Journal of Nonlinear and Convex Analysis
Volume 16, Number 5, 2015, 855–871
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

COMMON FIXED POINT RESULTS IN COMPLEX VALUED METRIC SPACE WITH APPLICATIONS TO SYSTEM OF INTEGRAL EQUATIONS

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Abstract. In this paper, we prove several common fixed point results by utilizing new control functions in the contractive inequalities. An example is also given to illustrate our main result. Moreover, we apply our main result to find unique common solution of system of integral equations.

1. Introduction and Preliminaries

Fixed point theory became one of the most interesting area of research in the last fifty years for its applications in optimization and control theory, differential and integral equations, economics etc. The fixed point theorem, generally known as the Banach contraction principle, appeared in explicit form in Banach's thesis in 1922 [12]. Since its simplicity and usefulness, it became a very popular tool in solving many problems in mathematical analysis. Several authors proved fixed point results in different metric spaces (see [1–9, 11, 14–20, 22–25]).

On the other hand, the study of metric spaces has expressed the most important role to many fields both in pure and applied sciences such as biology, medicine, physics, and computer science (see [28]). Azam et al. [10] introduced the concept of complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type condition. Subsequently, in [13], Bhatt et al. presented some common fixed point results of mappings satisfying rational inequality in the context of complex valued metric space. In the same way Rouzkard and Imdad [27] established some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces which generalize, unify and complement the results of Azam et al. [10]. Recently, Sintunavarat and Kumam [29] obtained common fixed point results by replacing constants of contractive condition to control functions. In this paper we generalize and improve all of the above mentioned results. As an application we will prove the existence of integrable solutions for an implicit system of integral equations. For the remainder of this section we gather some notations and preliminary facts. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2).$$

2010 *Mathematics Subject Classification.* 46S40, 47H10, 54H25.

Key words and phrases. Complex valued metric space, coincidence point, common fixed point, weakly compatible mappings, integral equations.

The second author gratefully acknowledges the support from the Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU) during this research.

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

$$\begin{aligned}
\text{(i) } \mathrm{Re}\,(z_1) &= \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2)\\
\text{(ii) } \mathrm{Re}\,(z_1) &< \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\\
\text{(iii) } \mathrm{Re}\,(z_1) &< \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2)\\
\text{(iv) } \mathrm{Re}\,(z_1) &= \mathrm{Re}\,(z_2)\,,\ \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\,.
\end{aligned}$$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$\begin{aligned}
0 &\precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|\,,\\
z_1 &\preceq z_2, z_2 \prec z_3 \implies z_1 \prec z_3.
\end{aligned}$$

Definition 1.1. Let X be a nonempty set. Suppose that the self-mapping $d : X \times X \to \mathbb{C}$ satisfies:

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$
(3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \smallsetminus \{x\}) \neq \phi$$

A is called open whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.

Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \longrightarrow x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space. Let X be a non empty set and $T, f : X \to X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i. e. $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. We require the following Lemmas:

Lemma 1.2 ([10])**.** *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.3 ([10]). *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$.*

Lemma 1.4 ([16]). *Let X be a non empty set and $f : X \to X$ be a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one.*

Lemma 1.5 ([22]). *Let X be a non empty set and the mappings $S,T,f : X \to X$ have a unique point of coincidence v in X. If (S,f) and (T,f) are weakly compatible, then S,T,f have a unique common fixed point.*

2. Main results

Now we state and prove our first main result.

Theorem 2.1. *Let (X,d) be a complete complex valued metric space and $S,T : X \to X$ be a self-mappings such that*

$$d(Sx,Ty) \preceq \begin{cases} \Lambda(x)d(x,y) + \Xi(x)\frac{d(x,Sx)d(y,Ty)+d(y,Sx)d(x,Ty)}{d(x,Sx)+d(y,Ty)} \\ +\Theta(x)\frac{d(x,Sx)d(x,Ty)+d(y,Sx)d(y,Ty)}{d(y,Sx)+d(x,Ty)}, \text{ if } A_1 \neq 0, A_2 \neq 0 \\ 0, \text{ if } A_1 = 0 \text{ or } A_2 = 0. \end{cases} \tag{2.1}$$

for all $x,y \in X$, where $A_1 = d(x,Sx) + d(y,Ty)$ and $A_2 = d(y,Sx) + d(x,Ty)$ and

$$\Lambda, \Xi, \Theta : X \to [0,1),$$

satisfying the following conditions,

(i) $\Lambda(Sx) \le \Lambda(x)$, $\Xi(Sx) \le \Xi(x)$ and $\Theta(Sx) \le \Theta(x)$;
(ii) $\Lambda(Tx) \le \Lambda(x)$, $\Xi(Tx) \le \Xi(x)$ and $\Theta(Tx) \le \Theta(x)$;
(iii) $(\Lambda + \Xi + \Theta)(x) < 1$.
Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_k\}$ as follows

$$x_{2k+1} = Sx_{2k} \text{ and } x_{2k+2} = Tx_{2k+1} \text{ for all } k \ge 0.$$

Now we have two cases.

Case 1: If $d(x_{2k},Sx_{2k})+d(x_{2k+1},Tx_{2k+1}) \neq 0$ and $d(x_{2k+1},Sx_{2k})+d(x_{2k},Tx_{2k+1}) \neq 0$ for $k \ge 0$, then

$$\begin{aligned} & d(x_{2k+1},x_{2k+2}) \\ = \; & d(Sx_{2k},Tx_{2k+1}) \\ \preceq \; & \Lambda(x_{2k})d(x_{2k},x_{2k+1}) \\ & +\Xi(x_{2k})\frac{d(x_{2k},Sx_{2k})d(x_{2k+1},Tx_{2k+1}) + d(x_{2k+1},Sx_{2k})d(x_{2k},Tx_{2k+1})}{d(x_{2k},Sx_{2k}) + d(x_{2k+1},Tx_{2k+1})} \\ & +\Theta(x_{2k})\frac{d(x_{2k},Sx_{2k})d(x_{2k},Tx_{2k+1}) + d(x_{2k+1},Sx_{2k})d(x_{2k+1},Tx_{2k+1})}{d(x_{2k+1},Sx_{2k}) + d(x_{2k},Tx_{2k+1})} \\ \preceq \; & \Lambda(x_{2k})d(x_{2k},x_{2k+1}) \end{aligned}$$

