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ON THE CONNECTEDNESS OF EFFICIENT SOLUTIONS FOR GENERALIZED KY FAN INEQUALITY PROBLEMS*

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Abstract. In this paper, we give a density theorem of efficient solution to generalized Ky Fan Inequality problem when f-solution is set-valued. By using the density result, we provide a sufficient condition for the connectedness of efficient solutions to the generalized Ky Fan Inequality problem without monotonicity. Our results are new and different from the corresponding ones in the literature. Some examples are given to illustrate our results.

1. Introduction

The Ky Fan Inequality is a very general mathematical format, which embraces the formats of several disciplines, as those for equilibrium problems of Mathematical Physics, those from Game Theory, those from (Vector) Optimization and (Vector) Variational Inequalities, and so on. It is well known that the famous Ky Fan inequality [8, 9] has proved to be fundamental for existence studies in many fields of mathematics, including optimization-related problems. Since then, it has been extended and generalized to vector-valued mappings. The Ky Fan Inequality for a vector valued mapping is known as the generalized Ky Fan Inequality. In the literature, existence results for various types of (generalized) Ky Fan Inequalities have been investigated intensively; see [3, 5, 6, 10, 11, 16, 20] and the references therein.

On the other hand, one of the most important problems for (generalized) Ky Fan Inequality is to investigate the properties of the solutions set. Among many desirable properties of the solutions set, the connectedness is of considerable interest, since it provides the possibility of continuously moving from one solution to any other solution. In [18], Lee et al. investigated the path-connectedness of the set of weakly efficient solutions and the set of efficient solutions for a class of Ky Fan Inequality in finite-dimensional spaces. In [7], Cheng discussed the connectedness of the set of weakly efficient solutions for a class of Ky Fan Inequality in finite-dimensional spaces by using scalarization method. In [12], Gong obtained the connectedness of the set of Henig efficient solutions and the set of weak efficient solutions to the vector-valued Hartman-Stampacchia variational inequality in normed spaces by using scalarization method. Recently, Gong [13] introduced the

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concepts of f-efficient solution, Henig efficient solution, globally efficient solution, weakly efficient solution, superefficient solution and cone-Benson efficient solution to Ky Fan Inequalities and gave some scalarization characterizations for various proper efficient solutions. By using the scalarization results, he investigated the connectedness of the Henig efficient solutions set, globally efficient solutions set, weakly efficient solutions set, superefficient solutions set and cone-Benson efficient solutions set for Ky Fan Inequalities in locally convex spaces. Very recently, Gong and Yao [14] introduced the concept of positive proper efficient solutions to a class of vector Ky Fan Inequality. They showed that, under some suitable conditions, the set of positive proper efficient solutions is dense in the set of efficient solutions to the vector Ky Fan Inequality. By virtue of the density result, they first discussed the connectedness of the set of efficient solutions for the Ky Fan Inequality with monotone bifunctions in real locally convex Hausdorff topological vector spaces.

In above mentioned works, the monotonicity plays an important role in deriving the connectedness of the sets of various (proper) efficient solutions to (generalized) Ky Fan Inequalities. We also observed that the density result and connectedness theorem of (weak) efficient solutions has been established under the critical assumption of C-strict/strong monotonicity, which implies that the f-solution set of the (Generalized) Ky Fan Inequalities ((G)KFIs, in short) is a singleton (eg. [13, 14]). However, it is well known that the f-solution set of the (G)KFI may be general, but not a singleton. In this paper, without assumption of monotonicity, we obtain a density result of positive proper efficient solutions for a class of GKFI. Then, by using density result, we discuss the connectedness of the efficient solutions set for the GKFI in metric spaces when the f-solution set is set-valued. Our main results extend and improve the corresponding ones of Gong [13, 14].

The rest of the paper is organized as follows. In Sect. 2, we introduce a class of generalized Ky Fan Inequality, and recall some concepts and their properties. In Sect. 3, we first give the density theorem of positive proper efficient solution sets to GKFI under the case that f-solution is a general set. Then, we discuss the connectedness of efficient solution mappings to the GKFI in metric spaces, and compare our main results with the corresponding ones in the recent literature ([13, 14, 21]). We also give some examples to illustrate our results. The final short Sect. 4 includes some concluding remarks.

2. Preliminaries

Throughout this paper, if not otherwise specified, $d(\cdot,\cdot)$ denote the metric in any metric space. Let $B(0,\delta)$ denote the open ball with radius $\delta > 0$ and center 0 in any metric linear spaces. Let X and Y be two real linear metric spaces. Let Y^* be the topological dual space of Y, and C be a closed, convex and pointed cone in Y with nonempty topological interior intC.

