Journal of Nonlinear and Convex Analysis
Volume 16, Number 5, 2015, 891–906

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

NEW CONVERGENCE CONDITIONS FOR THE SECANT METHOD

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Abstract. In this paper, we present new sufficient convergence conditions for the Secant method to a locally unique solution of a nonlinear equation in a Banach space. Our error bounds are finer than the earlier estimates. Some applications and examples for the main results are also provided.

1. Introduction

In this paper, we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a continuous operator defined on an open convex subset $\mathcal{D}$ of a Hilbert space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

Many problems in applied sciences can be expressed through the preceding equation. The solution methods for these equations are usually iterative. We note that, in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

We consider the Secant method (**SM**) in the following form:

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \quad (n \geq 0,\ x_{-1}, x_0 \in \mathcal{D}), \tag{1.2}$$

where $\delta F(x, y) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ for all $x, y \in \mathcal{D}$ is a consistent approximation of the Fréchet derivative of F ([4,11]). $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. The (**SM**) is an alternative to the well-known Newton method (**NM**):

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0,\ x_0 \in \mathcal{D}).$$

Argyros [4], Bosarge and Falb [8], Dennis [9], Hernández et al. [10], Potra [15]–[17] and others [5], [11], [14], [21] have provided sufficient convergence conditions for the (**SM**) based on Lipschitz–type conditions on δF (see also relevant works in [1, 2, 6, 13, 18, 19, 20, 22]).

In the previous mentioned studies, the conditions usually associated with the semilocal convergence of Secant method (1.2) are as follows:

- (H_1) F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;

2010 *Mathematics Subject Classification.* 65H10, 65B05, 65G99, 65N30, 47H17, 49M15.

Key words and phrases. Recurrent functions, semilocal convergence, Secant method, Banach space, majorizing sequence, divided difference, Fréchet–derivative.

This paper was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant Number: 2012-0008170).

(H_2) x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality

$$\| x_0 - x_{-1} \| \leq c;$$

(H_3) F is Fréchet–differentiable on $\mathcal{D}^0$ and there exists an operator $\delta F : \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that the linear operator $\mathcal{A} = \delta F(x_{-1}, x_0)$ is invertible, its inverse $\mathcal{A}^{-1}$ is bounded and

$$\| \mathcal{A}^{-1} F(x_0) \| \leq \eta;$$

$$\| \mathcal{A}^{-1} (\delta F(x, y) - F'(z)) \| \leq \ell (\| x - z \| + \| y - z \|);$$

$$\overline{U}(x_0, r) = \{x \in \mathcal{X} : \| x - x_0 \| \leq r\} \subseteq \mathcal{D}^0$$

for all $x, y, z \in \mathcal{D}$ and for some $r > 0$ depending on ℓ, c, η and

$$\ell\, c + 2 \sqrt{\ell\, \eta} \leq 1. \tag{1.3}$$

The sufficient convergence condition (1.3) is easily violated. Indeed, let $\ell = 1$, $\eta = 0.18$ and $c = 0.185$. Then (1.3) does not holds since

$$\ell\, c + 2 \sqrt{\ell\, \eta} = 1.033528137.$$

Moreover, our recently found corresponding conditions are also violated [7] (see Remark 2.4(c)). Hence there is no guarantee that the equation (1.1) under the information (ℓ, c, η) has a solution that can be found using the (**SM**).

In this paper, we are motivated by optimization considerations and the above observation. Here, using a combination of the Lipschitz and center–Lipschitz conditions, we provide a semilocal convergence analysis for the (**SM**). Our error bounds are tighter and our convergence conditions hold in cases where the corresponding hypotheses in the earlier results ([9, 10, 13, 17, 15, 17, 20, 22]) are violated. Also, some applications and examples are also provided.

2. Semilocal convergence analysis of the (SM)

We need the following result on the majorizing sequence for (**SM**) (1.2).

Lemma 2.1. *Let $l_0 > 0$, $l > 0$, $c > 0$ and $\eta > 0$ be given constants. Assume that*

$$l_0(c + \eta) < 1 \tag{2.1}$$

and, for

$$\delta = \frac{1}{2}\, \frac{-l + \sqrt{l^2 + 4\, l\, l_0}}{l_0}, \tag{2.2}$$

$$\max\left\{ \frac{\eta}{c}, \frac{l(c + \eta)}{1 - l_0(c + \eta)} \right\} \leq \delta \leq \frac{1 - l_0 c}{1 + l_0 c}. \tag{2.3}$$