$$
\begin{aligned}
&+\Xi(x_{2k})\frac{d(x_{2k},x_{2k+1})d(x_{2k+1},x_{2k+2})+d(x_{2k+1},x_{2k+1})d(x_{2k},x_{2k+2})}{d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})}\\
&+\Theta(x_{2k})\frac{d(x_{2k},x_{2k+1})d(x_{2k},x_{2k+2})+d(x_{2k+1},x_{2k+1})d(x_{2k+1},x_{2k+2})}{d(x_{2k+1},x_{2k+1})+d(x_{2k},x_{2k+2})}\\
\preceq\ &\Lambda(x_{2k})d(x_{2k},x_{2k+1})\\
&+\Xi(x_{2k})\frac{d(x_{2k},x_{2k+1})d(x_{2k+1},x_{2k+2})}{d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})}\\
&+\Theta(x_{2k})\frac{d(x_{2k},x_{2k+1})d(x_{2k},x_{2k+2})}{d(x_{2k},x_{2k+2})}.
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|d(x_{2k+1},x_{2k+2})| \ \leq\ & |\Lambda(x_{2k})||d(x_{2k},x_{2k+1})|\\
&+|\Xi(x_{2k})|\frac{|d(x_{2k},x_{2k+1})|.|d(x_{2k+1},x_{2k+2})|}{|d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})|}\\
&+|\Theta(x_{2k})||d(x_{2k},x_{2k+1})|.
\end{aligned}
$$

Since $|d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})|>|d(x_{2k},x_{2k+1})|$, therefore

$$
\begin{aligned}
|d(x_{2k+1},x_{2k+2})| \ \leq\ & |\Lambda(x_{2k})||d(x_{2k},x_{2k+1})|+|\Xi(x_{2k})||d(x_{2k+1},x_{2k+2})|\\
&+|\Theta(x_{2k})||d(x_{2k},x_{2k+1})|\\
=\ & |\Lambda(Tx_{2k-1})||d(x_{2k},x_{2k+1})|+|\Xi(Tx_{2k-1})||d(x_{2k+1},x_{2k+2})|\\
&+|\Theta(Tx_{2k-1})||d(x_{2k},x_{2k+1})|\\
\leq\ & |\Lambda(x_{2k-1})||d(x_{2k},x_{2k+1})|+|\Xi(x_{2k-1})||d(x_{2k+1},x_{2k+2})|\\
&+|\Theta(x_{2k-1})||d(x_{2k},x_{2k+1})|\\
=\ & |\Lambda(Sx_{2k-2})||d(x_{2k},x_{2k+1})|+|\Xi(Sx_{2k-2})||d(x_{2k+1},x_{2k+2})|\\
&+|\Theta(Sx_{2k-2})||d(x_{2k},x_{2k+1})|\\
\leq\ & |\Lambda(x_{2k-2})||d(x_{2k},x_{2k+1})|+|\Xi(x_{2k-2})||d(x_{2k+1},x_{2k+2})|\\
&+|\Theta(x_{2k-2})||d(x_{2k},x_{2k+1})|\\
&\vdots\\
\leq\ & |\Lambda(x_0)||d(x_{2k},x_{2k+1})|+|\Xi(x_0)||d(x_{2k+1},x_{2k+2})|\\
&+|\Theta(x_0)||d(x_{2k},x_{2k+1})|
\end{aligned}
$$

so that

$$
|d(x_{2k+1},x_{2k+2})|\leq\left|\frac{\Lambda(x_0)+\Theta(x_0)}{1-\Xi(x_0)}||d(x_{2k},x_{2k+1})\right|. \tag{2.2}
$$

Now similarly we get

$$
\begin{aligned}
&d(x_{2k+2},x_{2k+3})\\
=&d(x_{2k+3},x_{2k+2})=d(Sx_{2k+2},Tx_{2k+1})\\
\preceq&\Lambda(x_{2k+2})d(x_{2k+2},x_{2k+1})\\
&+\Xi(x_{2k+2})\frac{d(x_{2k+2},Sx_{2k+2})d(x_{2k+1},Tx_{2k+1})+d(x_{2k+1},Sx_{2k+2})d(x_{2k+2},Tx_{2k+1})}{d(x_{2k+2},Sx_{2k+2})+d(x_{2k+1},Tx_{2k+1})}
\end{aligned}
$$

$$
\begin{aligned}
&+\Theta(x_{2k+2})\frac{d(x_{2k+2},Sx_{2k+2})d(x_{2k+2},Tx_{2k+1})+d(x_{2k+1},Sx_{2k+2})d(x_{2k+1},Tx_{2k+1})}{d(x_{2k+1},Sx_{2k+2})+d(x_{2k+2},Tx_{2k+1})}\\
\preceq&\Lambda(x_{2k+2})d(x_{2k+2},x_{2k+1})\\
&+\Xi(x_{2k+2})\frac{d(x_{2k+2},x_{2k+3})d(x_{2k+1},x_{2k+2})+d(x_{2k+1},x_{2k+3})d(x_{2k+2},x_{2k+2})}{d(x_{2k+2},x_{2k+3})+d(x_{2k+1},x_{2k+2})}\\
&+\Theta(x_{2k+2})\frac{d(x_{2k+2},x_{2k+3})d(x_{2k+2},x_{2k+2})+d(x_{2k+1},x_{2k+3})d(x_{2k+1},x_{2k+2})}{d(x_{2k+1},x_{2k+3})+d(x_{2k+2},x_{2k+2})}\\
\preceq&\Lambda(x_{2k+2})d(x_{2k+2},x_{2k+1})+\Xi(x_{2k+2})\frac{d(x_{2k+2},x_{2k+3})d(x_{2k+1},x_{2k+2})}{d(x_{2k+2},x_{2k+3})+d(x_{2k+1},x_{2k+2})}\\
&+\Theta(x_{2k+2})\frac{d(x_{2k+1},x_{2k+3})d(x_{2k+1},x_{2k+2})}{d(x_{2k+1},x_{2k+3})},
\end{aligned}
$$

so that

$$
\begin{aligned}
|d(x_{2k+2},x_{2k+3})| \leq\ & |\Lambda(x_{2k+2})||d(x_{2k+2},x_{2k+1})|\\
&+|\Xi(x_{2k+2})|\frac{|d(x_{2k+2},x_{2k+3})|.|d(x_{2k+1},x_{2k+2})|}{|d(x_{2k+2},x_{2k+3})+d(x_{2k+1},x_{2k+2})|}\\
&+|\Theta(x_{2k+2})||d(x_{2k+1},x_{2k+2})|.
\end{aligned}
$$

