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# THE ROLE OF REGULARITY TO REACH THE VECTOR VALUED VERSION OF CARISTI'S FIXED POINT THEOREM

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Abstract. In this paper we discuss on the vector valued version of Caristi's theorem. We show that the regularity of the cone is an essential condition to reach the vector valued version of Caristi's theorem in vector valued metric spaces. It is shown that how the absence of the regularity causes some previous Caristi type theorems and results fail to be correct. Our main result give a vector valued version of Caristi's theorem for correspondences with weakened hypotheses in comparison with previous results and generalize them.

## 1. Introduction and preliminaries

Caristi's fixed point theorem states that if $X$ is a complete metric space and $\varphi$ is a lower semi-continuous map (briefly, lsc) from $X$ into the nonnegative real numbers, then any map $T : X \to X$ satisfying

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \qquad (x \in X)$$

has a fixed point.

Several works published each to obtain a vector version of Caristi's theorem. Specifically, we focus on the the following theorems from [3], [4] and [6].

**Theorem 1.1** ([6, Lemma 3.3]). *Let $X$ be a complete vector valued metric space ($\kappa$-metric space) over a regular, minihedral cone with a nonempty interior in a separable space $\mathcal{E}$. Let $\varphi : X \to \mathcal{E}_+$ be an lsc map. Then any map $T : X \to X$ such that*

$$d(x, T(x)) \preceq \varphi(x) - \varphi(T(x)), \qquad (x \in X) \tag{1.1}$$

*has a fixed point.*

**Theorem 1.2** ([3, Theorem 2.4]). *Let $X$ be a complete vector valued metric space (cone metric space) over a normal and order complete (strongly minihedral) cone $\mathcal{E}_+$ with nonempty interior. Let $\varphi : X \to \mathcal{E}_+$ be an lsc map. Then any map $T$ satisfying (1.1) has a fixed point.*

**Theorem 1.3** ([4, Theorem 2]). *Let $X$ be a complete vector valued metric space over an order complete and order continuous Banach lattice $\mathcal{E}$, and let $\varphi : X \to \mathcal{E}_+$ be an lsc map. Then any map $T$ satisfying (1.1) has a fixed point.*

In this paper we aim to reformulate Caristi's fixed point theorem for correspondences in vector valued metric spaces and obtain a generalization of all theorems

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above. The outline of this paper is as follows. Theorem 2.1 is a multivalued version of Theorem 1.3 with some weakened hypotheses (order continuity and order completeness are replaced with regularity). Theorem 2.1 also generalizes Theorem 1.1 (where the conditions on the emptiness of the interior of the cone and on the separability of the space are dropped). Example 2.4 is a counterexample of Theorem 1.2. In fact, this example shows that the regularity of the cone is an essential assumption and may not be ignored in Theorem 1.2 and Theorem 2.1. From this point of view, Theorem 2.1 would generalize Theorem 1.2 if the regularity of the cone were one of the assumptions. Example 2.5 shows that the supremum property used in the proof of both Theorem 1.1 and Theorem 1.2 as well as [1, Lemma 1, (i) ] and [2, Lemma 9, (i)] is not necessarily valid in vector valued metric spaces. Yet, the statement of Theorem 1.1 remains true and Theorem 2.1 will be a generalization of it. Finally, Example 2.6 contradicts [1, Lemma 1, (i) ] and [2, Lemma 9, (i)]. In fact, this example shows that order completeness and normality are not sufficient conditions for regularity of a cone.

We start with some preliminaries which will be needed in this paper.

Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be an ordered Banach space (for short $\mathcal{E}$) with (positive) cone $\mathcal{E}_+ = \{c \in \mathcal{E} : c \succeq \theta\}$, where $\theta$ is the null vector.

A *lattice norm* $\|\cdot\|$ has the property that for $a, b \in \mathcal{E}$ that $|a| \preceq |b|$ we have $\|a\| \leq \|b\|$. A Riesz space equipped with a lattice norm is called a *normed Riesz space*. A complete normed Riesz space is called a *Banach lattice*. A lattice norm $\|\cdot\|$ on a Riesz space is *order continuous* if $\inf_\alpha a_\alpha = 0$ implies $\inf_\alpha \|a_\alpha\| = 0$, for every decreasing net $\{a_\alpha\}$. A set $B \subset \mathcal{E}$ is called *bounded* if there exists $z \in \mathcal{E}$ such that for all $b \in B$, $b \preceq z$. A Riesz space is called *order complete*, if every nonempty subset that is bounded from above (below) has a supremum (infemum). A cone is called *regular* if every nondecreasing (decreasing) sequence which is bounded from above (below) is convergent in norm. A cone is called *normal* if $\theta \preceq a \preceq b$ implies $\|a\| \leq M\|b\|$, for some $M \geq 1$.