Then the scalar sequence $\{t_n\}$ $(n \geq -1)$ given by

$$t_{-1} = 0, \quad t_0 = c, \quad t_1 = c + \eta,$$

$$t_{n+2} = t_{n+1} + \frac{l(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}{1 - l_0(t_{n+1} - t_0 + t_n)} \tag{2.4}$$

is non-decreasing, bounded from above by

$$t^{\star\star} = \frac{c}{1-\delta}, \tag{2.5}$$

and converges to its unique least upper bound $t^{\star}$ *such that*

$$0 \le t^{\star} \le t^{\star\star}. \tag{2.6}$$

Moreover, the following estimates hold

$$0 \le t_{n+2} - t_{n+1} \le \delta(t_{n+1} - t_n) \le \delta^{n+1} c \quad (n \ge -1) \tag{2.7}$$

and

$$t^{\star} - t_n \le \frac{\delta^n c}{1-\delta} \quad (n \ge 0). \tag{2.8}$$

Proof. From (2.1) and (2.3), we obtain $\delta \in [0,1)$. Now, we show, using mathematical induction on $k \ge -1$,

$$0 \le t_{k+1} - t_k \le \delta\ (t_k - t_{k-1}). \tag{2.9}$$

By (2.4) for $k = 0$, we must show

$$0 \le t_1 - t_0 \le \delta(t_0 - t_{-1}), \quad 0 \le \frac{l(t_1 - t_{-1})}{1 - l_0\ t_1} \le \delta,$$

which are true from (2.1) and (2.3).

Assume that (2.9) holds for $k \le n+1$. The induction hypothesis yields

$$\begin{aligned} t_{k+2} &\le t_{k+1} + \delta(t_{k+1} - t_k) \\ &\le t_k + \delta(t_k - t_{k-1}) + \delta(t_{k+1} - t_k) \\ &\le t_1 + \delta(t_1 - t_0) + \cdots + \delta(t_{k+1} - t_k) \\ &\le c + \delta\, c + \delta^2\, c + \cdots + \delta^{k+2}\, c \\ &= \frac{1-\delta^{k+3}}{1-\delta} c \\ &< \frac{c}{1-\delta} \\ &= t^{\star\star}. \end{aligned} \tag{2.10}$$

We must have

$$l(t_{k+2} - t_k) + \delta\, l_0\, (t_{k+2} - t_0 + t_{k+1}) \le \delta$$

or

$$l(\delta^{k+2} + \delta^{k+1})c + \frac{\delta\, l_0}{1-\delta}\left(2 - \delta^{k+1} - \delta^{k+2}\right) c - \delta l_0 c \le \delta \tag{2.11}$$

or

$$\begin{aligned} l(\delta^{k+1} + \delta^k)c + l_0\left(\left(1 + \delta + \cdots + \delta^{k+2}\right) + \left(1 + \delta + \cdots + \delta^{k+1}\right) - 1\right) c - 1 \\ \le 0. \end{aligned} \tag{2.12}$$

From the above inequality, we are motivated to define (for $\delta = s$) the functions as follows: for all $k \geq 0$ on $[0, 1)$,

$$f_k(s) = l(s^{k+1} + s^k)c + l_0\left(2\left(1 + s + \cdots + s^{k+1}\right) + s^{k+2} - 1\right)c - 1. \tag{2.13}$$

We need a relationship between two consecutive functions f_k. From the preceding equation, we obtain

$$f_{k+1}(s) = g(s)s^k c + f_k(s), \tag{2.14}$$

where

$$g(s) = l_0 s^3 + (l_0 + l)s^2 - l. \tag{2.15}$$

Note that δ (given by (2.2)) is the unique positive root of the polynomial g. Instead of (2.11), it suffices to show

$$f_k(\delta) \leq 0 \quad (k \geq 0). \tag{2.16}$$

But, in view of (2.2), (2.3), (2.11), (2.14) and (2.15), we have

$$\begin{aligned} f_0(\delta) = f_1(\delta) = \cdots = f_k(\delta) = \cdots = f_\infty(\delta) = \lim_{k\to\infty} f_k(\delta) \\ = c\,l_0\left[\frac{2}{1-\delta} - 1\right] - 1 < 0, \end{aligned} \tag{2.17}$$

which shows (2.16) for all $k \geq 0$. Hence we showed that the sequence $\{t_n\}$ $(n \geq -1)$ is non-decreasing and bounded from above by $t^{\star\star}$, so that the estimate (2.7) holds. It follows that there exists $t^\star \in [0, t^{\star\star}]$ such that $\lim_{n\to\infty} t_n = t^\star$. The estimate (2.8) follows from (2.7) by using the standard majorization techniques ([4, 5, 11]). This completes the proof. □

Additionally, we provide the following alternative to Lemma 2.1:

Lemma 2.2. *Let $l_0 > 0$, $l > 0$, $c > 0$ and $\eta > 0$ be given constants. Assume that*

$$(\ell_0 + \ell)c < 1 \tag{2.18}$$

and

$$\max\left\{\frac{\eta}{c}, \frac{\ell(c+\eta)}{1 - \ell_0(c+\eta)}\right\} \leq \delta_+ \leq \delta, \tag{2.19}$$

where δ_+ is the only positive root, which is given as follows:

$$\delta_+ = \frac{-(\ell + 2\ell_0)c + \sqrt{((\ell + 2\ell_0)c)^2 + 4\ell_0\, c(1 - (\ell + \ell_0)c)}}{2\ell_0 c}, \tag{2.20}$$

of the polynomial

$$f_0(s) = \ell_0 c s^2 + (2\ell_0 + \ell)cs + (\ell + \ell_0)c - 1. \tag{2.21}$$

Then the conclusions of the Lemma 2.1 hold with δ_+ replacing δ.