Since $|d(x_{2k+2},x_{2k+3})+d(x_{2k+1},x_{2k+2})|>|d(x_{2k+1},x_{2k+2})|$, so we have

$$
\begin{aligned}
|d(x_{2k+2},x_{2k+3})| \leq\ & |\Lambda(x_{2k+2})||d(x_{2k+1},x_{2k+2})|+|\Xi(x_{2k+2})||d(x_{2k+2},x_{2k+3})|\\
&+|\Theta(x_{2k+2})||d(x_{2k+1},x_{2k+2})|\\
=\ & |\Lambda(Tx_{2k+1})||d(x_{2k+1},x_{2k+2})|+|\Xi(Tx_{2k+1})||d(x_{2k+2},x_{2k+3})|\\
&+|\Theta(Tx_{2k+1})||d(x_{2k+1},x_{2k+2})|\\
\leq\ & |\Lambda(x_{2k+1})||d(x_{2k+1},x_{2k+2})|+|\Xi(x_{2k+1})||d(x_{2k+2},x_{2k+3})|\\
&+|\Theta(x_{2k+1})||d(x_{2k+1},x_{2k+2})|\\
=\ & |\Lambda(Sx_{2k})||d(x_{2k+1},x_{2k+2})|+|\Xi(Sx_{2k})||d(x_{2k+2},x_{2k+3})|\\
&+|\Theta(Sx_{2k})||d(x_{2k+1},x_{2k+2})|\\
\leq\ & |\Lambda(x_{2k})||d(x_{2k+1},x_{2k+2})|+|\Xi(x_{2k})||d(x_{2k+2},x_{2k+3})|\\
&+|\Theta(x_{2k})||d(x_{2k+1},x_{2k+2})|\\
&\vdots\\
\leq\ & |\Lambda(x_0)||d(x_{2k+1},x_{2k+2})|+|\Xi(x_0)||d(x_{2k+2},x_{2k+3})|\\
&+|\Theta(x_0)||d(x_{2k+1},x_{2k+2})|,
\end{aligned}
$$

which implies that

$$
|d(x_{2k+2},x_{2k+3})| \leq \Big|\frac{\Lambda(x_0)+\Theta(x_0)}{1-\Xi(x_0)}||d(x_{2k+1},x_{2k+2})\Big|. \tag{2.3}
$$

Since $|(\Lambda+\Xi+\Theta)(x)|<1$, so we set $\lambda=|\frac{\Lambda(x_0)+\Theta(x_0)}{1-\Xi(x_0)}|<1$, it follows by (2.2) and (2.3) that

$$
\begin{aligned}
|d(x_n,x_{n+1})| \leq\ & \lambda|d(x_{n-1},x_n)|\\
\leq\ & \lambda^2|d(x_{n-2},x_{n-1})|
\end{aligned}
$$

$$\begin{aligned} &\vdots \\ \leq\ & \lambda^n|d(x_0,x_1)| \end{aligned}$$

for all $n \in \mathbb{N}$. Now, for any positive integer m and n with $m > n$, we have

$$\begin{aligned} |d(x_n,x_m)| &\leq |d(x_n,x_{n+1})| + |d(x_{n+1},x_{n+2})| + \cdots + |d(x_{m-1},x_m)| \\ &\leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}]|d(x_0,x_1)| \\ &\leq \left[\frac{\lambda^n}{1-\lambda}\right]|d(x_0,x_1)|, \end{aligned}$$

and so

$$|d(x_n,x_m)| \leq \frac{\lambda^n}{1-\lambda}|d(x_0,x_1)| \longrightarrow 0 \quad \text{as} \quad m,n \longrightarrow \infty.$$

Thus by Lemma 1.3, we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Next we claim that $z = Sz$. We suppose on the contrary that $z \neq Sz$ and $d(z,Sz) = u \neq 0$. Then by triangular inequality and given condition, we get

$$\begin{aligned} u &= d(z,Sz) \precsim d(z,Tx_{2k+1}) + d(Tx_{2k+1},Sz) = d(z,Tx_{2k+1}) + d(Sz,Tx_{2k+1}) \\ &\precsim d(z,Tx_{2k+1}) + \Lambda(z)d(z,x_{2k+1}) \\ &\quad +\Xi(z)\frac{d(z,Sz)d(x_{2k+1},Tx_{2k+1}) + d(x_{2k+1},Sz)d(z,Tx_{2k+1})}{d(z,Sz)+d(x_{2k+1},Tx_{2k+1})} \\ &\quad +\Theta(z)\frac{d(z,Sz)d(z,Tx_{2k+1}) + d(x_{2k+1},Sz)d(x_{2k+1},Tx_{2k+1})}{d(x_{2k+1},Sz)+d(z,Tx_{2k+1})} \\ &\precsim d(z,x_{2k+2}) + \Lambda(z)d(z,x_{2k+1}) \\ &\quad +\Xi(z)\frac{d(z,Sz)d(x_{2k+1},x_{2k+2}) + d(x_{2k+1},Sz)d(z,x_{2k+2})}{d(z,Sz)+d(x_{2k+1},x_{2k+2})} \\ &\quad +\Theta(z)\frac{d(z,Sz)d(z,x_{2k+2}) + d(x_{2k+1},Sz)d(x_{2k+1},x_{2k+2})}{d(x_{2k+1},Sz)+d(z,x_{2k+2})} \end{aligned}$$

which implies that

$$\begin{aligned} |u| &= |d(z,Sz)| \leq |d(z,x_{2k+2})| + |\Lambda(z)||d(z,x_{2k+1})| \\ &\quad +|\Xi(z)|\frac{|u||d(x_{2k+1},x_{2k+2})| + |d(x_{2k+1},Sz)||d(z,x_{2k+2})|}{|d(z,Sz)+d(x_{2k+1},x_{2k+2})|} \\ &\quad +|\Theta(z)|\frac{|u||d(z,x_{2k+2})| + |d(x_{2k+1},Sz)||d(x_{2k+1},x_{2k+2})|}{|d(x_{2k+1},Sz)+d(z,x_{2k+2})|}. \end{aligned}$$

Taking limit as $k \to \infty$, we get $|u| = 0$, which is a contradiction and hence $z = Sz$. Similarly it follows that $z = Tz$. Therefore z is the common fixed point S and T. Finally, we show that z is a unique common fixed point of S and T. Assume that there exists another common fixed point z^* that is $z^* = Sz^* = Tz^*$. Then

$$\begin{aligned} d(z,z^*) &= d(Sz,Tz^*) \\ &\precsim \Lambda(z)d(z,z^*) + \Xi(z)\frac{d(z,Sz)d(z^*,Tz^*) + d(z^*,Sz)d(z,Tz^*)}{d(z,Sz)+d(z^*,Tz^*)} \\ &\quad +\Theta(z)\frac{d(z,Sz)d(z,Tz^*) + d(z^*,Sz)d(z^*,Tz^*)}{d(z^*,Sz)+d(z,Tz^*)}, \end{aligned}$$

which implies that $d(z,z^*)=0$, so $z=z^*$. Thus S and T have a unique common fixed point.

