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# INEQUALITIES OF HERMITE-HADAMARD TYPE FOR EXTENDED $s$-CONVEX FUNCTIONS AND APPLICATIONS TO MEANS

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Abstract. In the paper, the authors introduce a new concept "extended $s$-convex functions", establish some new integral inequalities of Hermite-Hadamard type for this kind of functions, and apply these inequalities to derive some inequalities of special means.

## 1. Introduction

Throughout this paper, we use the following notations:

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty). \tag{1.1}$$

The following definitions are well known in the literature.

**Definition 1.1.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.2}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

**Definition 1.2** ([5])**.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be $P$-convex if

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y) \tag{1.3}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

**Definition 1.3** ([6])**.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be a Godunova-Levin function if $f$ is nonnegative and

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda} \tag{1.4}$$

holds for all $x, y \in I$ and $\lambda \in (0, 1)$.

**Definition 1.4** ([7])**.** Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R}_0 \to \mathbb{R}$ is said to be $s$-convex (in the second sense) if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.5}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

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In recent decades, a lot of inequalities of Hermite-Hadamard type for various kinds of convex functions have been established. Some of them may be recited as follows.

**Theorem 1.5** ([4]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)\big(|f'(a)|+|f'(b)|\big)}{8}. \tag{1.6}$$

**Theorem 1.6** ([9]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\qquad \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\big[|f'(a)|^q+|f'(b)|^q\big]^{1/q}. \end{aligned} \tag{1.7}$$

**Theorem 1.7** ([8]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$, then*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p} \\ &\times\left\{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[|f'(b)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\}, \end{aligned} \tag{1.8}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$.*

**Theorem 1.8** ([12]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|$ is $s$-convex on $[a, b]$ for some $s \in (0, 1]$, then*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\qquad \le \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\big(|f'(a)|+|f'(b)|\big), \end{aligned} \tag{1.9}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$.*

Some inequalities of Hermite-Hadamard type were also obtained in [1, 2, 3, 10, 11, 13, 14, 15, 16, 17, 18] and related references therein.

In this paper, we will introduce a new concept "extended $s$-convex functions", establish some new integral inequalities of Hermite-Hadamard type for extended $s$-convex functions, and apply these newly established integral inequalities to derive some inequalities of special means. These results generalize inequalities stated in Theorems 1.5 to 1.8.

## 2. Definition and lemmas

We first define the concept "extended $s$-convex functions" and establish an integral identity.

**Definition 2.1.** For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended $s$-convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{2.1}$$

holds for all $x, y \in I$ and $\lambda \in (0, 1)$.

It is obvious that the extended 1-convex function, 0-convex function, and $-1$-convex function are just the usually convex function in Definition 1.1, the $P$-convex functions in Definition 1.2, and Godunova-Levin convex function in Definition 1.3, respectively. It is also clear that Definition 2.1 extends Definition 1.4.

For establishing new integral inequalities of Hermite-Hadamard type for extended $s$-convex functions, we need the following integral identity.

**Lemma 2.2** ([17, Lemma 2.1])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I^\circ$ and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$ and $\lambda, \mu \in \mathbb{R}$, then*

$$\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$
$$= \frac{b-a}{4}\int_0^1 \left[(1-\lambda-t)f'\left(ta + (1-t)\frac{a+b}{2}\right) + (\mu - t)f'\left(t\frac{a+b}{2} + (1-t)b\right)\right]\mathrm{d}t.$$

**Lemma 2.3.** *Let $s > -1$, $0 \le \xi \le 1$, $\omega \in \mathbb{R} \setminus \{0\}$, $\eta \ge 0$, and $\omega + \eta \ge 0$. Then*

$$\int_0^1 |\xi - t|(\omega t + \eta)^s\,\mathrm{d}t$$
$$= \frac{2(\omega\xi + \eta)^{s+2} - [\eta + (s+2)\omega\xi]\eta^{s+1} - [2\omega\xi + \eta + s\omega(\xi - 1) - \omega](\omega + \eta)^{s+1}}{\omega^2(s+1)(s+2)}. \tag{2.2}$$

*In particular, if $(\omega, \eta) = (1, 0)$, $(1, 1)$, $(-1, 1)$, or $(-1, 2)$ respectively, then*

$$\int_0^1 |\xi - t| t^s\,\mathrm{d}t = \frac{2\xi^{s+2} - (s+2)\xi + s + 1}{(s+1)(s+2)},$$
$$\int_0^1 |\xi - t|(1+t)^s\,\mathrm{d}t = \frac{2(\xi+1)^{s+2} - [(s+2)\xi - s]2^{s+1} - (s+2)\xi - 1}{(s+1)(s+2)},$$
$$\int_0^1 |\xi - t|(1-t)^s\,\mathrm{d}t = \frac{2(1-\xi)^{s+2} + (s+2)\xi - 1}{(s+1)(s+2)},$$
$$\int_0^1 |\xi - t|(2-t)^s\,\mathrm{d}t = \frac{2(2-\xi)^{s+2} + [(s+2)\xi - 2]2^{s+1} + (s+2)\xi - s - 3}{(s+1)(s+2)}.$$

*Proof.* These follow from straightforward computation of definite integrals. □

## 3. Some integral inequalities of Hermite-Hadamard type

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for differentiable extended $s$-convex functions.