**Definition 1.4** ([4]). Let $X$ be a nonempty set and $\mathcal{E}$ be an ordered Banach space with order $\preceq$. If a map $d : X \times X \to \mathcal{E}$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$,

then $(X, d)$ is called a vector valued metric space (vvms for short) over $\mathcal{E}$.

Let $(X, d)$ be a vvms. A net $\{x_\alpha\}$ of $X$ is called *convergent* to $x \in X$, denoted by $x_\alpha \to x$, if $\|d(x_\alpha, x)\| \to 0$. A net $\{x_\alpha\}$ of $X$ is called *Cauchy net*, if $\|d(x_\alpha, x_\beta)\| \to 0$. If every Cauchy net of $X$ is convergent to a point of $X$, then $X$ is called a *complete* vvms. A map $\varphi : X \to \mathcal{E}$ is called lower semi-continuous (lsc) if for each net $\{x_\alpha\}$ of $X$ such that $x_\alpha \to x$ we have $\varphi(x) \preceq \liminf_\alpha \varphi(x_\alpha)$.

## 2. Vector version of Caristi's theorem

Hereafter it is supposed that $\mathcal{E}_+$ is a closed regular cone of ordered Banach space $\mathcal{E}$ with order $\preceq$ (the interior of $\mathcal{E}_+$ is not necessary to be nonempty), unless othewise stated. A correspondence $f : X \twoheadrightarrow X$ maps $x \in X$ to a nonempty subset of $X$. A point $a \in X$ is called fixed point of $f$ if $a \in f(a)$.

**Theorem 2.1.** *Let $(X, d)$ be a complete vvms over $\mathcal{E}$. Let $\varphi : X \to \mathcal{E}_+$ be an lsc map. Then every correspondence $f : X \twoheadrightarrow X$ such that for each $x \in X$ there exists $y \in f(x)$ satisfying*

$$d(x, y) \preceq \varphi(x) - \varphi(y),$$

*has a fixed point.*

*Proof.* By assumption, for each $x \in X$ the set $\{y \in f(x) : d(x, y) \preceq \varphi(x) - \varphi(y)\}$ is nonempty. By the axiom of choice, there is a single valued map $T : X \to X$ such that $d(x, T(x)) \preceq \varphi(x) - \varphi(T(x))$, for each $x \in X$. Define an order $\preceq_\varphi$ on $X$ by

$$x \preceq_\varphi y \Longleftrightarrow d(x, y) \preceq \varphi(y) - \varphi(x),$$

for any $x, y \in X$. We shall show that $(X, \preceq_\varphi)$ has a minimal element and $T$ fixes the minimal element. Let $C$ be a chain in $X$. The chain $C$ can be considered as a directed set induced by the reverse of $\preceq_\varphi$ and therefore it may be regarded as a decreasing net $C = \{x_\alpha\}_{\alpha = x_\alpha \in C}$. For $\alpha \preceq_\varphi \beta$ we have

$$\theta \preceq d(x_\alpha, x_\beta) \preceq \varphi(x_\beta) - \varphi(x_\alpha). \tag{2.1}$$

This implies that the net $\{\varphi(x_\alpha)\}_{\alpha \in C}$ is a decreasing net and bounded from below in $\mathcal{E}_+$. Since every closed regular cone is normal (see e.g., [9]; without loss of generality let the normal constant $M$ is equal to 1), the net $\{\|\varphi(x_\alpha)\|\}_{\alpha \in C}$ is decreasing and bounded below in $\mathbb{R}$ and therefore has an infimum. This allows us to find a countable subnet $\{\|\varphi(x_{\alpha_n})\|\}$ of $\{\|\varphi(x_\alpha)\|\}_{\alpha \in C}$. Now consider countable subnet $\{\varphi(x_{\alpha_n})\}$ of $\{\varphi(x_\alpha)\}_{\alpha \in C}$. From the regularity of the cone, $\{\varphi(x_{\alpha_n})\}$ is convergent in norm and therefore it is Cauchy in norm too. We claim that $\{\varphi(x_\alpha)\}_{\alpha \in C}$ is Cauchy. To see this, let $\epsilon > 0$ be given. There exists $\alpha_{n_0}$ such that for all $\alpha_n$, $\alpha_m \preceq_\varphi \alpha_{n_0}$,

$$\|\varphi(x_{\alpha_n}) - \varphi(x_{\alpha_m})\| < \epsilon/3.$$

Let $\alpha, \beta \preceq_\varphi \alpha_{n_0}$. Since $\{\varphi(x_\alpha)\}$ is decreasing, $\varphi(x_\alpha) \preceq \varphi(x_{\alpha_{n_0}})$. Also, $\{\varphi(x_{\alpha_n})\}$ is a subnet of $\{\varphi(x_\alpha)\}$, therefore there is a term $\varphi(x_{\alpha_m})$ in the sequence $\{\varphi(x_{\alpha_n})\}$ with $\alpha_m \preceq_\varphi \alpha$ such that $\varphi(x_{\alpha_m}) \preceq \varphi(x_\alpha)$. Thus $\varphi(x_{\alpha_m}) \preceq \varphi(x_\alpha) \preceq \varphi(x_{\alpha_{n_0}})$. This along with the compatibility of "$\preceq$" with "$+$" in $\mathcal{E}$ imply that