Case 2: If $d(x_{2k},Sx_{2k})+d(x_{2k+1},Tx_{2k+1})=0$ or $d(x_{2k+1},Sx_{2k})+d(x_{2k},Tx_{2k+1})=0$ (for any $k\geq 0$), then

$$(d(x_{2k},Sx_{2k})+d(x_{2k+1},Tx_{2k+1}))\times(d(x_{2k+1},Sx_{2k})+d(x_{2k},Tx_{2k+1}))=0,$$

implies that $d(Sx_{2k},Tx_{2k+1})=0$. Now, if

$$d(x_{2k},Sx_{2k})+d(x_{2k+1},Tx_{2k+1})=0,$$

then $x_{2k}=Sx_{2k}=x_{2k+1}=Tx_{2k+1}=x_{2k+2}$. Since we have $x_{2k+1}=Sx_{2k}=x_{2k}$, so there exist k_1 and l_1 such that $l_1=Sk_1=k_1$. Also, since $x_{2k+2}=Tx_{2k+1}=x_{2k+1}$, so there exist k_2 and l_2 such that $l_2=Sk_2=k_2$. As

$$d(x_{2k},Sx_{2k})+d(x_{2k+1},Tx_{2k+1})=0,$$

so we have

$$d(k_1,Sk_1)+d(k_2,Tk_2)=0,$$

which implies that

$$d(Sk_1,Tk_2)=0,$$

so that, $l_1=Sk_1=Tk_2=l_2$ which in turn yields that $l_1=Sk_1=Sl_1$. Similarly, one can also have $l_2=Tl_2$. As $l_1=l_2$, implies that $Sl_1=Tl_1=l_1$, so $l_1=l_2$, is a common fixed point of S and T. We now prove that S and T have a unique common fixed point. For this, let l_1^* be another common fixed point of S and T. Then we have $l_1^*=Sl_1^*=Tl_1^*$. Since

$$A_1=d(l_1,Sl_1)+d(l_1^*,Tl_1^*)=0,$$

implies that $d(l_1,l_1^*)=d(Sl_1,Tl_1^*)=0$. This implies that $l_1=l_1^*$. This completes the proof. □

Remark 2.2. By setting $\Lambda(x)=\lambda$, $\Xi(x)=\mu$ and $\Theta(x)=\gamma$ in Theorem 2.1, we get Theorem 2.11 of [27].

Remark 2.3. By setting $S=T$, $\Lambda(x)=\lambda$, $\Xi(x)=\mu$ and $\Theta(x)=\gamma$ in Theorem 2.1, we get Corollary 2.12 of [27].

Remark 2.4. By setting $\Lambda(x)=\Xi(x)=0$ and $\Theta(x)=a$ in Theorem 2.1, we get Theorem 2.1 of [13].

Corollary 2.5. *Let (X,d) be a complete complex valued metric space and $T:X\to X$ be a self-mapping such that*

$$d(Tx,Ty)\preceq\begin{cases}\Lambda(x)d(x,y)+\Xi(x)\frac{d(x,Tx)d(y,Ty)+d(y,Tx)d(x,Ty)}{d(x,Tx)+d(y,Ty)}\\ +\Theta(x)\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}, \text{ if } A_1\neq 0, A_2\neq 0\\ 0, \text{ if } A_1=0 \text{ or } A_2=0\end{cases}$$

for all $x,y\in X$, where $A_1=d(x,Tx)+d(y,Ty)$ and $A_2=d(y,Tx)+d(x,Ty)$ and

$$\Lambda,\Xi,\Theta:X\to[0,1),$$

satisfying the following conditions,

(i) $\Lambda(Tx)\leq\Lambda(x)$, $\Xi(Tx)\leq\Xi(x)$ *and* $\Theta(Tx)\leq\Theta(x)$;

(ii) $(\Lambda + \Xi + \Theta)(x) < 1$.
Then T has a unique fixed point.

Corollary 2.6. *Let (X, d) be a complete complex valued metric space and $T : X \to X$ be a self-mapping such that*

$$d(T^n x, T^n y) \preceq \begin{cases} \Lambda(x)d(x,y) + \Xi(x)\frac{d(x,T^n x)d(y,T^n y)+d(y,T^n x)d(x,T^n y)}{d(x,T^n x)+d(y,T^n y)} \\ +\Theta(x)\frac{d(x,T^n x)d(x,T^n y)+d(y,T^n x)d(y,T^n y)}{d(y,T^n x)+d(x,T^n y)}, \text{ if } A_1 \neq 0, A_2 \neq 0 \\ 0, \text{ if } A_1 = 0 \text{ or } A_2 = 0 \end{cases}$$

for all $x, y \in X$, and for some $n \in \mathbb{N}$, where $A_1 = d(x, T^n x) + d(y, T^n y)$ and $A_2 = d(y, T^n x) + d(x, T^n y)$ and

$$\Lambda, \Xi, \Theta : X \to [0, 1),$$

satisfying the following conditions,
(i) $\Lambda(T^n x) \leq \Lambda(x), \Xi(T^n x) \leq \Xi(x)$ *and* $\Theta(T^n x) \leq \Theta(x)$;
(ii) $(\Lambda + \Xi + \Theta)(x) < 1$.
Then T has a unique fixed point.

Proof. From Corollary 2.5, we get T^n has a unique fixed point z. It follows from

$$T^n(Tz) = T(T^n z) = Tz,$$

that Tz is a fixed point of T^n. Therefore $Tz = z$ by the uniqueness of a fixed point of T^n and then z is also a fixed point of T. Since the fixed point of T is also fixed point of T^n, so the fixed point of T is unique. □

Example 2.7. Let $X = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$ be a set. Define a mapping $d : X \times X \to \mathbb{C}$ as follows