**Theorem 3.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I$ with $a < b$, $f' \in L[a,b]$, and $0 \le \lambda, \mu \le 1$. If $|f'(x)|^q$ for $q \ge 1$ is an extended $s$-convex function on $[a,b]$ for some fixed $s \in [-1,1]$, then*

(1) *when $-1 < s \le 1$, we have*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{2^{s/q+2}}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\left[\left\{2(2-\lambda)^{s+2}\right.\right.\right. \\
&\quad + [(s+2)\lambda-2]2^{s+1}+(s+2)\lambda-s-3\big\}|f'(a)|^q+\big\{2\lambda^{s+2}-(s+2)\lambda+s \\
&\quad +1\big\}|f'(b)|^q\big]^{1/q}+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\Big[\big(2\mu^{s+2}-(s+2)\mu+s+1\big)|f'(a)|^q \\
&\quad + \left.\left.\big\{2(2-\mu)^{s+2}+[(s+2)\mu-2]2^{s+1}+(s+2)\mu-s-3\big\}|f'(b)|^q\right]^{1/q}\right\};
\end{aligned}$$

(2) *when $s = -1$, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \qquad (3.1)\\
&\le \frac{b-a}{2^{3-2/q}}\left\{\left[(2\ln 2-1)|f'(a)|^q+|f'(b)|^q\right]^{1/q}+\left[|f'(a)|^q+(2\ln 2-1)|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

*Proof.* For $-1 < s \le 1$, since $|f'(x)|^q$ is extended $s$-convex on $[a,b]$, by Lemmas 2.2 and 2.3 and by Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\left[\int_0^1|1-\lambda-t|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right. \\
&\qquad \left.+\int_0^1|\mu-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}t\right] \\
&= \frac{b-a}{4}\left[\int_0^1|1-\lambda-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\mathrm{d}t\right. \\
&\qquad \left.+\int_0^1|\mu-t|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\mathrm{d}t\right] \\
&\le \frac{b-a}{2^{s/q+2}}\left\{\left(\int_0^1|1-\lambda-t|\,\mathrm{d}t\right)^{1-1/q}\right. \\
&\qquad \left[\int_0^1|1-\lambda-t|\big((1+t)^s|f'(a)|^q+(1-t)^s|f'(b)|^q\big)\,\mathrm{d}t\right]^{1/q} \\
&\qquad \left.+\left(\int_0^1|\mu-t|\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1|\mu-t|\big(t^s|f'(a)|^q+(2-t)^s|f'(b)|^q\big)\,\mathrm{d}t\right]^{1/q}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b-a}{2^{s/q+2}}\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\left[\frac{1}{(s+1)(s+2)}\big[\big(2(2-\lambda)^{s+2}+((s+2)\lambda-2)2^{s+1}\right.\right. \\
&\quad \left. +(s+2)\lambda-s-3\big)|f'(a)|^q+\big(2\lambda^{s+2}+s+1-(s+2)\lambda\big)|f'(b)|^q\big]\right]^{1/q} \\
&\quad +\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\left[\frac{1}{(s+1)(s+2)}\big[\big(2\mu^{s+2}+s+1-(s+2)\mu\big)|f'(a)|^q\right. \\
&\quad \left.\left. +\big(2(2-\mu)^{s+2}+((s+2)\mu-2)2^{s+1}+(s+2)\mu-s-3\big)|f'(b)|^q\big]\right]^{1/q}\right\}.
\end{aligned}
$$

For $s=-1$, since $|f'(x)|^q$ is extended $-1$-convex on $[a,b]$, by Lemma 2.2 and Hölder integral inequality, we have

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\left[\int_0^1\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|(1-t)\,\mathrm{d}t+\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}t\right] \\
&\le \frac{b-a}{2-/q}\left\{\left[\int_0^1(1-t)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1(1-t)\big((1+t)^{-1}|f'(a)|^q+(1-t)^{-1}|f'(b)|^q\big)\,\mathrm{d}t\right]^{1/q}\right. \\
&\quad \left. +\left(\int_0^1 t\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1 t\big(t^{-1}|f'(a)|^q+(2-t)^{-1}|f'(b)|^q\big)\,\mathrm{d}t\right]^{1/q}\right\} \\
&= \frac{b-a}{2^{3-2/q}}\big\{\big[(2\ln 2-1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}+\big[|f'(a)|^q+(2\ln 2-1)|f'(b)|^q\big]^{1/q}\big\}.
\end{aligned}
$$

Theorem 3.1 is proved. □

**Corollary 3.2.** *Under conditions of Theorem 3.1,*

(1) *if* $q=1$ *and* $-1<s\le 1$, *we have*

$$
\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{2^{s+2}(s+1)(s+2)}\big\{\big[2(2-\lambda)^{s+2}+2\mu^{s+2}+((s+2)\lambda-2)2^{s+1} \\
&\quad +(s+2)(\lambda-\mu)-2\big]|f'(a)|+\big[2\lambda^{s+2}+2(2-\mu)^{s+2} \\
&\quad +((s+2)\mu-2)2^{s+1}+(s+2)(\mu-\lambda)-2\big]|f'(b)|\big\};
\end{aligned}
\tag{3.2}
$$

(2) *if* $q=1$ *and* $s=-1$, *we have*

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le (b-a)(\ln 2)\big[|f'(a)|+|f'(b)|\big]. \tag{3.3}
$$

**Corollary 3.3.** *Under conditions of Theorem 3.1,*

(1) *when* $\lambda=\mu$ *and* $-1<s\le 1$, *we have*

$$
\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|
$$

$$\le \frac{b-a}{2^{s/q+2}}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\big\{\big[\big(2(2-\lambda)^{s+2} + ((s+2)\lambda-2)2^{s+1}+(s+2)\lambda-s-3\big)|f'(a)|^q+\big(2\lambda^{s+2}+s+1 - (s+2)\lambda\big)|f'(b)|^q\big]^{1/q}+\big[\big(2\lambda^{s+2}-(s+2)\lambda+s+1\big)|f'(a)|^q + \big(2(2-\lambda)^{s+2}+((s+2)\lambda-2)2^{s+1}+(s+2)\lambda-s-3\big)|f'(b)|^q\big]^{1/q}\big\};$$

