_{n+1}}u_n\|.\end{aligned}$$

Since $\alpha_n \to 0$, $\lambda_{n+1}-\lambda_n \to 0$ and $\liminf_{n\to\infty}\lambda_n > 0$, we obtain

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0.$$

From Lemma 2.3, we get

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Consequently, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_n - x_n\| = 0.$$

From (3.15) and (3.16), we have

$$\begin{aligned}&\|x_{n+1}-z\|^2 \\ \le\ & \beta_n\|x_n - z\|^2 + (1-\beta_n)\left\|J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big) - z\right\|^2 \\ \le\ & (1-\beta_n)\Big\{(1-\alpha_n)\Big(\|x_n - z\|^2 + \frac{\lambda_n}{(1-\alpha_n)^2}(\lambda_n - 2(1-\alpha_n)\alpha)\|Ax_n - Az\|^2\Big) \\ & +\alpha_n\|z\|^2\Big\} + \beta_n\|x_n - z\|^2 \\ =\ & [1-(1-\beta_n)\alpha_n]\|x_n - z\|^2 + \frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(\lambda_n - 2(1-\alpha_n)\alpha)\|Ax_n - Az\|^2 \\ & +(1-\beta_n)\alpha_n\|z\|^2 \\ \le\ & \|x_n - z\|^2 + \frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(\lambda_n - 2(1-\alpha_n)\alpha)\|Ax_n - Az\|^2 + (1-\beta_n)\alpha_n\|z\|^2.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(2(1-\alpha_n)\alpha - \lambda_n)\|Ax_n - Az\|^2 \\ \le\ & \|x_n - z\|^2 - \|x_{n+1}-z\|^2 + (1-\beta_n)\alpha_n\|z\|^2 \\ \le\ & (\|x_n - z\| - \|x_{n+1}-z\|)\|x_{n+1}-x_n\| + (1-\beta_n)\alpha_n\|z\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\liminf_{n\to\infty}\frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(2(1-\alpha_n)\alpha - \lambda_n) > 0$, we have

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.17}$$

Next, we show $\|x_n - J^B_{\lambda_n}((1-\alpha_n)x_n - \lambda_n Ax_n)\| \to 0$. By using the firm nonexpansivity of $J^B_{\lambda_n}$, we have

$$\left\|J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big) - z\right\|^2$$

$$
\begin{aligned}
&= \left\| J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - J^B_{\lambda_n}\Big(z - \lambda_n A z\Big)\right\|^2 \\
&\leq \Big\langle (1-\alpha_n)x_n - \lambda_n A x_n - (z-\lambda_n A z), J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - z\Big\rangle \\
&= \frac{1}{2}\Big(\|(1-\alpha_n)x_n - \lambda_n A x_n - (z-\lambda_n A z)\|^2 + \left\|J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - z\right\|^2 \\
&\quad -\left\|(1-\alpha_n)x_n - \lambda_n(A x_n - \lambda_n A z) - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big)\right\|^2\Big).
\end{aligned}
$$

From condition (iii) and the α-inverse strongly monotonicity of A, we know that $I - \lambda_n A/(1-\alpha_n)$ is nonexpansive. Hence

$$
\begin{aligned}
&\|(1-\alpha_n)x_n - \lambda_n A x_n - (z-\lambda_n A z)\|^2 \\
&= \|(1-\alpha_n)((x_n - \lambda_n A x_n/(1-\alpha_n)) - (z - \lambda_n A z/(1-\alpha_n))) + \alpha_n(-z)\|^2 \\
&\leq (1-\alpha_n)\|(x_n - \lambda_n A x_n/(1-\alpha_n)) - (z - \lambda_n A z/(1-\alpha_n))\|^2 + \alpha_n\|z\|^2 \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|z\|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\|J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - z\right\|^2 \\
&\leq \frac{1}{2}\Big((1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|z\|^2 + \left\|J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - z\right\|^2 \\
&\quad -\left\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - \lambda_n(A x_n - \lambda_n A z)\right\|^2\Big).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\left\|J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - z\right\|^2 \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|z\|^2 \\
&\quad -\left\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big) - \lambda_n(A x_n - \lambda_n A z)\right\|^2\Big) \\
&= (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|z\|^2 - \left\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big)\right\|^2 \\
&\quad +2\lambda_n\Big\langle (1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big), A x_n - A z\Big\rangle - \lambda_n^2\|A x_n - A z\|^2 \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|z\|^2 - \left\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big)\right\|^2 \\
&\quad +2\lambda_n\left\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big)\right\|\|A x_n - A z\|.
\end{aligned}
$$