$$\begin{aligned}
d\Big(\frac{1}{2}, \frac{1}{2}\Big) &= d\Big(\frac{1}{3}, \frac{1}{3}\Big) = d\Big(\frac{1}{4}, \frac{1}{4}\Big) = d\Big(\frac{1}{5}, \frac{1}{5}\Big) = d\Big(\frac{1}{6}, \frac{1}{6}\Big) = 0 \\
d\Big(\frac{1}{2}, \frac{1}{3}\Big) &= d\Big(\frac{1}{3}, \frac{1}{2}\Big) = (2, 3), d\Big(\frac{1}{2}, \frac{1}{4}\Big) = d(\frac{1}{4}, \frac{1}{2}\Big) = (3, 4) \\
d\Big(\frac{1}{2}, \frac{1}{5}\Big) &= d\Big(\frac{1}{5}, \frac{1}{2}\Big) = (3, 4), d\Big(\frac{1}{2}, \frac{1}{6}\Big) = d\Big(\frac{1}{6}, \frac{1}{2}\Big) = (4, 5) \\
d\Big(\frac{1}{3}, \frac{1}{4}\Big) &= d\Big(\frac{1}{4}, \frac{1}{3}\Big) = (2, 3), d\Big(\frac{1}{3}, \frac{1}{5}\Big) = d\Big(\frac{1}{5}, \frac{1}{3}\Big) = (2, 3) \\
d\Big(\frac{1}{3}, \frac{1}{6}\Big) &= d\Big(\frac{1}{6}, \frac{1}{3}\Big) = (2, 3), d\Big(\frac{1}{4}, \frac{1}{5}\Big) = d\Big(\frac{1}{5}, \frac{1}{4}\Big) = (3, 4) \\
d\Big(\frac{1}{4}, \frac{1}{6}\Big) &= d\Big(\frac{1}{6}, \frac{1}{4}\Big) = (3, 4), d\Big(\frac{1}{5}, \frac{1}{6}\Big) = d\Big(\frac{1}{6}, \frac{1}{5}\Big) = (4, 5).
\end{aligned}$$

Then (X, d) is a complex valued metric space. Define a self-mapping T on X as follows

$$T\Big(\frac{1}{2}\Big) = \frac{1}{3}, T\Big(\frac{1}{3}\Big) = \frac{1}{4}, T\Big(\frac{1}{4}\Big) = \frac{1}{5}, T\Big(\frac{1}{5}\Big) = \frac{1}{6} \text{ and } T\Big(\frac{1}{6}\Big) = \frac{1}{6}.$$

Now we define the control functions $\Lambda, \Xi, \Theta : X \to [0, 1)$ as $\Lambda(x) = \frac{x}{2}, \Xi(x) = \frac{4}{5}x$ and $\Theta(x) = \frac{x}{1+x}$ for all $x \in X$. By a routine calculation, one can easily verify that (i) $\Lambda(Tx) \leq \Lambda(x)$, $\Xi(Tx) \leq \Xi(x)$ and $\Theta(Tx) \leq \Theta(x)$; (ii) $(\Lambda + \Xi + \Theta)(x) < 1$. Also the map T satisfies all the conditions of Corollary 2.5. Notice that the point $\frac{1}{6} \in X$ remains fixed under T and is indeed unique.

Theorem 2.8. *Let (X,d) be a complete complex valued metric space and $S,T : X \to X$ be self-mappings such that*

$$d(Sx,Ty) \preceq \Lambda(x)d(x,y) + \frac{\Xi(x)d(x,Sx)d(y,Ty) + \Theta(x)d(y,Sx)d(x,Ty)}{1+d(x,y)} \tag{2.4}$$

for all $x, y \in X$,where the control functions $\Lambda, \Xi, \Theta : X \to [0,1)$ satisfy the following conditions,

(i) $\Lambda(Sx) \leq \Lambda(x)$, $\Xi(Sx) \leq \Xi(x)$ and $\Theta(Sx) \leq \Theta(x)$;
(ii) $\Lambda(Tx) \leq \Lambda(x)$, $\Xi(Tx) \leq \Xi(x)$ and $\Theta(Tx) \leq \Theta(x)$;
(iii) $(\Lambda + \Xi + \Theta)(x) < 1$.

Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define a sequence $\{x_k\}$ as follows

$$x_{2k+1} = Sx_{2k} \text{ and } x_{2k+2} = Tx_{2k+1} \text{ for all } k \geq 0.$$

From (2.4), we get

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\preceq \Lambda(x_{2k})d(x_{2k}, x_{2k+1}) + \frac{\Xi(x_{2k})d(x_{2k}, Sx_{2k})d(x_{2k+1}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} \\ &\quad +\Theta(x_{2k})\frac{d(x_{2k+1}, Sx_{2k})d(x_{2k}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} \\ &\preceq \Lambda(x_{2k})d(x_{2k}, x_{2k+1}) + \Xi(x_{2k})\frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k}, x_{2n+1})} \\ &\quad +\Theta(x_{2k})\frac{d(x_{2k+1}, x_{2k+1})d(x_{2k}, x_{2k+2})}{1+d(x_{2k}, x_{2n+1})} \\ &\preceq \Lambda(x_{2k})d(x_{2k}, x_{2k+1}) + \frac{\Xi(x_{2k})d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k}, x_{2k+1})}, \end{aligned}$$

so that

$$\begin{aligned} |d(x_{2k+1}, x_{2k+2})| &\leq |\Lambda(x_{2k})||d(x_{2k}, x_{2k+1})| \\ &\quad +|\Xi(x_{2k})||d(x_{2k+1}, x_{2k+2})|\left|\frac{d(x_{2k}, x_{2k+1})}{1+d(x_{2k}, x_{2k+1})}\right|. \end{aligned}$$

Since $|1 + d(x_{2k}, x_{2k+1})| > |d(x_{2k}, x_{2k+1})|$, therefore

$$\begin{aligned} |d(x_{2k+1}, x_{2k+2})| &\leq |\Lambda(x_{2k})||d(x_{2k}, x_{2k+1})| + |\Xi(x_{2k})||d(x_{2k+1}, x_{2k+2})| \\ &= |\Lambda(Tx_{2k-1})||d(x_{2k}, x_{2k+1})| + |\Xi(Tx_{2k-1})||d(x_{2k+1}, x_{2k+2})| \\ &\leq |\Lambda(x_{2k-1})||d(x_{2k}, x_{2k+1})| + |\Xi(x_{2k-1})||d(x_{2k+1}, x_{2k+2})| \\ &= |\Lambda(Sx_{2k-2})||d(x_{2k}, x_{2k+1})| + |\Xi(Sx_{2k-2})||d(x_{2k+1}, x_{2k+2})| \\ &\leq |\Lambda(x_{2k-2})||d(x_{2k}, x_{2k+1})| + |\Xi(x_{2k-2})||d(x_{2k+1}, x_{2k+2})| \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq |\Lambda(x_0)||d(x_{2k}, x_{2k+1})| + |\Xi(x_0)||d(x_{2k+1}, x_{2k+2})|, \end{aligned}$$

Since $\Lambda, \Xi, \Theta : X \to [0,1)$,so we have

$$|d(x_{2k+1}, x_{2k+2})| \leq \Big|\frac{\Lambda(x_0)}{1-\Xi(x_0)}\Big||d(x_{2k}, x_{2k+1})|. \tag{2.5}$$