Proof. Following the proof of Lemma 2.1, but with δ_+ replaced by δ, we must show the following (instead of (2.16)):

$$f_k(\delta_+) \leq 0 \quad (k \geq 0). \tag{2.22}$$

The estimate (2.22) holds for $k = 0$ by the choice of δ_+ as the equality. From (2.14) and (2.19), we obtain

$$f_1(\delta_+) = f_0(\delta_+) + g(\delta_+)c = 0 + g(\delta_+)c = g(\delta_+)c \leq 0.$$

Assume that (2.22) holds for all $m \leq k$. Then, again by (2.14), we obtain

$$f_{k+1}(\delta_+) = f_k(\delta_+) + g(\delta_+)\delta_+^k c \leq 0 + 0 = 0,$$

which completes the induction for (2.22). Note also that

$$f_\infty(\delta_+) = \lim_{k\to\infty} f_k(\delta_+) \leq \lim_{k\to\infty} 0 = 0.$$

This completes the proof. □

We can also show the following result about the convergence order $p = \frac{(1+\sqrt{5})}{2}$ of the majorizing sequence $\{t_n\}$:

Lemma 2.3. *Under the hypotheses of Lemma 2.1, further assume*

$$q = \alpha\, c < 1, \tag{2.23}$$

where

$$\alpha = \frac{\ell(1+\delta)}{1 - \ell_0 c\left(\dfrac{1+\delta}{1-\delta}\right)}. \tag{2.24}$$

Then the following estimates hold:

$$t_n - t_{n-1} \leq q^{\theta_{(n-1)}-1}\, \eta \quad (n \geq 2) \tag{2.25}$$

and

$$t^\star - t_n \leq e_n \eta \quad (n \geq 1), \tag{2.26}$$

where

$$p = \frac{1+\sqrt{5}}{2}, \tag{2.27}$$

$$e_n = \frac{\left(q^{1/\sqrt{5}}\right)^{p^n}}{q\left(1 - q^{p^n(p-1)/\sqrt{5}}\right)} \tag{2.28}$$

and $\{\theta_n\}$ is Fibonacci's sequence given as follows:

$$\theta_0 = \theta_1 = 1, \quad \theta_{n+1} = \theta_n + \theta_{n-1} \quad (n \geq 1). \tag{2.29}$$

Proof. It follows from (2.4), (2.5), (2.11) and (2.24) that

$$t_{n+2} - t_{n+1} \le \alpha(t_n - t_{n-1})(t_{n+1} - t_n). \tag{2.30}$$

From the initial conditions and (2.29), the estimate (2.25) holds for $n = 0$. Assume that the estimate (2.25) holds for all $k \le n - 1$. Then, using the induction hypotheses and (2.30), we obtain

$$\begin{aligned} t_{k+1} - t_k &\le q \cdot q^{\theta_{(k-2)}-1}(t_k - t_{k-1}) \\ &= q^{\theta_{(k-1)}}(t_k - t_{k-1}) \\ &\le q^{\theta_{(k-2)}} q^{\theta_{(k-1)}} \eta \\ &= q^{\theta_k}\eta, \end{aligned} \tag{2.31}$$

which shows that (2.25) holds for all n. We also have

$$\begin{aligned} t_{n+1} - t_1 &= (t_2 - t_1) + (t_3 - t_2) + \cdots + (t_{k+1} - t_k) \\ &\le \frac{\eta}{q}\left(q^{\theta_1} + q^{\theta_2} + \cdots + q^{\theta_k}\right) \\ &< t_0^\star = \frac{\eta}{q}\sum_{k=1}^{\infty} q^{\theta_k}. \end{aligned} \tag{2.32}$$

Moreover, we note that

$$\theta_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \ge \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k} = \frac{p^k}{\sqrt{5}}. \tag{2.33}$$

Consequently, for all $m \ge 1$, we have

$$\begin{aligned} t_{k+m} - t_k &= (t_{k+1} - t_k) + (t_{k+2} - t_{k+1}) + \cdots + (t_{k+m} - t_{k+m-1}) \\ &\le \frac{\eta}{q}\left(q^{\theta_k} + q^{\theta_{k+1}} + \cdots + q^{\theta_{k+m-1}}\right) \\ &\le \frac{\eta}{q}\left(q^{p^k/\sqrt{5}} + q^{p^{k+1}/\sqrt{5}} + \cdots + q^{p^{k+m-1}/\sqrt{5}}\right). \end{aligned} \tag{2.34}$$

Using Bernoulli's inequality and the preceding inequality, we obtain

$$\begin{aligned} t_{k+m} - t_k &\le \frac{\eta}{q} q^{p^k/\sqrt{5}}\left(1 + q^{\frac{p^{k+1}-p^k}{\sqrt{5}}} + \cdots + q^{\frac{p^{k+m-1}-p^k}{\sqrt{5}}}\right) \\ &\le \frac{\eta}{q} q^{p^k/\sqrt{5}}\left(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(1+(m-1)(p-1)-1)}{\sqrt{5}}}\right) \\ &= \frac{\eta}{q} q^{p^k/\sqrt{5}}\left(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + \cdots + \left[q^{\frac{p^k(p-1)}{\sqrt{5}}}\right]^{m-1}\right) \\ &= \frac{\eta}{q} q^{p^k/\sqrt{5}}\left[\frac{1 - q^{\frac{p^k(p-1)m}{\sqrt{5}}}}{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}}\right], \end{aligned} \tag{2.35}$$

which shows (2.26) if we let $m \to \infty$. This completes the proof. □

Similarly, using the hypotheses of Lemma 2.2, we have the following:

Lemma 2.4. *Under the hypotheses of Lemma 2.2, further, assume that* (2.23) *holds, but δ is replaced by δ_+ in* (2.24). *Then the conclusions of Lemma 2.3 hold.*

Next, we study the (**SM**) for triplets (F, x_{-1}, x_0) belonging to the class $\mathcal{C} = \mathcal{C}(\ell_0, \ell, \eta, c, \delta)$ as follows:

Definition 2.5. Let $\ell_0, \ell, \eta, c, \delta$ be non-negative constants satisfying the hypotheses of Lemma 2.1, Lemma 2.2 or Lemma 2.3 or Lemma 2.4. A triplet (F, x_{-1}, x_0) belongs to the class $\mathcal{C}(\ell_0, \ell, \eta, c, \delta)$ if

($\mathcal{A}_1$) F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;

($\mathcal{A}_2$) x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality

$$\| x_0 - x_{-1} \| \leq c;$$

($\mathcal{A}_3$) F is Fréchet–differentiable on $\mathcal{D}^0$ and there exists an operator $\delta F : \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{A}^{-1} = \delta F(x_{-1}, x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and, for all $x, y, z \in \mathcal{D}$,

$$\| \mathcal{A}^{-1} F(x_0) \| \leq \eta,$$

$$\| \mathcal{A}^{-1} (\delta F(x, y) - F'(z)) \| \leq \ell \, (\| x - z \| + \| y - z \|),$$

$$\| \mathcal{A}^{-1} (\delta F(x, y) - F'(x_0)) \| \leq \ell_0 \, (\| x - x_0 \| + \| y - x_0 \|);$$

($\mathcal{A}_4$)

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D}_c = \{x \in \mathcal{D} \,:\, F \text{ is continuous at } x\} \subseteq \mathcal{D},$$

where $t^\star$ is given in Lemma 2.1.

The semilocal convergence theorem for (**SM**) is as follows:

Theorem 2.6. *If $(F, x_{-1}, x_0) \in \mathcal{C}(l, l_0, \eta, c, \delta)$, then the sequence $\{x_n\}$ $(n \geq -1)$ generated by the* (***SM***) *is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of the equation* (1.1). *Moreover, the following estimates hold: for all $n \geq 0$,*

$$\| x_n - x_{n-1} \| \leq t_n - t_{n-1} \tag{2.36}$$

and

$$\| x_n - x^\star \| \leq t^\star - t_n, \tag{2.37}$$

where $\{t_n\}$ $(n \geq 0)$ is given by (2.4). *Furthermore, if there exists a number R such that*

$$\overline{U}(x_0, R) \subseteq \mathcal{D}, \quad R \geq t^\star - c,$$

$$\ell_0 \, (t^\star - c + R) + \left\| \mathcal{A}^{-1}(F^{-1}(x_0) - \mathcal{A}) \right\| \leq 1, \tag{2.38}$$

then the solution $x^\star$ is unique in $U(x_0, R)$.

Proof. First, we show that $\mathcal{L} = \delta F(x_k, x_{k+1})$ is invertible for $x_k, x_{k+1} \in \overline{U}(x_0, t^\star)$. By (2.11), $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, we have

$$\begin{aligned} \| I - \mathcal{A}^{-1}\mathcal{L} \| &= \| \mathcal{A}^{-1}(\mathcal{L} - \mathcal{A}) \| \\ &\le \| \mathcal{A}^{-1}(\mathcal{L} - F'(x_0)) \| + \| \mathcal{A}^{-1}(F'(x_0) - \mathcal{A}) \| \\ &\le \ell_0 (\| x_k - x_0 \| + \| x_{k+1} - x_0 \| + \| x_0 - x_{-1} \|) \\ &\le \ell_0 (t_k - t_0 + t_{k+1} - t_0 + c) < 1. \end{aligned} \tag{2.39}$$

Using the Banach Lemma on invertible operators [4, 12] and (2.39), $\mathcal{L}$ is invertible and

$$\| \mathcal{L}^{-1}\mathcal{A} \| \le \Big(1 - \ell_0 (t_{k+1} - t_k - t_0)\Big)^{-1}. \tag{2.40}$$

By $(\mathcal{A}_3)$, we have

$$\| \mathcal{A}^{-1}(F'(u) - F'(v)) \| \le 2\,\ell\, \| u - v \| \quad (u, v \in \mathcal{D}^0). \tag{2.41}$$

We can write the identity

$$F(x) - F(y) = \int_0^1 F'(y + t(x - y))\,\mathrm{d}t\,(x - y) \tag{2.42}$$

and then, for all $x, y, u, v \in \mathcal{D}^0$, we obtain

$$\begin{aligned} &\| \mathcal{A}^{-1}(F(x) - F(y) - F'(u)(x - y)) \| \\ &\quad \le \ell\,(\| x - u \| + \| y - u \|)\ \| x - y \| \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} &\| \mathcal{A}^{-1}(F(x) - F(y) - \delta F(u, v)\,(x - y)) \| \\ &\quad \le \ell\,(\| x - v \| + \| y - v \| + \| u - v \|)\ \| x - y \|\,. \end{aligned} \tag{2.44}$$

By a continuity argument, (2.41)–(2.44) remain valid if x and/or y belong to $\mathcal{D}_c$.