This together with (3.16) imply that

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - z\|^2 + (1-\beta_n)\alpha_n\|z\|^2 \\
&\quad -(1-\beta_n)\left\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big)\right\|^2 \\
&\quad +2\lambda_n(1-\beta_n)\left\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n A x_n\Big)\right\|\|A x_n - A z\| \\
&= [1-(1-\beta_n)\alpha_n]\|x_n - z\|^2 + (1-\beta_n)\alpha_n\|z\|^2
\end{aligned}
$$

$$-(1-\beta_n)\Big\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\|^2$$
$$+2\lambda_n(1-\beta_n)\Big\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\|\|Ax_n - Az\|.$$

Hence,

$$\begin{aligned}
&(1-\beta_n)\Big\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\|^2 \\
\leq\ & \|x_n - z\|^2 - \|x_{n+1} - z\|^2 - (1-\beta_n)\alpha_n\|x_n - z\|^2 + (1-\beta_n)\alpha_n\|z\|^2 \\
&+2\lambda_n(1-\beta_n)\Big\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\|\|Ax_n - Az\| \\
\leq\ & (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\| + (1-\beta_n)\alpha_n\|z\|^2 \\
&+2\lambda_n(1-\beta_n)\Big\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\|\|Ax_n - Az\|.
\end{aligned}$$

Since $\limsup_{n\to\infty}\beta_n < 1$, $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and $\|Ax_n - Az\| \to 0$, we deduce

$$\lim_{n\to\infty}\Big\|(1-\alpha_n)x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| = 0.$$

This implies that

$$\lim_{n\to\infty}\Big\|x_n - J^B_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big)\Big\| = 0. \tag{3.18}$$

Put $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$ (i.e, $\tilde{x}$ is the minimum norm element in $F(S)\cap(A+B)^{-1}0$). We will finally show that $x_n \to \tilde{x}$.

Setting $v_n = x_n - \frac{\lambda_n}{1-\alpha_n}(Ax_n - A\tilde{x})$ for all n. Taking $z = \tilde{x}$ in (3.17) to get $\|Ax_n - A\tilde{x}\| \to 0$. First, we prove $\limsup_{n\to\infty}\langle\tilde{x}, v_n - \tilde{x}\rangle \geq 0$. We take a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle\tilde{x}, v_n - \tilde{x}\rangle = \lim_{i\to\infty}\langle\tilde{x}, v_{n_i} - \tilde{x}\rangle.$$

It is clear that $\{v_{n_i}\}$ is bounded due to the boundedness of $\{x_n\}$ and $\|Ax_n - A\tilde{x}\| \to 0$. Then, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to some point $w \in C$. Hence, $\{x_{n_{i_j}}\}$ and $\{y_{n_{i_j}}\}$ also converge weakly to w because of $\|v_{n_{i_j}} - x_{n_{i_j}}\| \to 0$ and $\|x_{n_{i_j}} - y_{n_{i_j}}\| \to 0$. At the same time, from (3.18) and $\|y_{n_{i_j}} - x_{n_{i_j}}\| = \|SJ^B_{\lambda_{n_{i_j}}}((1-\alpha_{n_{i_j}})x_{n_{i_j}} - \lambda_{n_{i_j}}Ax_{n_{i_j}}) - x_{n_{i_j}}\| \to 0$, we have

$$\lim_{j\to\infty}\|x_{n_{i_j}} - Sx_{n_{i_j}}\| = 0. \tag{3.19}$$

By the demi-closedness principle of the nonexpansive mapping (see Lemma 2.2) and (3.19), we deduce $w \in F(S)$. Furthermore, by the similar argument as that of Theorem 3.1, we can show that w is also in $(A+B)^{-1}0$. Hence, we have $w \in F(S)\cap(A+B)^{-1}0$. This implies that

$$\limsup_{n\to\infty}\langle\tilde{x}, v_n - \tilde{x}\rangle = \lim_{j\to\infty}\langle\tilde{x}, v_{n_{i_j}} - \tilde{x}\rangle = \langle\tilde{x}, w - \tilde{x}\rangle.$$