Similarly we get

$$\begin{aligned}
d(x_{2k+2},&x_{2k+3}) = d(x_{2k+3}, x_{2k+2}) = d(Sx_{2k+2}, Tx_{2k+1}) \\
&\preceq \Lambda(x_{2k+2})d(x_{2k+2}, x_{2k+1}) + \frac{\Xi(x_{2k+2})d(x_{2k+2}, Sx_{2k+2})d(x_{2k+1}, Tx_{2k+1})}{1+d(x_{2k+2}, x_{2k+1})} \\
&\quad + \Theta(x_{2k+2})\frac{d(x_{2k+1}, Sx_{2k+2})d(x_{2k+2}, Tx_{2k+1})}{1+d(x_{2k+2}, x_{2k+1})} \\
&\preceq \Lambda(x_{2k+2})d(x_{2k+2}, x_{2k+1}) + \frac{\Xi(x_{2k+2})d(x_{2k+2}, x_{2k+3})d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k+2}, x_{2k+1})} \\
&\quad + \Theta(x_{2k+2})\frac{d(x_{2k+1}, x_{2k+3})d(x_{2k+2}, x_{2k+2})}{1+d(x_{2k+2}, x_{2k+1})} \\
&\preceq \Lambda(x_{2k+2})d(x_{2k+2}, x_{2k+1}) + \frac{\Xi(x_{2k+2})d(x_{2k+2}, x_{2k+3})d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k+2}, x_{2k+1})}.
\end{aligned}$$

Hence

$$\begin{aligned}
|d(x_{2k+2}, x_{2k+3})| \leq\ & |\Lambda(x_{2k+2})||d(x_{2k+1}, x_{2k+2})| \\
& + |\Xi(x_{2k+2})||d(x_{2k+2}, x_{2k+3})|\Big|\frac{d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k+1}, x_{2k+2})}\Big|.
\end{aligned}$$

Since $|1+d(x_{2k+1}, x_{2k+2})| > |d(x_{2k+1}, x_{2k+2})|$, therefore

$$\begin{aligned}
|d(x_{2k+2}, x_{2k+3})| &\leq |\Lambda(x_{2k+2})||d(x_{2k+1}, x_{2k+2})| + |\Xi(x_{2k+2})||d(x_{2k+2}, x_{2k+3})| \\
&= |\Lambda(Tx_{2k+1})||d(x_{2k+1}, x_{2k+2})| + |\Xi(Tx_{2k+1})||d(x_{2k+2}, x_{2k+3})| \\
&\leq |\Lambda(x_{2k+1})||d(x_{2k+1}, x_{2k+2})| + |\Xi(x_{2k+1})||d(x_{2k+2}, x_{2k+3})| \\
&= |\Lambda(Sx_{2k})||d(x_{2k+1}, x_{2k+2})| + |\Xi(Sx_{2k})||d(x_{2k+2}, x_{2k+3})| \\
&\leq |\Lambda(x_{2k})||d(x_{2k+1}, x_{2k+2})| + |\Xi(x_{2k})||d(x_{2k+2}, x_{2k+3})| \\
&\ \ \vdots \\
&\leq |\Lambda(x_0)||d(x_{2k+1}, x_{2k+2})| + |\Xi(x_0)||d(x_{2k+2}, x_{2k+3})|
\end{aligned}$$

Since $\Lambda, \Xi, \Theta : X \to [0,1)$, so we have

$$|d(x_{2k+2}, x_{2k+3})| \leq \Big|\frac{\Lambda(x_0)}{1-\Xi(x_0)}\Big||d(x_{2k}, x_{2k+1})|. \tag{2.6}$$

Now, by setting $\lambda = \Big|\frac{\Lambda(x_0)}{1-\Xi(x_0)}\Big| < 1$, and using (2.5) and (2.6), we get

$$\begin{aligned}
|d(x_n, x_{n+1})| &\leq \lambda|d(x_{n-1}, x_n)| \\
&\leq \lambda^2|d(x_{n-2}, x_{n-1})| \\
&\ \ \vdots \\
&\leq \lambda^n|d(x_0, x_1)|
\end{aligned}$$

for all $n \in \mathbb{N}$. Now, for any positive integer m and n with $m > n$, we have

$$\begin{aligned} |d(x_n, x_m)| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \\ &\leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}]|d(x_0, x_1)| \\ &\leq \left[\frac{\lambda^n}{1-\lambda}\right]|d(x_0, x_1)|. \end{aligned}$$

Hence

$$|d(x_n, x_m)| \leq \frac{\lambda^n}{1-\lambda}|d(x_0, x_1)| \longrightarrow 0 \quad \text{as} \quad m, n \longrightarrow \infty.$$

Thus by Lemma 1.3, we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Next we claim that $z = Sz$. We suppose on the contrary that $z \neq Sz$ and $d(z, Sz) = u \neq 0$. Then by triangular inequality and given condition, we get

$$\begin{aligned} u &= d(z, Sz) \preceq d(z, Tx_{2k+1}) + d(Tx_{2k+1}, Sz) \\ &= d(z, Tx_{2k+1}) + d(Sz, Tx_{2k+1}) \\ &\preceq d(z, Tx_{2k+1}) + \Lambda(z)d(z, x_{2k+1}) \\ &\quad + \frac{\Xi(z)d(z, Sz)d(x_{2k+1}, Tx_{2k+1}) + \Theta(z)d(x_{2k+1}, Sz)d(z, Tx_{2k+1})}{1 + d(z, x_{2k+1})} \\ &\preceq d(z, x_{2k+2}) + \Lambda(z)d(z, x_{2k+1}) \\ &\quad + \frac{\Xi(z)u.d(x_{2k+1}, x_{2k+2}) + \Theta(z)d(x_{2k+1}, Sz)d(z, x_{2k+2})}{1 + d(z, x_{2k+1})}, \end{aligned}$$

which implies that

$$\begin{aligned} |u| &= |d(z, Sz)| \leq |d(z, x_{2k+2})| + |\Lambda(z)||d(z, x_{2k+1})| + \frac{|\Xi(z)||u|.|d(x_{2k+1}, x_{2k+2})|}{|1 + d(z, x_{2k+1})|} \\ &\quad + |\Theta(z)|\frac{|d(x_{2k+1}, Sz)||d(z, x_{2k+2})|}{|1 + d(z, x_{2k+1})|}. \end{aligned}$$