Now, we show (2.36). If (2.36) holds for all $n \le k$ and $\{x_n\}$ $(n \ge 0)$ is well defined for $n = 0, 1, 2, \cdots, k$, then we have

$$\| x_n - x_0 \| \le t_n - t_0 < t^\star - t_0 \quad (n \le k). \tag{2.45}$$

That is, (**??**) is well defined for $n = k + 1$. For $n = -1$ and $n = 0$, (2.36) reduces to $\| x_{-1} - x_0 \| \le c$ and $\| x_0 - x_1 \| \le \eta$. Suppose that (2.36) holds for $n = -1, 0, 1, \cdots, k$ $(k \ge 0)$. By (2.40), (2.44) and

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k)\ (x_{k+1} - x_k), \tag{2.46}$$

we obtain the following estimate:

$$\begin{aligned} \| x_{k+2} - x_{k+1} \| &= \| \delta F(x_k, x_{k+1})^{-1} F(x_{k+1}) \| \\ &\le \| \delta F(x_k, x_{k+1})^{-1} \mathcal{A} \| \, \| \mathcal{A}^{-1} F(x_{k+1}) \| \\ &\le \frac{\ell\,(\| x_{k+1} - x_k \| + \| x_k - x_{k-1} \|)}{1 - \ell_0 (\| x_{k+1} - x_0 \| + \| x_k - x_0 \| + c)} \| x_{k+1} - x_k \| \\ &\le \frac{\ell\,(t_{k+1} - t_k + t_k - t_{k-1})}{1 - \ell_0 (t_{k+1} - t_0 + t_k - t_0 + t_0 - t_{-1})} (t_{k+1} - t_k) \\ &= t_{k+2} - t_{k+1} \end{aligned} \tag{2.47}$$

and the induction for (2.36) is completed. It follows from (2.36) and Lemma 2.1 that $\{x_n\}$ $(n \geq -1)$ is a Cauchy sequence in a Banach space $\mathcal{X}$ and and so it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \to \infty$ in (2.47), we obtain $F(x^\star) = 0$. The estimate (2.37) follows from (2.36) by using standard majoration techniques [4, 6, 12].

Finally, to show the uniqueness in $\overline{U}(x_0, R)$, let $y^\star \in \overline{U}(x_0, R)$ be a solution of the equation (1.1). Set

$$\mathcal{M} = \int_0^1 F'(y^\star + t\ (y^\star - x^\star))\ \mathrm{d}t.$$

Then it follows from $(\mathcal{A}_3)$ and (2.38) that

$$\begin{aligned} \parallel \mathcal{A}^{-1}\,(\mathcal{A} - \mathcal{M})\parallel &= \ell_0\,(\parallel y^\star - x_0 \parallel + \parallel x^\star - x_0 \parallel) + \parallel \mathcal{A}^{-1}(F'(x_0) - \mathcal{A})\parallel \\ &\leq \ell_0\,[(t^\star - t_0) + R] + \parallel \mathcal{A}^{-1}(F'(x_0) - \mathcal{A})\parallel \qquad (2.48) \\ &< 1. \end{aligned}$$

It follows from (2.48) and the Banach lemma on invertible operators that $\mathcal{M}^{-1}$ exists on $U(x_0, t^\star)$. Using the identity:

$$F(x^\star) - F(y^\star) = \mathcal{M}\ (x^\star - y^\star), \qquad (2.49)$$

we deduce $x^\star = y^\star$. This completes the proof. □

Remark 2.7. (1) The point $t^{\star\star}$ given in closed form by (2.5) can replace $t^\star$ in the hypotheses of Theorem 2.6.

(2) In the uniqueness part (see (2.38)), we can replace $\|\mathcal{A}^{-1}(F'(x_0) - \mathcal{A})\|$ by the less tight $\ell_0 c$ since, by $(\mathcal{A}_3)$,

$$\|\mathcal{A}^{-1}(F'(x_0) - \mathcal{A})\| \leq l_0\|x_0 - x_{-1}\| \leq l_0\, c. \qquad (2.50)$$

In fact, according to (2.39), the majorizing sequence $\{t_n\}$ can be replaced by the following iteration, which is at least as tight as the sequence defined by (2.4):

$$\begin{aligned} &t_1 = 0, \quad t_0 = c, \quad t_1 = c + \eta, \\ &t_{n+2} = t_{n+1} + \frac{\ell(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}{1 - (L + \ell_0(t_{n+1} + t_n - 2c))}. \qquad (2.51) \end{aligned}$$

This is also a majorizing sequence for $\{x_n\}$ (converging under the same hypotheses to some $\overline{t}^\star \leq t^\star$).