$$\theta \preceq \varphi(x_\alpha) - \varphi(x_{\alpha_m}) \preceq \varphi(x_{\alpha_{n_0}}) - \varphi(x_{\alpha_m}),$$

and since the cone is normal, we get

$$0 \leq \|\varphi(x_\alpha) - \varphi(x_{\alpha_m})\| \leq \|\varphi(x_{\alpha_{n_0}}) - \varphi(x_{\alpha_m})\|.$$

In particular, we have $\|\varphi(x_\alpha) - \varphi(x_{\alpha_m})\| < \epsilon/3$. With a similar argument, there is $\alpha_n$ with $\alpha_{n_0} \succeq_\varphi \alpha_n$ such that $\|\varphi(x_\beta) - \varphi(x_{\alpha_n})\| < \epsilon/3$. Thus

$$\begin{aligned}
\|\varphi(x_\alpha) - \varphi(x_\beta)\| &\leq \|\varphi(x_\alpha) - \varphi(x_{\alpha_m})\| + \|\varphi(x_{\beta_m}) - \varphi(x_{\alpha_n})\| \\
&+ \|\varphi(x_{\alpha_n}) - \varphi(x_\beta)\|, \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3, \\
&< \epsilon.
\end{aligned}$$

From (2.1) we have $\|d(x_\alpha, x_\beta)\| \leq \|\varphi(x_\alpha) - \varphi(x_\beta)\|$ and therefore $\|d(x_\alpha, x_\beta)\| < \epsilon$. Thus, $\{x_\alpha\}$ is Cauchy and since $(X, d)$ is complete, $\{x_\alpha\}$ is convergent. Let $\lim_\alpha x_\alpha = z$. Now, we show that $z$ is a lower bound for $(C, \preceq_\varphi)$. Let $x_\beta \in (C, \preceq_\varphi)$ be fixed. For $x_\alpha \in C$, we have

$$d(x_\alpha, x_\beta) \preceq d(x_\alpha, z) + d(z, x_\beta),$$

and

$$d(z, x_\beta) \preceq d(z, x_\alpha) + d(x_\alpha, x_\beta).$$

Therefore

$$\theta \preceq d(z, x_\alpha) + d(x_\alpha, x_\beta) - d(z, x_\beta) \preceq 2d(z, x_\alpha),$$

and the normality yields

$$\|d(z, x_\alpha) + d(x_\alpha, x_\beta) - d(z, x_\beta)\| \leq 2\|d(z, x_\alpha)\|.$$

Thus

$$\begin{aligned} \|d(x_\alpha, x_\beta) - d(z, x_\beta)\| &\leq \|d(x_\alpha, x_\beta) - d(z, x_\beta) + d(z, x_\alpha)\| + \|d(z, x_\alpha)\|, \\ &\leq 3\|d(z, x_\alpha)\|. \end{aligned}$$

This implies that $\lim_\alpha d(x_\alpha, x_\beta) = d(z, x_\beta)$. For $x_\alpha \preceq_\varphi x_\beta$, since $\varphi$ is lsc, and $\mathcal{E}_+$ is closed, we have

$$\varphi(z) \preceq \liminf_\alpha \varphi(x_\alpha) = \lim_\alpha \varphi(x_\alpha) \preceq \lim_\alpha(\varphi(x_\beta) - d(x_\alpha, x_\beta)) \preceq \varphi(x_\beta) - d(z, x_\alpha),$$

that is $z \preceq_\varphi x_\beta$. Because $x_\beta$ was arbitrary, $z$ is a lower bound for $C$. Zorn's lemma will ensure the existence of a minimal element $w$ for $X$. Since for each $x \in X$ we have $T(x) \preceq_\varphi x$, therefore $T(w) = w$. □

The contraction given in the preceding theorem was also investigated in [7] for correspondences defined on cone metric spaces.

**Corollary 2.2.** *Let $(X, d)$ be a complete vvms over $\mathcal{E}$, and let $\varphi : X \to \mathcal{E}_+$ be an lsc map. Then any map $T : X \to X$ such that*

$$d(x, T(x)) \preceq \varphi(x) - \varphi(y),$$

*for any $x \in X$, has a fixed point.*

The following remark says why the Corollary 2.2 is a generalization of Theorem 1.3.