Note that $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$. Then, $\langle\tilde{x}, w - \tilde{x}\rangle \geq 0, w \in F(S)\cap(A+B)^{-1}0$. Therefore,

$$\limsup_{n\to\infty}\langle\tilde{x}, v_n - \tilde{x}\rangle \geq 0.$$

From (3.12), we have

$$\begin{aligned}
& \|x_{n+1}-\tilde{x}\|^2 \\
\leq\ & \beta_n\|x_n-\tilde{x}\|^2+(1-\beta_n)\|SJ^B_{\lambda_n}u_n-\tilde{x}\|^2 \\
\leq\ & \beta_n\|x_n-\tilde{x}\|^2+(1-\beta_n)\|J^B_{\lambda_n}u_n-\tilde{x}\|^2 \\
=\ & \beta_n\|x_n-\tilde{x}\|^2+(1-\beta_n)\|J^B_{\lambda_n}u_n-J^B_{\lambda_n}(\tilde{x}-\lambda_nA\tilde{x})\|^2 \\
\leq\ & \beta_n\|x_n-\tilde{x}\|^2+(1-\beta_n)\|u_n-(\tilde{x}-\lambda_nA\tilde{x})\|^2 \\
=\ & \beta_n\|x_n-\tilde{x}\|^2+(1-\beta_n)\|(1-\alpha_n)x_n-\lambda_nAx_n-(\tilde{x}-\lambda_nA\tilde{x})\|^2 \\
=\ & (1-\beta_n)\Big\|(1-\alpha_n)\Big((x_n-\frac{\lambda_n}{1-\alpha_n}Ax_n)-(\tilde{x}-\frac{\lambda_n}{1-\alpha_n}A\tilde{x})\Big)-\alpha_n\tilde{x}\Big\|^2 \\
& +\beta_n\|x_n-\tilde{x}\|^2 \\
=\ & (1-\beta_n)\Big((1-\alpha_n)^2\Big\|(x_n-\frac{\lambda_n}{1-\alpha_n}Ax_n)-(\tilde{x}-\frac{\lambda_n}{1-\alpha_n}A\tilde{x})\Big\|^2 \\
& -2\alpha_n(1-\alpha_n)\Big\langle\tilde{x},(x_n-\frac{\lambda_n}{1-\alpha_n}Ax_n)-(\tilde{x}-\frac{\lambda_n}{1-\alpha_n}A\tilde{x})\Big\rangle+\alpha_n^2\|\tilde{x}\|^2\Big) \\
& +\beta_n\|x_n-\tilde{x}\|^2 \\
\leq\ & \beta_n\|x_n-\tilde{x}\|^2+(1-\beta_n)\Big((1-\alpha_n)^2\|x_n-\tilde{x}\|^2 \\
& -2\alpha_n(1-\alpha_n)\Big\langle\tilde{x},x_n-\frac{\lambda_n}{1-\alpha_n}(Ax_n-A\tilde{x})-\tilde{x}\Big\rangle+\alpha_n^2\|\tilde{x}\|^2\Big) \\
\leq\ & [1-(1-\beta_n)\alpha_n]\|x_n-\tilde{x}\|^2 \\
& +(1-\beta_n)\alpha_n\Big\{-2(1-\alpha_n)\langle\tilde{x},v_n-\tilde{x}\rangle+\alpha_n\|\tilde{x}\|^2\Big\}.
\end{aligned}$$

It is clear that $\sum_n(1-\beta_n)\alpha_n=\infty$ and $\limsup_{n\to\infty}(-2(1-\alpha_n)\langle\tilde{x},v_n-\tilde{x}\rangle+\alpha_n\|\tilde{x}\|^2)\leq 0$. We can therefore apply Lemma 2.4 to conclude that $x_n\to\tilde{x}$. This completes the proof. □

Remark 3.3. From the listed references, there exist a large number of problems which need to find the minimum norm solution. A useful path to circumvent this problem is to use projection. Bauschke and Browein [2] and Censor and Zenios [7] provide reviews of the field. The main difficult is in computation. The present paper provides some methods which do not use projection for finding the minimum norm solution problem. On the other hand, our suggested algorithms (3.1) and (3.12) are very simple in compared with the algorithm introduced in [28].