(3) A more popular hypotheses used, instead of the first inequality in $(\mathcal{A}_3)$, is

$$\|\mathcal{A}^{-1}\,(\delta F(x, y) - \delta F(u, v))\| \leq \bar{l}\,(\|x - u\| + \|y - v\|) \qquad (2.52)$$

for all $x, y, u, v \in \mathcal{D}$. Note that (2.52) implies both the hypotheses in $(\mathcal{A}_3)$, but not necessarily vice versa. Note also that

$$\ell_0 \leq \overline{\ell}, \quad \ell \leq \overline{\ell}$$

and $\overline{\ell}/\ell_0$, $\overline{\ell}/\ell$ can be arbitrarily large ([1-6]).

(4) Our sufficient convergence conditions differ from the ones in [2-10, 12-16, 19, 20].

Remark 2.8. (1) Let us define the majorizing sequence $\{w_n\}$, which are used in [9, 10, 13, 15, 17, 20, 22] (under the condition (1.3)):

$$w_{-1} = 0,\ w_0 = c,\ w_1 = c + \eta,$$

$$w_{n+2} = w_{n+1} + \frac{\ell\,(w_{n+1} - w_{n-1})\,(w_{n+1} - w_n)}{1 - \ell\,(w_{n+1} - w_0 + w_n)}. \tag{2.53}$$

Note that, in general,

$$\ell_0 \leq \ell \tag{2.54}$$

holds and ℓ/ℓ_0 can be arbitrarily large. In the case, $\ell_0 = \ell$ and then $t_n = w_n$ $(n \geq -1)$. Otherwise, a simple inductive argument shows ([4, 6]) that

$$t_n < w_n, \quad t_{n+1} - t_n \leq w_{n+1} - w_n, \tag{2.55}$$

$$0 \leq t^\star - t_n \leq w^\star - w_n, \quad w^\star = \lim_{n \longrightarrow \infty} w_n. \tag{2.56}$$

Note also that the strict inequality holds in (2.55) for all $n \geq 1$, if $\ell_0 < \ell$.

Note that the only difference in the proofs is that the conditions of Lemma 2.1 are used here, instead of the ones in [4, 6]. However, this makes no difference between the proofs.

Finally, note that (1.3) is the sufficient convergence condition for the sequence (2.34).

(2) It turns out from the proof of Theorem 2.6 that the sequences $\{v_n\}$ given by

$$v_{-1} = 0,\ v_0 = c,\ v_1 = c + \eta,$$

$$v_{n+2} = v_{n+1} + \frac{\ell_1\,(v_{n+1} - v_{n-1})\,(v_{n+1} - v_n)}{1 - \ell_0\,(v_{n+1} - v_0 + v_n)}, \tag{2.57}$$

where

$$\ell_1 = \begin{cases} \ell_0 & \text{if} \quad n = 0, \\ \ell & \text{if} \quad n > 0 \end{cases}$$

is a finer majorizing sequence for $\{x_n\}$ than $\{t_n\}$ if $\ell_0 < \ell$.

Moreover, we have

$$v_n < t_n, \quad v_{n+1} - v_n < t_{n+1} - t_n, \tag{2.58}$$

$$0 \leq v^\star - v_n \leq t^\star - t_n, \quad v^\star = \lim_{n \longrightarrow \infty} v_n. \tag{2.59}$$

We also have the following useful extension of Lemma 2.1:

Lemma 2.9. *Let $N = 0, 1, 2, \ldots$ be fixed. Assume that*

$$t_{-1} \leq t_0 \leq t_1 \leq \cdots \leq t_N \leq t_{N+1},$$
$$\ell_0(t_{N+1} - t_{N-1}) < 1$$

and

$$\max\left(\frac{t_{N+1} - t_N}{t_N - t_{N-1}}, \frac{\ell(t_{N+1} - t_{N-1})}{1 - \ell_0(t_{N+1} - t_{N-1})}\right) \leq \delta \leq \frac{1 - \ell_0(t_N - t_{N-1})}{1 + \ell_0(t_N - t_{N-1})}.$$

Then the conclusions of Lemma 2.1 for the sequence $\{t_n\}$ hold with c replaced by $t_N - t_{N-1}$.

Remark 2.10. If $N = 0$, then Lemma 2.9 reduces to Lemma 2.1. Clearly, the hypotheses of Lemma 2.9 can replace the hypotheses of Lemma 2.1 in Theorem 2.6. Similarly, we can provide an extension of Lemma 2.2 or the extension using the tighter sequence $\{v_n\}$.

3. Examples

In this section, we present some numerical examples.

Example 3.1. In the following table, we validate (1) and (2) of Remark 2.8. The constants are selected as follows:

$$\ell = 1, \quad \ell_0 = 0.9, \quad c = 0.185, \quad \eta = 0.115.$$

The table shows that our error bounds $v_{n+1} - v_n$ and $t_{n+1} - t_n$ are finer than $w_{n+1} - w_n$ given in [9, 10, 13, 15, 17, 20, 22].