(2) *when* $\lambda=\mu$, $-1<s\le 1$, *and* $q=1$,

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)\{(2-\lambda)^{s+2}+\lambda^{s+2}+[(s+2)\lambda-2]2^s-1\}(|f'(a)|+|f'(b)|)}{(s+1)(s+2)2^{s+1}}; \tag{3.4}$$

(3) *when* $-1<s\le 1$ *and* $\lambda=\mu=1$, *we have*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{8}\left[\frac{2}{(s+1)(s+2)}\right]^{1/q}\left\{\left[\left(2s+\frac{1}{2^s}\right)|f'(a)|^q+\frac{|f'(b)|^q}{2^s}\right]^{1/q}\right.\\ &\quad\left.+\left[\left(2s+\frac{1}{2^s}\right)|f'(b)|^q+\frac{|f'(a)|^q}{2^s}\right]^{1/q}\right\}\\ &\le \frac{(b-a)\big[|f'(a)|^q+|f'(b)|^q\big]^{1/q}}{4}\left[\frac{4+(1/2)^{s-1}}{(s+1)(s+2)}\right]^{1/q}.\end{aligned} \tag{3.5}$$

**Remark 3.4.** The inequality (1.7) is a special case of (3.5) applied to $0<s\le 1$. The inequality (1.9) can be deduced from (3.4) applied to $\lambda=\mu=\frac{1}{3}$ and $0<s\le 1$. These show that Theorem 3.1 and its corollaries generalize some main results obtained in [9, 12].

**Corollary 3.5.** *Under conditions of Theorem 3.1,*

(1) *when* $s=1$, *we have*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2^{1/q+2}}\left(\frac{1}{6}\right)^{1/q} \times\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\big[\big(4-9\lambda+12\lambda^2-2\lambda^3\big)|f'(a)|^q+\big(2-3\lambda+2\lambda^3\big)|f'(b)|^q\big]^{1/q} + \left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\big[\big(2-3\mu+2\mu^3\big)|f'(a)|^q+(4-9\mu+12\mu^2-2\mu^3)|f'(b)|^q\big]^{1/q}\right\};$$

(2) *when* $s=1$ *and* $q=1$, *we have*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{48}\big\{\big(6-9\lambda+12\lambda^2-2\lambda^3-3\mu+2\mu^3\big)|f'(a)|+\big(6-3\lambda+2\lambda^3-9\mu+12\mu^2-2\mu^3\big)|f'(b)|\big\};$$

(3) *when* $s=1$ *and* $\lambda=\mu$,

$$\left|\frac{\lambda[f(a)+f(b)]}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4}\left(\frac{1}{12}\right)^{1/q}\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}$$
$$\times\left\{\left[\left(4-9\lambda+12\lambda^2-2\lambda^3\right)|f'(a)|^q+\left(2-3\lambda+2\lambda^3\right)|f'(b)|^q\right]^{1/q}\right.$$
$$\left.+\left[\left(2-3\lambda+2\lambda^3\right)|f'(a)|^q+\left(4-9\lambda+12\lambda^2-2\lambda^3\right)|f'(b)|^q\right]^{1/q}\right\};$$

(4) *when* $s=1$, $q=1$, *and* $\lambda=\mu$, *we have*

$$\left|\frac{\lambda[f(a)+f(b)]}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{8}\left(1-2\lambda+2\lambda^2\right)\left[|f'(a)|+|f'(b)|\right]. \tag{3.6}$$

**Remark 3.6.** Letting $\lambda=1$ in (3.6) yields the inequality (1.6) in [4].

**Corollary 3.7.** *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be differentiable on* $I^\circ$, $a,b\in I$ *with* $a<b$, *and* $f'\in L[a,b]$. *If* $|f'(x)|^q$ *is convex on* $[a,b]$ *for* $q\ge1$, *then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{16}\left\{\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{1/q}+\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{1/q}\right\},$$
$$\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{5(b-a)}{72}\left\{\left[\frac{37|f'(a)|^q+8|f'(b)|^q}{45}\right]^{1/q}+\left[\frac{8|f'(a)|^q+37|f'(b)|^q}{45}\right]^{1/q}\right\},$$
$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{5(b-a)}{72}\left\{\left[\frac{61|f'(a)|^q+29|f'(b)|^q}{90}\right]^{1/q}+\left[\frac{29|f'(a)|^q+61|f'(b)|^q}{90}\right]^{1/q}\right\}.$$

**Theorem 3.8.** *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be differentiable on* $I^\circ$, $a,b\in I$ *with* $a<b$, *and* $f'\in L[a,b]$. *If* $|f'(x)|^q$ *for* $q\ge1$ *is an extended* $s$*-convex function on* $[a,b]$, *then for* $s\in(-1,1]$ *and* $0\le\lambda,\mu\le1$,

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\right.$$
$$\times\left[\left(2(1-\lambda)^{s+2}+(s+2)\lambda-1\right)|f'(a)|^q\right.$$

$$
\begin{aligned}
&\quad + \left(2\lambda^{s+2}+s+1-(s+2)\lambda\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\Bigg]^{1/q} \\
&+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q} \\
&\qquad\times\Bigg[\left(2\mu^{s+2}+s+1-(s+2)\mu\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&\qquad + \left(2(1-\mu)^{s+2}+(s+2)\mu-1\right)|f'(b)|^q\Bigg]^{1/q}\Bigg\} \\
&\le \frac{b-a}{2^{s/q+2}}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\Bigg\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\big[\left((1-\lambda)^{s+2}2^{s+1}+2\lambda^{s+2}\right. \\
&\left. + s+1+((s+2)\lambda-1)2^s-(s+2)\lambda\right)|f'(a)|^q \\
&+\left(2\lambda^{s+2}-(s+2)\lambda+s+1\right)|f'(b)|^q\big]^{1/q} \\
&+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\big[\left(2\mu^{s+2}-(s+2)\mu+s+1\right)|f'(a)|^q \\
&+\left((1-\mu)^{s+2}2^{s+1}+2\mu^{s+2}+\left((s+2)\mu-1\right)2^s-(s+2)\mu+s+1\right)|f'(b)|^q\big]^{1/q}\Bigg\}.
\end{aligned}
$$