Let

$$C^* := \{f \in Y^* : f(y) \geq 0,\ \forall y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e.,

$$C^\sharp := \{f \in Y^* : f(y) > 0,\ \forall y \in C \setminus \{0\}\},$$

and assume $C^\sharp \neq \emptyset$ in the paper.

Let A be a nonempty subset of X and $F : A \times A \to Y$ be a vector-valued mapping. We consider the following generalized Ky Fan Inequality (GKFI) of finding $x \in A$ such that

$$\text{(GKFI)} \qquad F(x, y) \notin -K, \text{ for all } y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y.

Definition 2.1 ([12])**.** A vector $x \in A$ is called a weak efficient solution to the *(GKFI)*, iff

$$F(x, y) \notin -\text{int}C, \text{ for all } y \in A.$$

The set of the weak efficient solutions to the (GKFI) is denoted by $V_w(A, F)$.

Definition 2.2 ([12, 13])**.** Let $f \in C^* \setminus \{0\}$. A vector $x \in A$ is called a f-solution to the *(GKFI)*, iff

$$f(F(x, y)) \geq 0, \ \forall y \in A.$$

The set of the f-solutions to the (GKFI) is denoted by $V_f(A, F)$.

Definition 2.3 ([14])**.** (i) A vector $x \in A$ is called a efficient solution to the *(GKFI)*, iff

$$F(x, y) \notin -C \setminus \{0\}, \ \forall y \in A.$$

The set of the efficient solutions to the *(GKFI)* is denoted by $V(A, F)$.

(ii) A vector $x \in A$ is called a positive proper efficient solution to the *(GKFI)* if there exists $f \in C^\sharp$ such that

$$f(F(x, y)) \geq 0, \ \forall y \in A.$$

Special case:

If for any $x, y \in A$, $F(x, y) := \varphi(x, y) + \psi(y) - \psi(x)$, where $\varphi : A \times A \to Y$ and $\psi : A \to Y$ are two vector-valued maps, the (GKFI) reduces to the vector equilibrium problem (VEP) considered in [12, 13, 14, 21].

Throughout this paper, we always assume $V(A, F) \neq \emptyset$ and $V_f(A, F) \neq \emptyset$ in A. This paper aims at investigating the connectedness of efficient solutions for (GKFI).

Now we recall some basic definitions and their properties which are needed in this paper.

Definition 2.4. Let $F : X \times X \to Y$ be a vector-valued mapping.

(i) $F(\cdot, \cdot)$ is called C-monotone on $A \times A$, iff for each $x, y \in A$ $F(x, y) + F(y, x) \in -C$.

(ii) $F(\cdot, \cdot)$ is called C-strongly monotone (i.e., C-strictly monotone in [13]) on $A \times A$, iff F is C-monotone on $A \times A$, and for any each $x, y \in A$ with $x \neq y$, $F(x, y) + F(y, x) \in -intC$.

(iii) $F(x, \cdot)$ is called C-convex(C-concave) on convex set A if, for each $x_1, x_2 \in A$ and $t \in [0, 1], tF(x, x_1) + (1-t)F(x, x_2) \in F(x, tx_1 + (1-t)x_2) + C(F(x, tx_1 + (1-t)x_2) \in tF(x, x_1) + (1-t)F(x, x_2) + C)$.

(iv) A set $D \subset Y$ is called a C-convex set, iff $D + C$ is a convex set in Y.

Definition 2.5 ([1, 2])**.** Let $F : \Lambda \rightrightarrows X$ be a set-valued mapping, and given $\bar{\lambda} \in \Lambda$.

(i) F is called lower semicontinuous(l.s.c, in short) at $\bar{\lambda}$, iff for any open set V satisfying $V \bigcap F(\bar{\lambda}) \neq \emptyset$, there exists $\delta > 0$, such that for every $\lambda \in B(\bar{\lambda}, \delta)$, $V \bigcap F(\lambda) \neq \emptyset$.

(ii) F is called upper semicontinuous(u.s.c, in short) at $\bar{\lambda}$, iff for any open set V satisfying $F(\bar{\lambda}) \subset V$, there exists $\delta > 0$, such that for every $\lambda \in B(\bar{\lambda}, \delta), F(\lambda) \subset V$.