Let us validate the estimates (2.6), (2.7) and (2.8). From the equations (2.2) and (2.5), we get

$$\delta = 6{,}359\,784 \cdot 10^{-01}, \quad t^{\star\star} = 5{,}082\,116 \cdot 10^{-01}.$$

From the inequality (2.3), we get

$$4{,}109\,589 \cdot 10^{-01} \le \delta (= 6{,}359\,784 \cdot 10^{-01}) \le 7{,}145\,306 \cdot 10^{-01}.$$

Thus the inequality (2.3) is satisfied. To obtain $t^\star$, we use the value t_{20} (obtained from the sequence (2.4)), which is

$$t^\star \approx t_{20} = 3{,}619\,319 \cdot 10^{-01}.$$

Comparing the values of $t^\star$ and $t^{\star\star}$, we observe that the estimate (2.6) holds. We verify the estimates (2.7) and (2.8) through the Table 2. In the Table 2, we notice that the estimates (2.7) and (2.8) hold.

Example 3.2. Define the scalar function F by $F(x) = c_0\,x + c_1 + c_2\,\sin e^{c_3\,x}$, $x_0 = 0$, where c_i, $i = 0, 1, 2, 3$, are the given parameters. Define a linear operator $\delta F(x, y)$ by

$$\delta F(x, y) = \int_0^1 F'(y + t\,(x - y))\,\mathrm{d}t = c_0 + c_2\,\frac{\sin e^{c_3\,x} - \sin e^{c_3\,y}}{x - y}.$$

Then it can easily be seen that, for c_3 large and c_2 sufficiently small, $\dfrac{\ell}{\ell_0}$ can be arbitrarily large. That is, (2.3)) may be satisfied, but not (1.3).

Example 3.3 ([4])**. (Newton's method case)** Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be a space of real–valued continuous functions defined on the interval $[0, 1]$ equipped with the max–norm $\|\cdot\|$. Let $\theta \in [0, 1]$ be a given parameter. Consider the "Cubic" Chandrasekhar integral equation:

$$u(s) = u^3(s) + \lambda\,u(s) \int_0^1 q(s, t)\,u(t)\,\mathrm{d}t + y(s) - \theta. \tag{3.1}$$

Here, the kernel $q(s, t)$ is a continuous function of two variables defined on $[0, 1] \times [0, 1]$. The parameter λ in (3.1) is a real number called the "albedo" for scattering and $y(s)$ is a given continuous function defined on $[0, 1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0, 1]$. For the simplicity, we choose $u_0(s) = y(s) = 1$ and

$q(s,t) = \dfrac{s}{s+t}$ for all $s \in [0,1]$ and $t \in [0,1]$ with $s+t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$ and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, \mathrm{d}t + y(s) - \theta \tag{3.2}$$

for all $s \in [0,1]$, then every zero of F satisfies the equation (3.1). We have the estimates

$$\max_{0 \le s \le 1} \left| \int_0^1 \frac{s}{s+t}\, dt \right| = \ln 2.$$

Therefore, if we set $\xi = \| F'(u_0)^{-1} \|$, then the hypotheses of Theorem 2.6 (see $(\mathcal{A}_3)$) correspond to the usual Lipschitz and center–Lipschitz conditions for the (**NM**) (see [7, Theorem 3.4]) such that

$$\eta = \xi\, (|\lambda|\, \ln 2 + 1 - \theta),$$

$$\ell = 2\, \xi\, (|\lambda|\, \ln 2 + 3\,(2-\theta)) \quad and \quad \ell_0 = \xi\, (2\,|\lambda|\, \ln 2 + 3\,(3-\theta)).$$

It follows from an equivalent Theorem for the (NM) to Theorem 2.6 that, if the condition

$$h_A = \frac{1}{8} \left(\ell + 4\,\ell_0 + \sqrt{\ell^2 + 8\,\ell\,\ell_0} \right) \eta \le \frac{1}{2}$$

holds, then the problem (3.1) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis.

Note also that $\ell_0 < \ell$ for all $\theta \in [0,1]$.

Example 3.4. (Secant method case) Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ equipped with the norm $\| x \| = \max_{0 \le s \le 1} |x(s)|$. Consider the following nonlinear boundary value problem ([4]):

$$\begin{cases} u'' = -u^3 - \gamma\, u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation:

$$u(s) = s + \int_0^1 Q(s,t)\, (u^3(t) + \gamma\, u^2(t))\, dt, \tag{3.3}$$

where Q is the Green function:

$$Q(s,t) = \begin{cases} t\,(1-s), & t \le s, \\ s\,(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \le s \le 1} \int_0^1 |Q(s,t)|\, dt = \frac{1}{8}.$$

Then the problem (3.3) is in the form (1.1)), where $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is an operator defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 Q(s,t)\, (x^3(t) + \gamma\, x^2(t))\, dt.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v]\,(s) = v(s) - \int_0^1 Q(s,t)\ (3\ x^2(t) + 2\ \gamma\ x(t))\ v(t)\ dt.$$