*Proof.* By similar arguments as in the proof of Theorem 3.1 and by the extended $s$-convexity of the function $|f'(x)|^q$, we have

$$
\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\Bigg\{\left(\int_0^1|1-\lambda-t|\,\mathrm{d}t\right)^{1-1/q} \\
&\qquad\left[\int_0^1|1-\lambda-t|\left((1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+t^s|f'(a)|^q\right)\mathrm{d}t\right]^{1/q} \\
&+\left(\int_0^1|\mu-t|\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1|\mu-t|\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s|f'(b)|^q\right)\mathrm{d}t\right]^{1/q}\Bigg\} \\
&=\frac{b-a}{4}\Bigg\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\Bigg[\frac{1}{(s+1)(s+2)}\Big(\left(2(1-\lambda)^{s+2}+(s+2)\lambda-1\right)|f'(a)|^q \\
&+\left(2\lambda^{s+2}-(s+2)\lambda+s+1\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\Big)\Bigg]^{1/q} \\
&+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\Bigg[\frac{1}{(s+1)(s+2)}\Big(\left(2\mu^{s+2}+s+1-(s+2)\mu\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&+\left(2(1-\mu)^{s+2}+(s+2)\mu-1\right)|f'(b)|^q\Big)\Bigg]^{1/q}\Bigg\}.
\end{aligned}
$$

Combining this with

$$\left|f'\left(\frac{a+b}{2}\right)\right|^q \le \left(\frac{1}{2}\right)^s \left[|f'(a)|^q + |f'(b)|^q\right]$$

leads to Theorem 3.8. □

**Corollary 3.9.** *Under conditions of Theorem 3.8, when $q=1$, we have*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4(s+1)(s+2)}\Big\{\big[2(1-\lambda)^{s+2} + (s+2)\lambda - 1\big]|f'(a)| + \big[2\lambda^{s+2} + 2\mu^{s+2} \\
&\quad + (s+2)(1-\lambda-\mu) + s\big]\left|f'\left(\frac{a+b}{2}\right)\right| + \big[2(1-\mu)^{s+2} + (s+2)\mu - 1\big]|f'(b)|\Big\} \\
&\le \frac{b-a}{2^{s+2}(s+1)(s+2)}\big\{\big[(1-\lambda)^{s+2}2^{s+1} + 2\lambda^{s+2} + 2\mu^{s+2} + ((s+2)\lambda-1)2^s \\
&\quad + (s+2)(1-\lambda-\mu) + s\big]|f'(a)| + \big[2\lambda^{s+2} + (1-\mu)^{s+2}2^{s+1} \\
&\quad + 2\mu^{s+2} + (s+2)(1-\lambda-\mu) + ((s+2)\mu-1)2^s + s\big]|f'(b)|\big\}.
\end{aligned}$$

**Corollary 3.10.** *Under conditions of Theorem 3.8, if $\lambda=\mu$, then*

$$\begin{aligned}
&\left|\frac{\lambda[f(a)+f(b)]}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\Bigg\{\Bigg[\big(2(1-\lambda)^{s+2} + (s+2)\lambda - 1\big)|f'(a)|^q \\
&\quad + \big(2\lambda^{s+2} + s + 1 - (s+2)\lambda\big)\left|f'\left(\frac{a+b}{2}\right)\right|^q\Bigg]^{1/q} + \Bigg[\big(2\lambda^{s+2} + s + 1 \\
&\quad - (s+2)\lambda\big)\left|f'\left(\frac{a+b}{2}\right)\right|^q + \big(2(1-\lambda)^{s+2} + (s+2)\lambda - 1\big)|f'(b)|^q\Bigg]^{1/q}\Bigg\} \\
&\le \left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\frac{b-a}{2^{s/q+2}}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\big\{\big[\big(2^{s+1}(1-\lambda)^{s+2} + 2\lambda^{s+2} \\
&\quad - (s+2)\lambda + ((s+2)\lambda-1)2^s + s + 1\big)|f'(a)|^q \\
&\quad + \big(2\lambda^{s+2} - (s+2)\lambda + s + 1\big)|f'(b)|^q\big]^{1/q} \\
&\quad + \big[\big(2\lambda^{s+2} - (s+2)\lambda + s + 1\big)|f'(a)|^q + \big(2\lambda^2 + (1-\lambda)^{s+2}2^{s+1} \\
&\quad + ((s+2)\lambda-1)2^s - (s+2)\lambda + s + 1\big)|f'(b)|^q\big]^{1/q}\big\}.
\end{aligned}$$