(iii) F is said to be closed if $\text{Graph}(F) = \{(\lambda, x) : \lambda \in \Lambda \text{ and } x \in F(\lambda)\}$ is a closed set in $\Lambda \times X$.

We say F is l.s.c(resp. u.s.c) on Λ, iff it is l.s.c(resp. u.s.c) at each $\lambda \in \Lambda$. F is said to be continuous on Λ, iff it is both l.s.c and u.s.c on Λ.

Definition 2.6. A vector-valued mapping $F(\cdot)$ is called C-convexlike on A, iff for any $x_1, x_2 \in A$ and any $t \in [0, 1]$, there exists $x_3 \in A$ such that $tF(x_1) + (1 - t)F(x_2) \in F(x_3) + C$.

Remark 2.7. (i) F is C-convexlike on X if and only if $F(X) + C$ is convex.

(ii) From the definitions, we can obtain immediately the following implications for the map F:

$$C - convexity \Rightarrow C - convexlikeness$$

However, one simple example in [19] (Example 3.2) show that the converse implication is generally not valid. Hence, the class of C-convexlike maps is larger than the class of C-convex maps.

Lemma 2.8 ([2, 4])**.** *Let X and Y be topological spaces, $T : X \rightrightarrows Y$ be a set-valued mapping. T is l.s.c at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in T(x_0)$, there exists $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \to y_0$.*

Lemma 2.9 ([1])**.** *Let $S : K \rightrightarrows Y$ be a set-valued mapping. If S is closed and Y is compact, then S is upper semicontinuous.*

3. Connectedness of eficient solutions for (GKFI)

In this section, we obtain an important density result for (GKFI), then we further discuss the connectedness of efficient solutions to the (GKFI).

Define set-valued mapping $H : C^* \setminus \{0\} \rightrightarrows A$ by

$$H(f) := V_f(A, F), \ f \in C^* \setminus \{0\}.$$

Firstly, we establish the following lemma.

Lemma 3.1. *Let $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied*

(i) *A is a nonempty compact set;*

(ii) *$F(\cdot, \cdot)$ is continuous on $A \times A$;*

(iii) *For each $x \in A \setminus V_f(A, F)$, there exists $y \in V_f(A, F)$ such that*

$$F(x, y) + F(y, x) + B(0, d^r(x, y)) \subset -C,$$

where $\gamma > 0$ is a positive constant.

Let us define set-valued mapping $H : C^* \setminus \{0\} \rightrightarrows A$ by

$$H(f) := V_f(A, F), \ f \in C^* \setminus \{0\}.$$

Then we have $H(\cdot)$ is l.s.c on $C^* \setminus \{0\}$.

Proof. Suppose to the contrary that there exists $f_0 \in C^* \setminus \{0\}$, such that $H(f)$ be not l.s.c at f_0. Then, there exist a sequence $\{f_n\} \subset C^* \setminus \{0\}$ with $f_n \to f_0$ and $x_0 \in H(f_0)$ such that for any $x_n \in H(f_n), x_n \not\to x_0$.

Since $x_0 \in A$ and A is nonempty compact, then there exists $\bar{x}_n \in A$, such that $\bar{x}_n \to x_0$. Obviously, $\bar{x}_n \in A \setminus H(f_n)$. By (iii), there exists $y_n \in H(f_n)$ such that

$$F(\bar{x}_n, y_n) + F(y_n, \bar{x}_n) + B(0, d^r(\bar{x}_n, y_n)) \subset -C, \tag{3.1}$$

where $\gamma > 0$ is a positive constant.

For $y_n \in H(f_n)$ implies $y_n \in A$, because A is nonempty compact, there exist $y_0 \in A$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$, such that $y_{n_k} \to y_0$. In particular, for (3.1), we have

$$F(\bar{x}_{n_k}, y_{n_k}) + F(y_{n_k}, \bar{x}_{n_k}) + B(0, d^r(\bar{x}_{n_k}, y_{n_k})) \subset -C. \tag{3.2}$$