Let

$$\delta F(x,y) = \int_0^1 F'(y + t\,(x-y))\,dt.$$

If we set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R)$, then, since $\parallel u_0 \parallel = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\ \gamma < 5$ and then (see [4])

$$\parallel I - F'(u_0) \parallel \leq \frac{3 + 2\ \gamma}{8}, \quad \parallel F'(u_0)^{-1} \parallel \leq \frac{8}{5 - 2\ \gamma},$$

$$\parallel F(u_0) \parallel \leq \frac{1+\gamma}{8}, \quad \parallel F(u_0)^{-1}\ F(u_0) \parallel \leq \frac{1+\gamma}{5 - 2\ \gamma}.$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v]\,(s) = -\int_0^1 Q(s,t)\,(3\,x^2(t) - 3\,y^2(t) + 2\,\gamma\,(x(t) - y(t)))\,v(t)\,\mathrm{d}t.$$

Consequently, we have (see [4])

$$\parallel F'(x) - F'(y) \parallel \leq \frac{\gamma + 6\,R + 3}{4}\ \parallel x - y \parallel,$$

$$\parallel F'(x) - F'(u_0) \parallel \leq \frac{2\,\gamma + 3\,R + 6}{8}\ \parallel x - u_0 \parallel .$$

Define a linear operator $\delta F(x,y)$ by

$$\delta F(x,y) = \int_0^1 F'(y + t\,(x-y))\,dt.$$

Then the conditions of Theorem 2.6 hold with

$$\eta = \frac{1+\gamma}{5 - 2\,\gamma}, \quad \ell = \frac{\gamma + 6\,R + 3}{8}, \quad \ell_0 = \frac{2\,\gamma + 3\,R + 6}{16}.$$

Note also that $\ell_0 < \ell$.

4. Conclusions

We provided new sufficient convergence conditions for the (**SM**) to a locally unique solution of a nonlinear equation in a Banach space. Using our new concept of recurrent functions and combining the Lipschitz and center–Lipschitz conditions on the divided difference operator, we obtained the semilocal convergence analysis of the (**SM**). Our error bounds are more precise than earlier ones and, under our convergence hypotheses, we can cover cases where earlier conditions are violated [9, 10, 13, 15, 17, 20, 22]. Applications and numerical examples are also provided in this study.

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Manuscript received March 10, 2013
revised April 21, 2015

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n	Sequence (2.4) $t_{n+1} - t_n$	Sequence (2.53) $w_{n+1} - w_n$	Sequence (2.57) $v_{n+1} - v_n$
-1	1.850000×10^{-01}	1.850000×10^{-01}	1.850000×10^{-01}
0	1.150000×10^{-01}	1.150000×10^{-01}	1.150000×10^{-01}
1	4.726027×10^{-02}	4.928571×10^{-02}	4.253425×10^{-02}
2	1.313170×10^{-02}	1.511429×10^{-02}	1.139133×10^{-02}
3	1.497413×10^{-03}	2.065204×10^{-03}	1.138225×10^{-03}
4	4.241644×10^{-05}	7.812470×10^{-05}	2.698940×10^{-05}
5	1.268085×10^{-07}	3.704647×10^{-07}	5.963361×10^{-08}
6	1.047497×10^{-11}	6.434798×10^{-11}	3.058827×10^{-12}
7	2.579352×10^{-18}	5.275975×10^{-17}	3.459238×10^{-19}
8	5.246109×10^{-29}	7.512485×10^{-27}	2.006539×10^{-30}

TABLE 1. Comparison among scalar sequences (2.4), (2.53) and (2.57).

n	$t_{n+2} - t_{n+1}$	$\delta\,(t_{n+1} - t_n)$	$\delta^{n+1}\,c$	$t^\star - t_n$	$\dfrac{\delta^n\,c}{1-\delta}$
-1	1.150000×10^{-01}	1.176560×10^{-01}	1.850000×10^{-01}	3.619319×10^{-01}	$4.319470 \times 10^{+00}$
0	4.726027×10^{-02}	7.313751×10^{-02}	1.176560×10^{-01}	1.769319×10^{-01}	$2.747089 \times 10^{+00}$
1	1.313170×10^{-02}	3.005651×10^{-02}	7.482667×10^{-02}	6.193193×10^{-02}	$1.747089 \times 10^{+00}$
2	1.497413×10^{-03}	8.351479×10^{-03}	4.758814×10^{-02}	1.467166×10^{-02}	$1.111111 \times 10^{+00}$
3	4.241644×10^{-05}	9.523225×10^{-04}	3.026503×10^{-02}	1.539957×10^{-03}	7.066426×10^{-01}
4	1.268085×10^{-07}	2.697594×10^{-05}	1.924790×10^{-02}	4.254326×10^{-05}	4.494094×10^{-01}
5	1.047497×10^{-11}	8.064746×10^{-08}	1.224125×10^{-02}	1.268190×10^{-07}	2.858147×10^{-01}
6	2.579352×10^{-18}	6.661853×10^{-12}	7.785170×10^{-03}	1.047497×10^{-11}	1.817719×10^{-01}
7	5.246109×10^{-29}	1.640412×10^{-18}	4.951200×10^{-03}	2.579352×10^{-18}	1.156030×10^{-01}
8	2.627373×10^{-46}	3.336412×10^{-29}	3.148856×10^{-03}	5.246109×10^{-29}	7.352102×10^{-02}

TABLE 2. Verification of the estimates (2.7) and (2.8).