**Remark 3.11.** The inequality (1.8) can be deduced from letting $\lambda=\mu=0$ in Corollary 3.10.

**Corollary 3.12.** *Under conditions of Theorem 3.8, when $s=1$, we have*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$

$$
\begin{aligned}
&\le \frac{b-a}{4}\left(\frac{1}{6}\right)^{1/q}\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\left[\left(1-3\lambda+6\lambda^2-2\lambda^3\right)|f'(a)|^q\right.\right.\\
&\quad\left.+\left(2\lambda^3-3\lambda+2\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\\
&\quad+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\left[\left(2\mu^3-3\mu+2\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.\\
&\quad\left.\left.+\left(1-3\mu+6\mu^2-2\mu^3\right)|f'(b)|^q\right]^{1/q}\right\}\\
&\le \frac{b-a}{2^{1/q+2}}\left(\frac{1}{6}\right)^{1/q}\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\left[\left(4-9\lambda+12\lambda^2-2\lambda^3\right)|f'(a)|^q\right.\right.\\
&\quad\left.+\left(2\lambda^3-3\lambda+2\right)|f'(b)|^q\right]^{1/q}+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\left[\left(2\mu^3-3\mu+2\right)|f'(a)|^q\right.\\
&\quad\left.\left.+\left(4-9\mu+12\mu^2-2\mu^3\right)|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}
$$

**Theorem 3.13.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ for $q \ge 1$ is an extended $s$-convex function on $[a, b]$, then for $s \in (-1, 1]$ and $0 \le \lambda, \mu \le 1$,*

(1) *when $q = 1$, we have*

$$
\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2^{s+2}(s+1)}\\
&\times\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)\left[|f'(b)|+(2^{s+1}-1)|f'(a)|\right]+\left(\frac{1}{2}-\mu+\mu^2\right)\left[(2^{s+1}-1)|f'(b)|+|f'(a)|\right]\right\};
\end{aligned}
$$

(2) *when $q > 1$, we have*

(3.7)
$$
\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2^{s/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\
&\times\left(\frac{1}{s+1}\right)^{1/q}\left\{\left[(1-\lambda)^{(2q-1)/(q-1)}+\lambda^{(2q-1)/(q-1)}\right]^{1-1/q}\left[(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q\right]^{1/q}\right.\\
&\quad\left.+\left[\mu^{(2q-1)/(q-1)}+(1-\mu)^{(2q-1)/(q-1)}\right]^{1-1/q}\left[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}
$$

*Proof.* For $q > 1$, by the extended $s$-convexity of $|f'(x)|^q$ on $[a, b]$, Lemma 2.2, and Hölder integral inequality, we have

$$
\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le \frac{b-a}{4}\left[\int_0^1|1-\lambda-t|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right.\\
&\qquad\left.+\int_0^1|\mu-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}t\right]
\end{aligned}
$$

$$\le \frac{b-a}{2^{s/q+2}}\left\{\left(\int_0^1 |1-\lambda-t|^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1 \big((1+t)^s|f'(a)|^q+(1-t)^s|f'(b)|^q\big)\,\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left(\int_0^1 |\mu-t|^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1 \big(t^s|f'(a)|^q+(2-t)^s|f'(b)|^q\big)\,\mathrm{d}t\right]^{1/q}\right\}.$$

A direct calculation yields

$$\int_0^1 |1-\lambda-t|^{q/(q-1)}\,\mathrm{d}t=\frac{q-1}{2q-1}\big[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\big],$$
$$\int_0^1 |\mu-t|^{q/(q-1)}\,\mathrm{d}t=\frac{q-1}{2q-1}\big[\mu^{(2q-1)/(q-1)}+(1-\mu)^{(2q-1)/(q-1)}\big].$$

A straightforward computation gives

$$\int_0^1 \big[(1+t)^s|f'(a)|^q+(1-t)^s|f'(b)|^q\big]\,\mathrm{d}t=\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{s+1},$$
$$\int_0^1 \big[t^s|f'(a)|^q+(2-t)^s|f'(b)|^q\big]\,\mathrm{d}t=\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{s+1}.$$

Substituting the last four equalities into the first inequality and simplifying establish the inequality (3.7).

For $q=1$, utilizing the extended $s$-convexity of $|f'(x)|^q$ on $[a,b]$, Lemma 2.2, and Hölder integral inequality, we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4}\left[\int_0^1 |1-\lambda-t|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right.$$
$$\left.+\int_0^1 |\mu-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}t\right]$$
$$\le\frac{b-a}{2^{s+2}}\left\{\left(\int_0^1 |1-\lambda-t|\,\mathrm{d}t\right)\int_0^1\big[(1+t)^s|f'(a)|+(1-t)^s|f'(b)|\big]\,\mathrm{d}t\right.$$
$$\left.+\left(\int_0^1 |\mu-t|\,\mathrm{d}t\right)\int_0^1\big[t^s|f'(a)|+(2-t)^s|f'(b)|\big]\,\mathrm{d}t\right\}$$
$$=\frac{b-a}{2^{s+2}(s+1)}\left\{\left(\frac12-\lambda+\lambda^2\right)\big[|f'(b)|+(2^{s+1}-1)|f'(a)|\big]\right.$$
$$\left.+\left(\frac12-\mu+\mu^2\right)\big[(2^{s+1}-1)|f'(b)|+|f'(a)|\big]\right\}.$$

Theorem 3.13 is thus proved. □

**Corollary 3.14.** *Under conditions of Theorem 3.13,*

(1) *when* $\lambda=\mu$ *and* $q>1$, *we have*

$$\left|\frac{\lambda[f(a)+f(b)]}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{2^{s/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}\big[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\big]^{1-1/q}$$
$$\times\big\{\big[(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}+\big[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q\big]^{1/q}\big\};$$

(2) *when* $\lambda=\mu=0,1$ *and* $q>1$, *we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{2^{s/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}$$
$$\times\big\{\big[(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}+\big[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q\big]^{1/q}\big\}$$

*and*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{2^{s/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}$$
$$\times\big\{\big[(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}+\big[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q\big]^{1/q}\big\}.$$