Corollary 3.4. *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J^B_\lambda=(I+\lambda B)^{-1}$ be the resolvent of B for $\lambda>0$ such that $(A+B)^{-1}0\neq\emptyset$. Let λ be a constant satisfying $a\leq\lambda\leq b$ where $[a,b]\subset(0,2\alpha)$. For $t\in(0,1-\frac{\lambda}{2\alpha})$, let $\{x_t\}\subset C$ be a net generated by*

$$x_t=J^B_\lambda\Big((1-t)x_t-\lambda Ax_t\Big).$$

Then the net $\{x_t\}$ *converges strongly, as* $t \to 0+$, *to a point* $\tilde{x} = P_{(A+B)^{-1}0}(0)$ *which is the minimum norm element in* $(A+B)^{-1}0$.

Corollary 3.5. *Let* C *be a closed and convex subset of a real Hilbert space* H. *Let* A *be an* α*-inverse strongly-monotone mapping of* C *into* H *and let* B *be a maximal monotone operator on* H, *such that the domain of* B *is included in* C. *Let* $J_\lambda^B = (I + \lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$ *such that* $(A+B)^{-1}0 \neq \emptyset$. *For given* $x_0 \in C$, *let* $\{x_n\} \subset C$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J_{\lambda_n}^B \Big((1-\alpha_n)x_n - \lambda_n A x_n \Big)$$

for all $n \geq 0$, *where* $\{\lambda_n\} \subset (0, 2\alpha)$,$\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(*iii*) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$ *where* $[a,b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ *converges strongly to a point* $\tilde{x} = P_{(A+B)^{-1}0}(0)$ *which is the minimum norm element in* $(A+B)^{-1}0$.

4. Applications

Next, we consider the problem for finding the minimum norm solution of a mathematical model related to equilibrium problems. Let C be a nonempty, closed and convex subset of a Hilbert space and let $G : C \times C \to R$ be a bifunction satisfying the following conditions:

(E1) $G(x,x) = 0$ for all $x \in C$;

(E2) G is monotone, i.e., $G(x,y) + G(y,x) \leq 0$ for all $x, y \in C$;

(E3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} G(tz + (1-t)x, y) \leq G(x,y)$;

(E4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

Then, the mathematical model related to equilibrium problems (with respect to C) is to find $\tilde{x} \in C$ such that

$$G(\tilde{x}, y) \geq 0 \tag{4.1}$$

for all $y \in C$. The set of such solutions $\tilde{x}$ is denoted by $EP(G)$. The following lemma appears implicitly in Blum and Oettli [4]:

Lemma 4.1. *Let* C *be a nonempty, closed and convex subset of* H *and let* G *be a bifunction of* $C \times C$ *into* R *satisfying (E1)-(E4). Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

The following lemma was given in Combettes and Hirstoaga [10]:

Lemma 4.2. *Assume that* $G : C \times C \to R$ *satisfies (E1)-(E4). For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \{z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. *Then, the following hold:*

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive mapping, i.e., for all* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(G)$;

(4) $EP(G)$ *is closed and convex.*

We call such T_r the resolvent of G for $r > 0$. Using Lemmas 4.1 and 4.2, we have the following lemma. See [1] for a more general result.

Lemma 4.3. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $G : C \times C \to R$ *satisfy (E1)-(E4). Let* A_G *be a multivalued mapping of* H *into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, x \in C, \\ \emptyset, x \notin C. \end{cases}$$

Then, $EP(G) = A_G^{-1}(0)$ *and* A_G *is a maximal monotone operator with* $dom(A_G) \subset C$. *Further, for any* $x \in H$ *and* $r > 0$, *the resolvent* T_r *of* G *coincides with the resolvent of* A_G; *i.e.,*

$$T_r x = (I + rA_G)^{-1} x.$$

Form Lemma 4.3, Theorems 3.1 and 3.2, we have the following results.

Theorem 4.4. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* G *be a bifunction from* $C \times C \to R$ *satisfying (E1)-(E4) and let* T_r *be the resolvent of* G *for* $r > 0$. *Let* S *be a nonexpansive mapping from* C *into itself, such that* $F(S) \cap EP(G) \neq \emptyset$. *For* $t \in (0, 1)$, *let* $\{x_t\} \subset C$ *be a net generated by*

$$x_t = ST_r\Big((1 - t)x_t\Big), t \in (0, 1).$$

Then the net $\{x_t\}$ *converges strongly, as* $t \to 0+$, *to a point* $\tilde{x} = P_{F(S)\cap EP(G)}(0)$ *which is the minimum norm element in* $F(S) \cap EP(G)$.