**Remark 2.3.** In every order continuous Banach lattice, the (positive) cone $\mathcal{E}_+$ is closed and regular. In fact, let $\{a_n\}$ be a decreasing sequence in $\mathcal{E}$ which is bounded from below. Since every order continuous Banach lattice $\mathcal{E}$ is order complete (see e.g., [5, Corollary 9.24]), $\{a_n\}$ has an infimum, say, $a$. Since $\inf_n(a_n - a) = 0$, the order continuity implies taht $\inf_n \|a_n - a\| = 0$. Thus $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to $a$. Since $\{a_n - a\}$ is decreasing and $\mathcal{E}$ is a Banach lattice, we have $\|a_n - a\| \to 0$. That is the cone $\mathcal{E}_+$ is regular. To see the closeness of $\mathcal{E}_+$, let $\{a_n\}$ be a sequence in $\mathcal{E}_+$ which is convergent to $b$. Since $a_n \succeq 0$ and $\mathcal{E}$ is order complete, infimum $\{a_n\}$ exists. Let $\inf_n a_n = a \succeq \theta$. A similar reasoning shows that $\{a_n\}$ has a subsequence which is convergent to $a$ and therefore convergent to $b$.

The following example shows that the regularity of $\mathcal{E}_+$ is essential in Theorem 2.1 and may not be dropped. Also, this example contradicts the Theorem 1.2.

**Example 2.4.** Consider the Banach space $l_\infty(\mathbb{R})$ with usual pointwise order induced by $\mathbb{R}$. Let $\mathcal{E}_+ = \{(x_n) \in l_\infty(\mathbb{R}) : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$ be its cone. It is not difficult to see that $l_\infty(\mathbb{R})$ is order complete and $\mathcal{E}_+$ is normal with nonempty interior. Let $B$ be a subset of $l_\infty(\mathbb{R})$ consisting of all $(x_n)$ which are nondecreasing and converging to 1 with $1/2 \leq x_n \leq 1$, for all $n \in \mathbb{N}$. Define $d : B \times B \to \mathcal{E}_+$ by

$$d((x_n), (y_n)) = (|x_1 - y_1|, \ldots, |x_n - y_n|, \ldots),$$

for every $(x_n), (y_n) \in B$. It is not hard to check that $(B, d)$ is a complete vvms. Now define the map $T : B \to B$ by

$$T((x_n)) = (1/2, x_1, x_2, \ldots),$$

for every $(x_n) \in B$. Let $\varphi : B \to \mathcal{E}_+$ be the inclusion map. It is clear that $\varphi$ is lsc and $T$ satisfies (1.1). Indeed,

$$\begin{aligned} d((x_n), T((x_n))) &= (|x_1 - 1/2|, \ldots, |x_{i+1} - x_i| \ldots), \\ &= (x_1 - 1/2, \ldots, x_{i+1} - x_i, \ldots) \\ &= \varphi((x_n)) - \varphi(T((x_n))), \end{aligned}$$

for every $(x_n) \in B$. Thus all assumptions of Theorem 1.2 and Theorem 2.1 are fulfilled but $T$ is a fixed point free map on $B$ since the possible fixed point of $T$ is $(1/2, 1/2, \ldots)$ which does not belong to $B$.

The next example shows that the supremum property is not necessarily valid in vvms's. It clarifies the existed gap in the proof of Theorem 1.1 and Theorem 1.2.

**Example 2.5.** Consider the Banach lattice $\mathcal{E} = \mathbb{R}^2$ with $\mathcal{E}_+ = \{(a, b) \in \mathbb{R}^2 : a, b \in [0, +\infty)\}$. The subset $A = \{(1/n, -1/n) : n \in \mathbb{N}\}$ of $\mathbb{R}^2$ has supremum $(1, 0)$. The element $c = (1/3, 1/3)$ is an interior point of $\mathcal{E}_+$ but there is no element $(a, b)$ of $A$ such that $(a, b) \succeq (2/3, -1/3) = \sup A - c$. Therefore the Banach lattice $\mathbb{R}^2$ does not have supremum property.

Hence, the statement of Theorem 1.1 remains true because of Theorem 2.1 which is a generalization of it. Theorem 2.1 would be a generalization of Theorem 1.2 if in the latter the normality could be replaced with the regularity of the cone.

The following example contradicts [1, Lemma 1, (i) ] and [2, Lemma 9, (i)]. It shows that order completeness and normality are not sufficient conditions for regularity of a cone.

**Example 2.6.** Consider the Banach space $l_\infty(\mathbb{R})$ and $\mathcal{E}_+$ as given in Example 2.4. Suppose that the sequence $\{a_n\}$ is defined as

$$a_n = (\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots),$$

for each $n \in \mathbb{N}$. Although $\{a_n\}$ is nondecreasing and bounded from above but it is not convergent.

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