**Corollary 3.15.** *Under conditions of Theorem 3.13,*

(1) *when* $q=1$ *and* $s=1$, *we have*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{16}\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)\big[|f'(b)|+3|f'(a)|\big]+\left(\frac{1}{2}-\mu+\mu^2\right)\big[3|f'(b)|+|f'(a)|\big]\right\};$$

(2) *when* $q>1$ *and* $s=1$, *we have*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{2^{2/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\big\{\big[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\big]^{1-1/q}\big[3|f'(a)|^q+|f'(b)|^q\big]^{1/q}$$
$$+\big[\mu^{(2q-1)/(q-1)}+(1-\mu)^{(2q-1)/(q-1)}\big]^{1-1/q}\big[|f'(a)|^q+3|f'(b)|^q\big]^{1/q}\big\};$$

(3) *when* $q=1$, $\lambda=\mu$, *and* $s=1$, *we have*

$$\left|\frac{\lambda[f(a)+f(b)]}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\big\{\big(1-2\lambda+2\lambda^2\big)\big[|f'(a)|+|f'(b)|\big]\big\};\tag{3.8}$$

(4) *when* $q>1$, $\lambda=\mu$, *and* $s=1$, *we have*

(3.9)
$$\left|\frac{\lambda[f(a)+f(b)]}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\left(\frac{q-1}{2q-1}\right)^{1-1/q}\frac{b-a}{2^{2/q}}$$

$$\times\big[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\big]^{1-1/q}\big[|f'(a)|^q+|f'(b)|^q\big]^{1/q}.$$

**Theorem 3.16.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable on $I^\circ$, $a,b\in I$ with $a<b$, and $f'\in L[a,b]$. If $|f'(x)|^q$ for $q\ge 1$ is an extended $s$-convex function on $[a,b]$, then, for $s\in(-1,1]$ and $0\le\lambda,\mu\le 1$,*

(1) *when $q=1$, we have*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{b-a}{4(s+1)}\left\{\left(\frac12-\lambda+\lambda^2\right)\left[|f'(a)|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]\right.\\
&\quad\left.+\left(\frac12-\mu+\mu^2\right)\left[\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right]\right\}\\
&\le\frac{b-a}{2^{s+2}(s+1)}\left\{\left(\frac12-\lambda+\lambda^2\right)[(2^s+1)|f'(a)|+|f'(b)|]\right.\\
&\quad\left.+\left(\frac12-\mu+\mu^2\right)[|f'(a)|+(2^s+1)|f'(b)|]\right\};
\end{aligned}$$

(2) *when $q>1$, we have*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}\left\{\big[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\big]^{1-1/q}\right.\\
&\quad\times\left[|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\big[\mu^{(2q-1)/(q-1)}+(1-\mu)^{(2q-1)/(q-1)}\big]^{1-1/q}\\
&\quad\left.\times\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}\right\}\\
&\le\frac{b-a}{2^{s/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}\\
&\quad\times\Big\{\big[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\big]^{1-1/q}\big[(2^s+1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}\\
&\quad+\big[\mu^{(2q-1)/(q-1)}+(1-\mu)^{(2q-1)/(q-1)}\big]^{1-1/q}\big[|f'(a)|^q+(2^s+1)|f'(b)|^q\big]^{1/q}\Big\}.
\end{aligned}$$

*Proof.* For $q>1$, since $|f'(x)|^q$ is extended $s$-convex on $[a,b]$, by Lemma 2.2 and Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{b-a}{4}\left[\int_0^1|1-\lambda-t|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right.\\
&\quad\left.+\int_0^1|\mu-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}t\right]
\end{aligned}$$

$$\le \frac{b-a}{4}\Biggl\{\biggl(\int_0^1 |1-\lambda-t|^{q/(q-1)}\,\mathrm{d}\,t\biggr)^{1-1/q}$$

$$\biggl[\int_0^1 \biggl(t^s|f'(a)|^q+(1-t)^s\biggl|f'\biggl(\frac{a+b}{2}\biggr)\biggr|^q\biggr)\,\mathrm{d}\,t\biggr]^{1/q}$$

$$+\biggl(\int_0^1 |\mu-t|^{q/(q-1)}\,\mathrm{d}\,t\biggr)^{1-1/q}\biggl[\int_0^1 \biggl(t^s\biggl|f'\biggl(\frac{a+b}{2}\biggr)\biggr|^q+(1-t)^s|f'(b)|^q\biggr)\,\mathrm{d}\,t\biggr]^{1/q}\Biggr\}.$$

If $q=1$, we have

$$\biggl|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\biggl(\frac{a+b}{2}\biggr)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\biggr|$$

$$\le \frac{b-a}{4}\biggl[\int_0^1 |1-\lambda-t|\biggl|f'\biggl(ta+(1-t)\frac{a+b}{2}\biggr)\biggr|\,\mathrm{d}\,t$$

$$+\int_0^1 |\mu-t|\biggl|f'\biggl(t\frac{a+b}{2}+(1-t)b\biggr)\biggr|\,\mathrm{d}\,t\biggr]$$

$$\le \frac{b-a}{4}\Biggl\{\biggl(\int_0^1 |1-\lambda-t|\,\mathrm{d}\,t\biggr)\biggl[\int_0^1 \biggl(t^s|f'(a)|+(1-t)^s\biggl|f'\biggl(\frac{a+b}{2}\biggr)\biggr|\biggr)\,\mathrm{d}\,t\biggr]$$

$$+\biggl(\int_0^1 |\mu-t|\,\mathrm{d}\,t\biggr)\biggl[\int_0^1 \biggl(t^s\biggl|f'\biggl(\frac{a+b}{2}\biggr)\biggr|+(1-t)^s|f'(b)|\biggr)\,\mathrm{d}\,t\biggr]\Biggr\}.$$