Taking the limit as $n_k \to +\infty$, it follows from the continuity of F we have

$$F(x_0, y_0) + F(y_0, x_0) + B(0, d^r(x_0, y_0)) \subset -C. \tag{3.3}$$

Assume that $x_0 \neq y_0$, by (3.3), we can obtain $F(x_0, y_0) + F(y_0, x_0) \in -\text{intC}$. Thus, it follows from $f_0 \in C^* \setminus \{0\}$, we have

$$f_0(F(x_0, y_0) + F(y_0, x_0)) < 0. \tag{3.4}$$

Noting that $x_0 \in H(f_0)$ and $y_0 \in A$, we have

$$f_0(F(x_0, y_0)) \geq 0. \tag{3.5}$$

Moreover, since $y_{n_k} \in H(f_{n_k})$ and $\bar{x}_{n_k} \in A$, it follows from the continuity of f_0 and F that

$$f_0(F(y_0, x_0)) \geq 0. \tag{3.6}$$

By (3.5), (3.6) and the linearity of f_0, we have

$$f_0(F(x_0, y_0) + F(y_0, x_0)) \geq 0.$$

which contradicts (3.4). Therefore $x_0 = y_0$. This is impossible by the contradiction assumption. Therefore, $H(\cdot)$ is l.s.c on $C^* \setminus \{0\}$. The proof is complete. ☐

Theorem 3.2. *Let $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied:*

(i) *A is a nonempty compact set;*
(ii) *$F(\cdot, \cdot)$ is continuous on $A \times A$;*
(iii) *For each $x \in A \setminus V_f(A, F)$, there exists $y \in V_f(A, F)$ such that*

$$F(x, y) + F(y, x) + B(0, d^r(x, y)) \subset -C,$$

where $\gamma > 0$ is a positive constant.

(iv) *For each $x \in A$, $F(x, \cdot)$ is C-convexlike on A.*

Then,

$$\bigcup_{f \in C^\sharp} V_f(A, F) \subset V(A, F) \subset cl\Big(\bigcup_{f \in C^\sharp} V_f(A, F)\Big).$$

Proof. By definition, we can easily obtain

$$\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset V_w(A,F). \tag{3.7}$$

Since for any $x \in A, F(x,\cdot)$ is C-convexlike, then $F(x,A)+C$ is a convex set. From Lemma 2.1 in [12], we have

$$V_w(A,F) = \bigcup_{f\in C^*\setminus\{0\}} V_f(A,F). \tag{3.8}$$

By (3.7) and (3.8), we can get

$$\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset \bigcup_{f\in C^*\setminus\{0\}} V_f(A,F). \tag{3.9}$$

Hence, we need to prove that

$$\bigcup_{f\in C^*\setminus\{0\}} V_f(A,F) \subset cl(\bigcup_{f\in C^\sharp} V_f(A,F)).$$

By the definition of set-valued mapping $H(f) = V_f(A,F)$ and by virtue of Lemma 3.1, we know that $H(\cdot)$ is lower semicontinuous on $C^* \setminus \{0\}$.

Let $x_0 \in \bigcup_{f\in C^*\setminus\{0\}} V_f(A,F)$. Then, there exists $f_0 \in C^* \setminus \{0\}$ such that

$$x_0 \in V_{f_0}(A,F) = H(f_0).$$

Since $C^\sharp \neq \emptyset$, let $g \in C^\sharp$ and set

$$f_n = f_0 + (1/n)g.$$

Then, $f_n \in C^\sharp$. We show that $\{f_n\}$ weak* converges to f_0 with respect to the topology $\beta(Y^*,Y)$.

For any neighborhood U of 0 with respect to $\beta(Y^*,Y)$, there exist bounded subsets $B_i \subset Y (i = 1,2,\dots,m)$ and $\epsilon > 0$ such that

$$\bigcap_{i=1}^{m}\{f \in Y^* : \sup_{y\in B_i} |f(y)| < \epsilon\} \subset U.$$

Since B_i is bounded and $g \in Y^*, |g(B_i)|$ is bounded for $i = 1,\dots,m$. Thus, there exists N such that

$$\sup_{y\in B_i} |(1/n)g(y)| < \epsilon,\ i = 1,\dots,m,\ n \geq N.$$

Hence $(1/n)g \in U$, that is, $f_n - f_0 \in U$. This means that $\{f_n\}$ weak* converges to f_0 with respect to $\beta(Y^*,Y)$.