Taking limit as $k \to \infty$, we get $|u| = 0$, which is a contradiction and hence $z = Sz$. Similarly it follows that $z = Tz$. Therefore z is the common fixed point S and T. Finally, we show that z is a unique common fixed point of S and T. Assume that there exists another common fixed point z^* that is $z^* = Sz^* = Tz^*$. Then

$$\begin{aligned} d(z, z^*) &= d(Sz, Tz^*) \\ &\preceq \Lambda(z)d(z, z^*) + \frac{\Xi(z)d(z, Sz)d(z^*, Tz^*) + \Theta(z)d(z^*, Sz)d(z, Tz^*)}{1 + d(z, z^*)} \\ &\preceq \Lambda(z)d(z, z^*) + \frac{\Xi(z)d(z, z)d(z^*, z^*) + \Theta(z)d(z^*, z)d(z, z^*)}{1 + d(z, z^*)} \\ &\preceq \Lambda(z)d(z, z^*) + \frac{\Theta(z)d(z^*, z)d(z, z^*)}{1 + d(z, z^*)}, \end{aligned}$$

so that

$$|d(z, z^*)| \leq |\Lambda(z)||d(z, z^*)| + \frac{|\Theta(z)||d(z^*, z)||d(z, z^*)|}{|1 + d(z, z^*)|}.$$

Since

$$|1 + d(z, z^*)| > |d(z, z^*)|,$$

therefore

$$|d(z,z^*)| \le |\Lambda(z)+\Theta(z)||d(z,z^*),$$

which is a contradiction so that $z = z^*$. This completes the proof of the theorem. □

Remark 2.9. By choosing $\Theta(x) = 0$ in Theorem 2.8, we get Theorem 3.1 of Sintunavarat and Kumam [29].

Remark 2.10. By setting $\Lambda(x) = \lambda$, $\Xi(x) = \mu$ and $\Theta(x) = \gamma$, in Theorem 2.8, we get the Theorem 2.1 of [27].

Remark 2.11. By setting $\Lambda(x) = \lambda$, $\Xi(x) = \mu$ and $\Theta(x) = 0$, in Theorem 2.8, we get Theorem 4 of [10].

By setting $S = T$ in Theorem 2.8, we get the following:

Corollary 2.12. *Let (X, d) be a complete complex valued metric space and $T : X \to X$ be a self-mapping such that*

$$d(Tx,Ty) \precsim \Lambda(x)d(x,y) + \frac{\Xi(x)d(x,Tx)d(y,Ty) + \Theta(x)d(y,Tx)d(x,Ty)}{1+d(x,y)}$$

for all $x, y \in X$ and

$$\Lambda, \Xi, \Theta : X \to [0,1),$$

satisfying the following conditions,

(i) $\Lambda(Tx) \le \Lambda(x)$, $\Xi(Tx) \le \Xi(x)$ *and* $\Theta(Tx) \le \Theta(x)$;
(ii) $(\Lambda + \Xi + \Theta)(x) < 1$.

Then T has a unique fixed point.

Remark 2.13. By setting $S = T$, and $\Theta(x) = 0$, in Theorem 2.8, we get Corollary 3.3 of [29].

Remark 2.14. By setting $S = T$, $\Lambda(x) = \lambda$, $\Xi(x) = \mu$ and $\Theta(x) = \gamma$, in Theorem 2.8, we get the Corollary 2.3 of [27].

Remark 2.15. By setting $S = T$, $\Lambda(x) = \lambda$, $\Xi(x) = \mu$ and $\Theta(x) = 0$, in Theorem 2.8, we get the Corollary 5 of [10].

Corollary 2.16. *Let (X, d) be a complete complex valued metric space and $T : X \to X$ be a self-mapping such that*

$$d(T^nx,T^ny) \precsim \Lambda(x)d(x,y) + \frac{\Xi(x)d(x,T^nx)d(y,T^ny) + \Theta(x)d(y,T^nx)d(x,T^ny)}{1+d(x,y)}$$

for all $x, y \in X$ and

$$\Lambda, \Xi, \Theta : X \to [0,1)$$

satisfying the following conditions,

(i) $\Lambda(T^nx) \le \Lambda(x)$, $\Xi(T^nx) \le \Xi(x)$ *and* $\Theta(T^nx) \le \Theta(x)$,
(ii) $(\Lambda + \Xi + \Theta)(x) < 1$.

Then T has a unique fixed point.

Theorem 2.17. *Let (X, d) be a complete complex valued metric space and $S, T, f : X \to X$ be self-mappings such that $SX \cup TX \subset fX$. Assume that the following conditions holds:*

(2.7)
$$d(Sx, Ty) \preceq \Lambda(fx)d(fx, fy) + \frac{\Xi(fx)d(fx, Sx)d(fy, Ty) + \Theta(fx)d(fy, Sx)d(fx, Ty)}{1 + d(fx, fy)}$$

for all $x, y \in X$,where the mappings $\Lambda, \Xi, \Theta : X \to [0, 1)$ satisfy the following conditions,

(i) $\Lambda(Sx) \leq \Lambda(fx)$, $\Xi(Sx) \leq \Xi(fx)$ *and* $\Theta(Sx) \leq \Theta(fx)$;
(ii) $\Lambda(Tx) \leq \Lambda(fx)$, $\Xi(Tx) \leq \Xi(fx)$ *and* $\Theta(Tx) \leq \Theta(fx)$;
(iii) $(\Lambda + \Xi + \Theta)(fx) < 1$.

If (S, f) and (T, f) are weakly compatible and $f(X)$ is closed subspace of X, then S, T, f have a unique common fixed point.

Proof. By Lemma 1.4, there exists $D \subseteq X$ such that $f(D) = f(X)$ and $f : D \to X$ is one-to-one. Now since $SX \cup TX \subset fX$, we define two mappings $\Gamma, \digamma : f(D) \to f(D)$ by

$$\Gamma(fx) = Sx \tag{2.8}$$

and

$$\digamma(fx) = Tx \tag{2.9}$$

respectively. Since f is one-to-one on D, then $\Gamma, \digamma$ are well-defined. Note that for $fx, fy \in f(D), fx \neq fy$, inequality (2.7) implies that

$$\begin{aligned} d(\Gamma(fx), \digamma(fy)) \;\preceq\; & \Lambda(fx)d(fx, fy) + \frac{\Xi(fx)d(fx, \Gamma(fx))d(fy, \digamma(fy))}{1 + d(fx, fy)} \\ & + \frac{\Theta(fx)d(fy, \Gamma(fx))d(fx, \digamma(fy))}{1 + d(fx, fy)}, \end{aligned}$$