Theorem 3.16 is thus proved. □

**Corollary 3.17.** *Under conditions of Theorem 3.16,*

(1) *when* $q=1$ *and* $\lambda=\mu$, *we have*

$$\biggl|\frac{\lambda[f(a)+f(b)]}{2}+(1-\lambda)f\biggl(\frac{a+b}{2}\biggr)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\biggr| \tag{3.10}$$

$$\le \frac{b-a}{4(s+1)}\biggl(\frac{1}{2}-\lambda+\lambda^2\biggr)\biggl[|f'(a)|+2\biggl|f'\biggl(\frac{a+b}{2}\biggr)\biggr|+|f'(b)|\biggr]$$

$$\le \frac{b-a}{2^{s+1}(s+1)}\biggl(\frac{1}{2}-\lambda+\lambda^2\biggr)(2^{s-1}+1)[|f'(a)|+|f'(b)|];$$

(2) *when* $q>1$ *and* $\lambda=\mu$, *we have*

$$\biggl|\frac{\lambda[f(a)+f(b)]}{2}+(1-\lambda)f\biggl(\frac{a+b}{2}\biggr)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\biggr|$$

$$\le \frac{b-a}{4}\biggl(\frac{q-1}{2q-1}\biggr)^{1-1/q}\biggl(\frac{1}{s+1}\biggr)^{1/q}\bigl[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\bigr]^{1-1/q}$$

$$\times\Biggl\{\biggl[|f'(a)|^q+\biggl|f'\biggl(\frac{a+b}{2}\biggr)\biggr|^q\biggr]^{1/q}+\biggl[\biggl|f'\biggl(\frac{a+b}{2}\biggr)\biggr|^q+|f'(b)|^q\biggr]^{1/q}\Biggr\}$$

$$\le \frac{b-a}{2^{s/q+2}}\biggl(\frac{q-1}{2q-1}\biggr)^{1-1/q}\biggl(\frac{1}{s+1}\biggr)^{1/q}\bigl[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\bigr]^{1-1/q}$$

$$\times\bigl\{\bigl[(2^s+1)|f'(a)|^q+|f'(b)|^q\bigr]^{1/q}+\bigl[|f'(a)|^q+(2^s+1)|f'(b)|^q\bigr]^{1/q}\bigr\}.$$

**Corollary 3.18.** *Under conditions of Theorem 3.16,*

(1) *when* $q=1$ *and* $s=1$, *we have*

$$\begin{aligned}&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le\frac{b-a}{8}\left\{\left(\frac12-\lambda+\lambda^2\right)\left[|f'(a)|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]\right.\\ &\quad\left.+\left(\frac12-\mu+\mu^2\right)\left[\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right]\right\}\\ &\le\frac{b-a}{16}\left\{\left(\frac12-\lambda+\lambda^2\right)[3|f'(a)|+|f'(b)|]+\left(\frac12-\mu+\mu^2\right)[|f'(a)|+3|f'(b)|]\right\};\end{aligned}$$

(2) *when* $q>1$ *and* $s=1$, *we have*

$$\begin{aligned}&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le\frac{b-a}{2^{1/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\ &\quad\left\{\left[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\right]^{1-1/q}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right]^{1/q}\right.\\ &\quad\left.+\left[\mu^{(2q-1)/(q-1)}+(1-\mu)^{(2q-1)/(q-1)}\right]^{1-1/q}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}\right\}\\ &\le\frac{b-a}{2^{2/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\right]^{1-1/q}\right.\\ &\quad\left[3|f'(a)|^q+|f'(b)|^q\right]^{1/q}+\left[\mu^{(2q-1)/(q-1)}\right.\\ &\qquad\left.\left.+(1-\mu)^{(2q-1)/(q-1)}\right]^{1-1/q}\left[|f'(a)|^q+3|f'(b)|^q\right]^{1/q}\right\}.\end{aligned}$$

## 4. Applications to means

Finally, we apply some inequalities of Hermite-Hadamard type for extended $s$-convex functions to construct some inequalities for means.

For two positive numbers $a,b\in\mathbb{R}_+$, let

(4.1)
$$A(a,b)=\frac{a+b}{2}\quad\text{and}\quad L_s(a,b)=\begin{cases}\left[\dfrac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right]^{1/s}, & s\ne0,-1\text{ and }a\ne b,\\ \dfrac{b-a}{\ln b-\ln a}, & s=-1\text{ and }a\ne b,\\ \dfrac1e\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & s=0\text{ and }a\ne b,\\ a, & a=b.\end{cases}$$

They are called the arithmetic and generalized logarithmic means of two positive numbers $a$ and $b$ respectively.

Let $f(x)=x^s$ for $x>0$, $s>0$, and $q\ge 1$. If $0\le (s-1)q\le 1$ and $0\le s-1\le 1$, then

$$\begin{aligned}|f'(tx+(1-t)y)|^q &\le s^q\big[t^{(s-1)q}x^{(s-1)q}+(1-t)^{(s-1)q}y^{(s-1)q}\big]\\ &\le t^{s-1}|f'(x)|^q+(1-t)^{s-1}|f'(y)|^q\end{aligned}$$

for $x,y>0$ and $t\in(0,1)$. If $-1<(s-1)q\le 0$ and $-1<s-1\le 0$, then

$$|f'(tx+(1-t)y)|^q\le t^{s-1}|f'(x)|^q+(1-t)^{s-1}|f'(y)|^q$$

for $x,y>0$ and $t\in(0,1)$. If $-1<(s-1)q\le 1$ and $-1<s-1\le 1$, then $|f'(x)|^q=s^qx^{(s-1)q}$ is an extended $(s-1)$-convex function on $[a,b]$.