Since $H(\cdot)$ is l.s.c at f_0, then for sequence $\{f_n\} \subset C^* \setminus \{0\}, f_n \to f_0$ and $x_0 \in H(f_0)$, there exists $x_n \in H(f_n) = V_{f_n}(A,F) \subset \bigcup_{f\in C^\sharp} V_f(A,F)$, such that $x_n \to x_0$. This means that

$$x_0 \in cl\Big(\bigcup_{f\in C^\sharp} V_f(A,F)\Big).$$

By the arbitrariness of $x_0 \in \bigcup_{f\in C^*\setminus\{0\}} V_{A,F}(\mu)$, we have

$$\bigcup_{f\in C^*\setminus\{0\}} V_f(A,F) \subset cl\Big(\bigcup_{f\in C^\sharp} V_f(A,F)\Big). \tag{3.10}$$

By (3.9) and (3.10), we obtain that

$$\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset cl\Big(\bigcup_{f\in C^\sharp} V_f(A,F)\Big).$$

The proof is completed. □

Remark 3.3. Theorem 3.2 improves and extends Theorem 2.1 of [14]. In [14], under the condition of C-strong monotonicity, the f-solution set for GKFI is confined to be a singleton (also see, Theorem 3.2 in [13]). In our paper, we use condition (iii) in Theorem 3.2 to weaken this condition. Moreover, the f-solution set may be a general set, but not a singleton. The following example is given to illustrate the case.

Example 3.4. Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+ := [0,+\infty)\times[0,+\infty), A = [-1,1]$. For each $x, y \in A$, define the mapping $F : X \times X \to Y$ by

$$F(x,y) = \Big(-x^3 + \frac{1}{2}x^2 - \frac{1}{3}x^{\frac{1}{3}}y - 2, 10x^{\frac{1}{3}}\Big(y^2 + \frac{1}{2}x^2\Big)\Big).$$

For any given $\mu > 0$, let $f((x,y)) = \frac{1}{\mu}y$. It follows from a direct computation that $V_f(A,F) = [0,1]$. Obviously, the f-solution set of GKFI is set-valued, but not a singleton.

Clearly, conditions (i) (ii) (iv) of Theorem 3.2 are satisfied. The assumption (iii) can be checked as follows:

For any $x \in A \setminus V_f(A,F) = [-1,0)$, there exists $y = 0 \in V_f(A,F)$ and $r = \frac{7}{3} > 0$ such that

$$\begin{aligned}
F(x,y) + F(y,x) + B(0,d^r(x,y)) =& \Big(-x^3 + \frac{1}{2}x^2 - \frac{1}{3}x^{\frac{1}{3}}y - 2, 10x^{\frac{1}{3}}\Big(y^2 + \frac{1}{2}x^2\Big)\Big) \\
&+ \Big(-y^3 + \frac{1}{2}y^2 - \frac{1}{3}y^{\frac{1}{3}}x - 2, 10y^{\frac{1}{3}}\Big(x^2 + \frac{1}{2}y^2\Big)\Big) \\
&+ B(0,d^r(x,y)) \\
=& \Big(-x^3 + \frac{1}{2}x^2 - 2, 5x^{\frac{7}{3}}\Big) + (-2,0) + B(0,d^r(x,0)) \\
=& \Big(-x^3 + \frac{1}{2}x^2 - 4, 5x^{\frac{7}{3}}\Big) + B(0,d^r(x,0)) \subset -C.
\end{aligned}$$

By Theorem 3.2, we can obtain that $\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset cl(\bigcup_{f\in C^\sharp} V_f(A,F))$.

However, the condition of C-strong monotonicity in [14](or called C-strict monotonicity in [13]) does not hold. Indeed, for any $x \in A \setminus V_f(A,F) = [-1,0)$, there exists $y = -x \in V_f(A,F) = [0,1]$, such that

$$\begin{aligned}
F(x,y) + F(y,x) =& \Big(-x^3 + \frac{1}{2}x^2 - \frac{1}{3}x^{\frac{1}{3}}y - 2, 10x^{\frac{1}{3}}\Big(y^2 + \frac{1}{2}x^2\Big)\Big) \\
&+ \Big(-y^3 + \frac{1}{2}y^2 - \frac{1}{3}y^{\frac{1}{3}}x - 2, 10y^{\frac{1}{3}}\Big(x^2 + \frac{1}{2}y^2\Big)\Big)
\end{aligned}$$

$$= \left(x^2 + \frac{2}{3}x^{\frac{4}{3}} - 4, 0\right) \in -\partial C.$$

where ∂C is the boundary of C. Obviously, $F(x,y)+F(y,x) \notin -\mathrm{int}C$, which implies $F(\cdot,\cdot)$ is not C-strongly monotone on $A\times A$. Then, the density Theorem 2.1 in [14] is not applicable.

Now, we establish a sufficient condition for connectedness of efficient solutions to (GKFI).