Since $f(D) = f(X)$ is complete, so there exists a unique common fixed point $z \in f(D)$ of Γ and $\digamma$, that is $z = \Gamma(z) = \digamma(z)$. Now there exists some $u \in D$, such that $z = fu$. Hence $z = fu = \Gamma(fu)$ and $z = fu = \digamma(fu)$ that is u is the coincidence point of f, S and T. This implies that z is the point of coincidence of (f, S) and (f, T) and u is the coincidence point of f,S and T. We suppose on the contrary that there exists $z^* \in f(D)$ such that $z^* = f(v) = S(v) = T(v)$ and $z \neq z^*$. Now from (2.7), we get

$$\begin{aligned} d(z, z^*) &= d(Su, Tv) \preceq \Lambda(fu)d(fu, fv) \\ &\quad + \frac{\Xi(fu)d(fu, Su)d(fv, Tv) + \Theta(fu)d(fv, Su)d(fu, Tv)}{1 + d(fu, fv)} \\ &\preceq \Lambda(z)d(z, z^*) + \frac{\Xi(z)d(z, z)d(z^*, z^*) + \Theta(z)d(z^*, z)d(z, z^*)}{1 + d(z, z^*)} \\ &\preceq \Lambda(z)d(z, z^*) + \frac{\Theta(z)d(z^*, z)d(z, z^*)}{1 + d(z, z^*)}, \end{aligned}$$

which implies that

$$|d(z,z^*)| \leq |\Lambda(z)||d(z,z^*)| + \frac{|\Theta(z)||d(z^*,z)||d(z,z^*)|}{|1+d(z,z^*)|},$$

since $|1+d(z,z^*)| > |d(z,z^*)|$, so that $|d(z,z^*)| \leq |\Lambda(z)+\Theta(z)||d(z,z^*)|$. Hence $|d(z,z^*)| = 0$ that is $z = z^*$. Thus the point of coincidence is unique. Now since (S,f) and (T,f) are weakly compatible, so by Lemma 1.5, S,T,f have a unique common fixed point. □

Similarly we get the following result.

Theorem 2.18. *Let (X,d) be a complete complex valued metric space and $S,T,f : X \to X$ be self-mappings such that $SX \cup TX \subset fX = gX$. Assume that the following conditions hold:*

(2.10)

$$d(Sx,Ty) \precsim \Lambda(fx)d(fx,gy) + \frac{\Xi(fx)d(fx,Sx)d(gy,Ty) + \Theta(fx)d(gy,Sx)d(fx,Ty)}{1+d(fx,gy)}$$

for all $x,y \in X$, where the control functions $\Lambda, \Xi, \Theta : X \to [0,1)$ satisfy the following conditions,

(i) $\Lambda(Sx) \leq \Lambda(fx)$, $\Xi(Sx) \leq \Xi(fx)$ *and* $\Theta(Sx) \leq \Theta(fx)$;
(ii) $\Lambda(Tx) \leq \Lambda(fx)$, $\Xi(Tx) \leq \Xi(fx)$ *and* $\Theta(Tx) \leq \Theta(fx)$;
(iii) $(\Lambda + \Xi + \Theta)(fx) < 1$.

If (S,f) and (T,g) are weakly compatible and $f(X)$ is closed subspace of X, then S,T,f,g have a unique common fixed point.

3. Applications

Fixed point theorems for operators in ordered Banach spaces are widely investigated and have found various applications in differential and integral equations (see [5, 20, 21, 26] and references therein). In this section, we apply Theorem 2.8 to the existence of common solution of the system of Urysohn integral equations.

Theorem 3.1. *Let $X = C([a,b],\mathbb{R}^n)$, $a > 0$ and $d : X \times X \to \mathbb{C}$ be defined as follows:*

$$d(x,y) = \max_{t\in[a,b]} \|x(t) - y(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}.$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))ds + g(t), \tag{3.1}$$

$$x(t) = \int_a^b K_2(t,s,x(s))ds + h(t), \tag{3.2}$$

where $t \in [a,b] \subset \mathbb{R}$, $x,g,h \in X$.

Suppose that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for each $x \in X$, where,

$$F_x(t) = \int_a^b K_1(t,s,x(s))ds, \quad G_x(t) = \int_a^b K_2(t,s,x(s))ds \text{ for all } t \in [a,b].$$

If there exist three mappings $\Lambda, \Xi, \Theta : X \to [0,1)$ with

(i) $\Lambda(F_x + g) \leq \Lambda(x)$, $\Xi(F_x + g) \leq \Xi(x)$ and $\Theta(F_x + g) \leq \Theta(x)$;
(ii) $\Lambda(G_x + h) \leq \Lambda(x)$, $\Xi(G_x + h) \leq \Xi(x)$ and $\Theta(G_x + h) \leq \Theta(x)$;
(iii) $(\Lambda + \Xi + \Theta)(x) < 1$,

such that for all $x, y \in X$ the following condition holds:

$$\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a} \precsim \Lambda(x) A(x,y)(t) + \Xi(x) B(x,y)(t) + \Theta(x) C(x,y)(t),$$

where

$$A(x,y)(t) = \|x(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$$

$$B(x,y)(t) = \frac{\|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + d(x,y)} \sqrt{1+a^2} e^{i \tan^{-1} a}$$

$$C(x,y)(t) = \frac{\|F_x(t) + g(t) - y(t)\|_\infty \|G_y(t) + h(t) - x(t)\|_\infty}{1 + d(x,y)} \sqrt{1+a^2} e^{i \tan^{-1} a},$$

then the system of integral equations (3.1) and (3.2) have a unique common solution.

Proof. Define $S, T : X \to X$ by

$$Sx = F_x + g, \quad Tx = G_x + h.$$

Then

$$d(Sx, Ty) = \max_{t \in [a,b]} \|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a},$$

$$d(x,y) = \max_{t \in [a,b]} A(x,y)(t),$$

$$d(x, Sx) = \max_{t \in [a,b]} \|F_x(t) + g(t) - x(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a},$$

$$d(y, Ty) = \max_{t \in [a,b]_\infty} \|G_y(t) + h(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$$

$$d(y, Sx) = \max_{t \in [a,b]} \|F_x(t) + g(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a},$$

$$d(x, Ty) = \max_{t \in [a,b]_\infty} \|G_y(t) + h(t) - x(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}.$$

It is easily seen that for all $x, y \in X$, we have

$$d(Sx, Ty) \preceq \Lambda(x) d(x,y) + \frac{\Xi(x) d(x, Sx) d(y, Ty) + \Theta(x) d(y, Sx) d(x, Ty)}{1 + d(x,y)},$$

and

(i) $\Lambda(Sx) \leq \Lambda(x)$, $\Xi(Sx) \leq \Xi(x)$ and $\Theta(Sx) \leq \Theta(x)$;
(ii) $\Lambda(Tx) \leq \Lambda(x)$, $\Xi(Tx) \leq \Xi(x)$ and $\Theta(Tx) \leq \Theta(x)$.

By Theorem 2.8, we get S and T have a common fixed point. Thus there exists a unique point $x \in X$ such that $x = Sx = Tx$. Therefore, we conclude that the system of Urysohn integral equations (3.1) and (3.2) have a unique common solution. □

Acknowledgements

The authors thank the editor and the referees for their valuable comments and suggestions which improved greatly the quality of this paper.

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Manuscript received April 7, 2013
revised February 22, 2015

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