Applying Corollary 3.3 to $|s|^qx^{(s-1)q}$ yields the following theorem.

**Theorem 4.1.** *Let $b>a>0$, $q\ge 1$, $0<s\le 2$, $-1<(s-1)q\le 1$, and $0\le\lambda\le 1$. Then*

(4.2)
$$\begin{aligned}\big|\lambda A(a^s,b^s)+(1-\lambda)A^s(a,b)-L_s^s(a,b)\big| &\le \frac{(b-a)s}{2^{(s-1)/q+2}}\left[\frac{1}{s(s+1)}\right]^{1/q}\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\\ &\times\big\{\big[\big(2(2-\lambda)^{s+1}+2^s((s+1)\lambda-2)+(s+1)\lambda-s-2\big)a^{(s-1)q}\\ &+\big(2\lambda^{s+1}+s-(s+1)\lambda\big)b^{(s-1)q}\big]^{1/q}+\big[\big(2\lambda^{s+1}+s-(s+1)\lambda\big)a^{(s-1)q}\\ &+\big(2(2-\lambda)^{s+1}+2^s((s+1)\lambda-2)+(s+1)\lambda-s-2\big)b^{(s-1)q}\big]^{1/q}\big\}.\end{aligned}$$

*Specially, if $q=1$, then*

$$\begin{aligned}&\big|\lambda A(a^s,b^s)+(1-\lambda)A^s(a,b)-L_s^s(a,b)\big|\\ &\le\frac{(b-a)s}{2^{s-1}s(s+1)}\big\{(2-\lambda)^{s+1}+\lambda^{s+1}+[(s+1)\lambda-2]2^{s-1}-1\big\}A(a^{s-1},b^{s-1}).\end{aligned}\tag{4.3}$$

Taking $f(x)=x^s$ for $x>0$ and $s>0$ in Corollary 3.10 derives the following inequalities for means.

**Theorem 4.2.** *Let $b>a>0$, $q\ge 1$, $0<s\le 2$, $-1<(s-1)q\le 1$, and $0\le\lambda\le 1$. Then*

$$\begin{aligned}&\big|\lambda A(a^s,b^s)+(1-\lambda)A^s(a,b)-L_s^s(a,b)\big|\le\frac{(b-a)s}{4}\left[\frac{1}{s(s+1)}\right]^{1/q}\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\\ &\times\big\{\big[\big(2(1-\lambda)^{s+1}+(s+1)\lambda-1\big)a^{(s-1)q}+\big(2\lambda^{s+1}-(s+1)\lambda+s\big)A^{(s-1)q}(a,b)\big]^{1/q}\\ &+\big[\big(2\lambda^{s+1}-(s+1)\lambda+s\big)A^{(s-1)q}(a,b)+\big(2(1-\lambda)^{s+1}+(s+1)\lambda-1\big)b^{(s-1)q}\big]^{1/q}\big\}.\end{aligned}$$

*In particular, if $q=1$, then*

$$\begin{aligned}&\left|\lambda A(a^s,b^s)+(1-\lambda)A^s(a,b)-L_s^s(a,b)\right|\\ &\le\frac{b-a}{2(s+1)}\big\{\big[2(1-\lambda)^{s+1}+(s+1)\lambda-1\big]A\big(a^{s-1},b^{s-1}\big)+\big[2\lambda^{s+1}+s-(s+1)\lambda\big]A^{s-1}(a,b)\big\}.\end{aligned}$$

Letting $f(x)=x^s$ for $x>0$ and $s>0$ in Corollary 3.14 generates inequalities below.

**Theorem 4.3.** *Let $b > a > 0$, $q \ge 1$, $0 < s \le 2$, and $0 \le \lambda \le 1$.*

(1) *If $q > 1$ and $-1 < (s-1)q <= 1$, then*

$$\left|\lambda A(a^s, b^s) + (1-\lambda)A^s(a,b) - L_s^s(a,b)\right| \le \frac{(b-a)s}{2^{s-1/q+2}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s}\right)^{1/q}\left[\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}\right]^{1-1/q}\left\{\left[(2^s-1)a^{(s-1)q} + b^{(s-1)q}\right]^{1/q} + \left[a^{(s-1)q} + (2^s-1)b^{(s-1)q}\right]^{1/q}\right\}.$$

(2) *If $q = 1$, then*

$$\left|\lambda A(a^s, b^s) + (1-\lambda)A^s(a,b) - L_s^s(a,b)\right| \le \frac{(b-a)s}{s+1}\left(\frac{1}{2} - \lambda + \lambda^2\right)A\left(a^{s-1}, b^{s-1}\right). \tag{4.4}$$

From Corollary 3.17, it follows that

**Theorem 4.4.** *Let $b > a > 0$, $q \ge 1$, $0 < s \le 2$, and $0 \le \lambda \le 1$.*

(1) *If $q > 1$ and $-1 < (s-1)q \le 1$, then*

$$\left|\lambda A(a^s, b^s) + (1-\lambda)A^s(a,b) - L_s^s(a,b)\right| \le \frac{(b-a)s}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s}\right)^{1/q} \left[\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}\right]^{1-1/q}\left\{\left[a^{(s-1)q} + A^{(s-1)q}(a,b)\right]^{1/q} + \left[A^{(s-1)q}(a,b) + b^{(s-1)q}\right]^{1/q}\right\}.$$

(2) *If $q = 1$, then*

$$\left|\lambda A(a^s, b^s) + (1-\lambda)A^s(a,b) - L_s^s(a,b)\right| \le \frac{b-a}{2}\left(\frac{1}{2} - \lambda + \lambda^2\right)\left[A\left(a^{s-1}, b^{s-1}\right) + A^{s-1}(a,b)\right]. \tag{4.5}$$

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