Theorem 3.5. *Let $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied:*

(i) *A is a nonempty compact convex set;*

(ii) *$F(\cdot, y)$ is C-concave on A and $F(\cdot,\cdot)$ is continuous on $A \times A$;*

(iii) *For each $x \in A \setminus V_f(A,F)$, there exists $y \in V_f(A,F)$ such that*

$$F(x,y) + F(y,x) + B(0, d^r(x,y)) \subset -C,$$

where $\gamma > 0$ is a positive constant.

(iv) *For each $x \in A$, $F(x,\cdot)$ is C-convexlike on A;*

(v) *$F(A,A)$ is a bounded subsets of Y.*

Then, $V(A,F)$ is a connected set.

Proof. The proof is divided into the following three steps :

Step 1 $V_f(A,F)$ is a convex set on A.

In fact, for any fixed $y \in A$, let $x_1, x_2 \in V_f(A,F)$ and $\lambda \in [0,1]$. Then, $\lambda x_1 + (1-\lambda)x_2 \in A$ and

$$f(F(x_1,y)) \geq 0, \tag{3.11}$$

$$f(F(x_2,y)) \geq 0. \tag{3.12}$$

Multiplying both side of (3.11) by λ and of (3.12) by $1-\lambda$, and together with the C-concavity of F with respect to the first argument yields

$$f(F(\lambda x_1 + (1-\lambda)x_2, y)) \geq \lambda f(F(x_1,y)) + (1-\lambda) f(F(x_2,y)) \geq 0.$$

It follows that $\lambda x_1 + (1-\lambda)x_2 \in V_f(A,F)$. Therefore, $V_f(A,F)$ is convex. It follows that for any $f \in C^*\backslash\{0\}$, $H(f)$ is a connected set.

Step 2 It is clear that $C^*\backslash\{0\}$ is convex, so it is a connected set.

Step 3 Now we show that $H(\cdot)$ is upper semicontinuous on $C^*\backslash\{0\}$.

Since A is compact, by Lemma 2.9, we need only to prove that H is closed. Let $\{(f_\alpha, x_\alpha) : \alpha \in I\}$ be a net such that

$$\{(f_\alpha, x_\alpha) : \alpha \in I\} \subset \mathrm{Graph}(H) = \{(f,x) \in (C^*\backslash\{0\}) \times A : x \in H(f)\}$$

and

$$(f_\alpha, x_\alpha) \to (f, x_0) \in (C^*\backslash\{0\}) \times A,$$

where $f_\alpha \to f$ means that $\{f_\alpha\}$ weak*converges to f with respect to the strong topology $\beta(Y^*, Y)$ in Y^*. Since $x_\alpha \in H(f_\alpha)$, $\alpha \in I$, one has

$$f_\alpha(F(x_\alpha, y)) \geq 0, \ \forall\, y \in A. \tag{3.13}$$

By assumption, $D = \{F(x,y) : x, y \in A\}$ are bounded subsets of Y. Define

$$P_D(y^*) := \sup\{|y^*(u)| : u \in D\}, \ y^* \in Y^*.$$

It is easy to see that P_D is a seminorm of Y^*. For arbitrary $\varepsilon > 0$,

$$U = \{y^* \in Y^* : P_D(y^*) < \varepsilon\}$$

is a neighborhood of zero with respect to $\beta(Y^*, Y)$. Since $f_\alpha - f \to 0$, there exists $\alpha_0 \in I$ such that $f_\alpha - f \in U$, $\forall\, \alpha \geq \alpha_0$. It follows that

$$P_D(f_\alpha - f) = \sup\{|(f_\alpha - f)(u)| : u \in D\} < \varepsilon, \text{ whenever } \alpha \geq \alpha_0.$$

Therefore, for any $y \in A$,

$$|(f_\alpha - f)(F(x_\alpha, y))| = |f_\alpha(F(x_\alpha, y)) - f(F(x_\alpha, y))| < \varepsilon. \tag{3.14}$$

It follows from (3.14) and the continuity of f and F, we can get that

$$\begin{aligned}\lim |f(F(x_0, y)) - f_\alpha(F(x_\alpha, y))| &\leq \lim |f(F(x_0, y)) - f(F(x_\alpha, y))| \\ &\quad + \lim |f(F(x_\alpha, y)) - f_\alpha(F(x_\alpha, y))| \\ &= 0.\end{aligned}$$

This fact together with (3.13) yields,

$$f(F(x_0, y)) \geq 0, \forall y \in Y.$$

It follows that $x_0 \in V_f(A, F) = H(f)$. Therefore, H is a closed mapping, and so H is upper semicontinuous on $C^*\backslash\{0\}$. From Theorem 3.1 in [17] (or Theorem 3.1 in [24]),

$$\bigcup_{f \in C^*\backslash\{0\}} V_f(A, F)$$

is a connected set. Using a similar method, with suitable modifications, we can get $\bigcup_{f \in C^\sharp} V_f(A, F)$ is a connected set.