Corollary 4.5. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* G *be a bifunction from* $C \times C \to R$ *satisfying (E1)-(E4) and let* T_r *be the resolvent of* G *for* $r > 0$. *Suppose* $EP(G) \neq \emptyset$. *For* $t \in (0, 1)$, *let* $\{x_t\} \subset C$ *be a net generated by*

$$x_t = T_r\Big((1 - t)x_t\Big), t \in (0, 1).$$

Then the net $\{x_t\}$ *converges strongly, as* $t \to 0+$, *to a point* $\tilde{x} = P_{EP(G)}(0)$ *which is the minimum norm element in* $EP(G)$.

Theorem 4.6. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* G *be a bifunction from* $C \times C \to R$ *satisfying (E1)-(E4) and let* T_λ *be the resolvent of* G *for* $\lambda > 0$. *Let* S *be a nonexpansive mapping from* C *into itself, such that* $F(S) \cap EP(G) \neq \emptyset$. *For given* $x_0 \in C$, *let* $\{x_n\} \subset C$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)ST_{\lambda_n}\Big((1 - \alpha_n)x_n\Big)$$

for all $n \geq 0$, *where* $\{\lambda_n\} \subset (0, \infty)$,$\{\alpha_n\} \subset (0, 1)$ *and* $\{\beta_n\} \subset (0, 1)$ *satisfy*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(*iii*) $a \le \lambda_n \le b$ *where* $[a,b] \subset (0,\infty)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ *converges strongly to a point* $\tilde{x} = P_{F(S)\cap EP(G)}(0)$ *which is the minimum norm element in* $F(S) \cap EP(G)$.

Corollary 4.7. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* G *be a bifunction from* $C \times C \to R$ *satisfying (E1)-(E4) and let* T_λ *be the resolvent of* G *for* $\lambda > 0$. *Suppose* $EP(G) \neq \emptyset$. *For given* $x_0 \in C$, *let* $\{x_n\} \subset C$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_{\lambda_n}\Big((1-\alpha_n)x_n\Big)$$

for all $n \ge 0$, *where* $\{\lambda_n\} \subset (0,\infty)$,$\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;
(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(*iii*) $a \le \lambda_n \le b$ *where* $[a,b] \subset (0,\infty)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ *converges strongly to a point* $\tilde{x} = P_{EP(G)}(0)$ *which is the minimum norm element in* $EP(G)$.

Remark 4.8. Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. Then, the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \le f(y), y \in H\}$$

for all $x \in H$. We know that ∂f is maximal monotone (see [24]). Let C be a closed and convex subset of H and let i_C be the indicator function of C. Then the subdifferential ∂i_C of i_C is a maximal monotone operator because of i_C is a proper lower semicontinuous convex function on H. So, we can define the resolvent $J_\lambda x$ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1}x, \forall x \in H.$$

It follows that

$$x \in (I + \lambda \partial i_C)J_\lambda x,$$

which is equivalent to

$$x \in (I + \lambda N_C)J_\lambda x, \tag{4.2}$$

where $N_C x$ is the normal cone to C at x, i.e.,

$$N_C x = \{z \in H : \langle z, u - x\rangle \le 0, \forall u \in C\}. \tag{4.3}$$

From (4.2), we have

$$\frac{x - J_\lambda x}{\lambda} \in N_C J_\lambda x.$$

This together with (4.3) imply that

$$\frac{1}{\lambda}\langle x - J_\lambda x, u - J_\lambda x\rangle \le 0, \forall u \in C.$$

Thus, $J_\lambda x = P_C x$. Moreover, we know that if C is a closed half space (i.e., $C = \{z \in H : \langle v, z\rangle \leq \rho\}$), then the metric projection P_C can be expressed by

$$P_C x = \begin{cases} x - \frac{\max\{0, \langle v,x\rangle - \rho\}}{\|v\|^2} v, (v \neq 0), \\ x, (v = 0), \end{cases}$$

for all $x \in H$. In this case, we can compute $J_\lambda x$ by

$$J_\lambda x = \begin{cases} x - \frac{\max\{0, \langle v,x\rangle - \rho\}}{\|v\|^2} v, (v \neq 0), \\ x, (v = 0). \end{cases}$$

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