Furthermore, by the Theorem 3.2, we have,

$$\bigcup_{f \in C^\sharp} V_f(A, F) \subset V(A, F) \subset cl\Big(\bigcup_{f \in C^\sharp} V_f(A, F)\Big).$$

So, we can obtain $V(A, F)$ is a connected set. This completes the proof. □

Remark 3.6. Theorem 3.5 generalizes and improves Theorem 2.2 of Gong and Yao [14]. In Theorem 3.5, the assumption of C-strong/strict monotonicity is removed by assumption (iii), where the f-solution set is not necessary a singleton, may be a general one. And the mapping F that is a C-convex mapping is extended to the C-likeconvex mapping.

Moreover, we also can see that the obtained result improves the ones of [13, 21], where the strong assumptions that $F(x, x) = \varphi(x, x) \geq 0$ and $F(0, 0) = 0$(or $\psi(0) = 0$ and $\varphi(0, 0) = 0$) is not necessary.

Now, we give an example to illustrate our result extends those of [13] and [14, 21].

Example 3.7. Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, A = [-1, 1]$, and let $F : X \times X \to Y$ be a vector-valued mapping defined by

$$F(x, y) = \Big(-x^2 + 3x + \frac{3}{2}y - \frac{11}{2}, \frac{7}{3}xy^2 + \frac{14}{3}x\Big), \ \ \forall x, y \in A$$

Let $f = (0, 2) \in C^* \setminus \{0\}$, it follows from a direct computation that $V_f(A, F) = [0, 1]$(not a singleton).

We can verify that A is a nonempty compact convex set, $F(\cdot, y)$ is C-concave on A and $F(\cdot,\cdot)$ is continuous on $A\times A$. Obviously, conditions (i) (ii) (iv) and (v) of Theorem 3.5 are satisfied.

For any $x \in A\setminus V_f(A,F)$, there exists $y=0\in V_f(A,F)$ and $r=1>0$ such that

$$\begin{aligned} & F(x,y)+F(y,x)+B(0,d^r(x,y)) \\ = & \Big(-x^2+3x+\frac{3}{2}y-\frac{11}{2},\frac{7}{3}xy^2+\frac{14}{3}x\Big) \\ & +\Big(-y^2+3y+\frac{3}{2}x-\frac{11}{2},\frac{7}{3}yx^2+\frac{14}{3}y\Big)+B(0,d^r(x,y)) \\ = & \Big(-x^2+3x-\frac{11}{2},\frac{14}{3}x\Big)+\Big(\frac{3}{2}x-\frac{11}{2},0\Big)+B(0,d^r(x,0)) \\ = & \Big(-x^2+\frac{9}{2}x-11,\frac{14}{3}x\Big)+B(0,d^r(x,0))\subset -C. \end{aligned}$$

Thus, the condition (iii) in Theorem 3.5 is satisfied. It is clear that $F(A,A)$ are also bounded subsets of Y. By virtue of Theorem 3.5, we conclude that $V(A,F)$ is a connected set.

However, Theorem 2.2 in [14] is not applicable because F (or φ) is not C-strongly/strictly monotone on $A\times A$. Indeed, for any $x\in A\setminus V_f(A,F)=[-1,0)$, there exists $y=-x\in V_f(A,F)=[0,1]$, such that

$$\begin{aligned} F(x,y)+F(y,x) &= \Big(-x^2+3x+\frac{3}{2}y-\frac{11}{2},\frac{7}{3}xy^2+\frac{14}{3}x\Big) \\ &\quad +\Big(-y^2+3y+\frac{3}{2}x-\frac{11}{2},\frac{7}{3}yx^2+\frac{14}{3}y\Big) \\ &= (-2x^2-11,0)\notin -intC, \end{aligned}$$

which means $F(\cdot,\cdot)$ is not C-strongly monotone on $A\times A$. Then, Theorem 2.2 of [14] is not applicable, and Theorem 3.2 in [13] is also invalid.

Moreover, we observe that $F(0,0)\neq 0$, i.e., the condition that $F(0,0)=0$(or $\psi(0)=0$ and $\varphi(0,0)=0$) in Theorem 3.2 of [21] is not satisfied. Meanwhile, the condition that $F(x,x)=\varphi(x,x)\geq 0$ in Theorems 4.1-4.5 of [13] is also not satisfied. Therefor, the corresponding results of [13, 21] are also inapplicable.

4. Conclusions

In this paper, we study some properties of efficient solutions set. The connectedness of efficient solutions for GFKI is established by using density theorem without monotonicity.

Now one open problem arises in a natural way: Can we establish the connectedness of Henig/super/weak efficient solutions set for GFKI or set-valued GFKI without using monotonicity ? This is a very interesting and valuable topic, and we will investigate it in our future work.

References

[1] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley, New York, 1984.
[2] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhanser, Boston, 1990.
[3] Q. H. Ansari and J. C. Yao, *An existence result for the generalized vector equilibrium*, Appl. Math. Lett. **12** (1999), 53–56.

[4] C. Berge, *Topological Spaces*, Oliver and Boyd, London, 1963.
[5] M. Bianchi, N. Hadjisavvas and S. Schaibles, *Vector equilibrium problems with generalized monotone bifunctions*, J. Optim. Theory Appl. **92** (1997), 527–542.
[6] H. Brezis, L. Nirenberg and G. Stampacchia, *A remark on Ky Fan's minimax principle*, Boll. Unione Mat. Ital. (III) VI (1972), 129–132.
[7] Y. H. Cheng, *On the connectedness of the solution set for the weak vector variational inequality*, J. Math. Anal. Appl. **260** (2001), 1–5.
[8] K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. **112** (1969), 234–240.
[9] K. Fan, *A minimax Inequality and Applications*, in Inequality III, Shihsha, O. (ed.), Academic Press, New York, 1972, pp. 103–113.
[10] F. Giannessi, *Vector Variational Inequilities and Vector Equilibria. In Mathematical Theories*, Kluwer, Dordrechet, 2000.
[11] F. Giannessi, A. Maugeri and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academi Publishers, Dordrecht, 2001.
[12] X. H. Gong, *Efficiency and Henig efficiency for vector equilibrium problems*, J. Optim. Theory Appl. **108** (2001), 139–154.
[13] X. H. Gong, *Connectedness of the solution sets and scalarization for vector equilibrium problems*, J. Optim. Theory Appl. **133** (2007), 151–161.
[14] X. H. Gong and J. C. Yao, *Connectedness of the set of efficient solutions for generalized systems*, J. Optim. Theory Appl. **138** (2008), 189–196.
[15] X. H. Gong, *Connectedness of Efficiency solution sets for set-valued maps in normed spaces*, J. Optim. Theory Appl. **83** (1994), 83–96.
[16] N. Hadjisavvas and S. Schaibles, *From scalar to vector equilibrium problems in the quasimonotone case*, J. Optim. Theory Appl. **96** (1998), 297–309.
[17] Y. D. Hu, *The efficiency Theory of Multiobjective Programming*, Shanghai: Shanghai Science and Technology Press, 1994.
[18] G. M. Lee, D. S. Kim, B. S. Lee and N. D. Yun, *Vector variational inequalities as a tool for studing vector optimization problems*, Nonlinear Anal. **34** (1998), 745–765.
[19] Z. H. Li, *Benson Proper Efficiency in the Vector Optimization of Set-Valued Maps*, J. Optim. Theory Appl. **98** (1998), 623–649.
[20] X. J. Long, N. J. Huang and K. L. Teo, *Existence and stability of solutions for generalized strong vector quasi-equilibrium problems*, Math, Comput. Modelling **47** (2008), 445–451.
[21] X. J. Long and J. W. Peng, *Connectedness and compactness of weak efficient solutions for vector equilibrium problems*, Bull. Korean Math. Soc. **48** (2011), 1225–1233.
[22] Q. Y. Liu, X. J. Long and N. J. Huang, *Connectedness of the sets of weak efficient solutions for generalized vector equilibrium problems*, Math. Slovaca. **62** (2012), 123–136.
[23] Q. S. Qiu and X. M. Yang, *Connectedness of Henig weakly efficient solution Set for set-valued optimization problems*, J. Optim. Theory Appl. **152** (2012), 439–449.
[24] A. R. Warburton, *Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives*, J. Optim. Theory Appl. **40** (1983), 537–557.
[25] R. Y. Zhong, N. J. Huang and M. M. Wong, *Connectedness and Path-Connectedness of solution sets to symmetric vector equilibrium problems*, Taiwan. J. Math. **13** (2009), 821–836.

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