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APPROXIMATING COMMON SOLUTION OF A SYSTEM OF EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. In this paper we introduce new Halpern-type iterative algorithms for finding a common solution of a system of equilibrium problems in Banach spaces. We prove strong convergence of a modified Halpern-type scheme to an element of the set of common solution of a system of equilibrium problems in a reflexive Banach space and provide an affirmative answer to an open question raised by Zegeye and Shahzad in their final remark of [Zegeye and Shahzad, Approximating common solution of variational inequality problems for two monotone mappings in Banach spaces, Optimization Letters, 5 (2011) 691-704]. This scheme has an advantage that we do not use any generalized projection of a point on the intersection of closed and convex sets which creates some difficulties in a practical calculation of the iterative sequence. Some application of our results to the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space is presented. Our results improve and generalize many known results in the current literature.

1. Introduction

The equilibrium problem, introduced by Blum and Oettli [4] in 1994, has been attracting a growing attention of researchers; see, e.g., [18, 25] and the references therein. Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. In order to approximate the solution to this problem, various types of iterative schemes have been proposed; see, for instance, [17, 19]. Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|.\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^*\rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

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exists. In this case, E is called *smooth*. If the limit (1.1) is attained uniformly for all $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [23, 24]. Recall that a Banach space E has the *Kadec-Klee property* if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ and $x \in E$, if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ as $n \to \infty$. For more information concerning the Kadec-Klee property the reader is referred to [8] and the references therein. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property; the Banach space E is uniformly smooth if and only if E^* is uniformly convex.

Let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote the set of fixed points of T by $F(T)$, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. The mapping T is called *demicontinuous* if $\{x_n\}_{n\in\mathbb{N}} \subset C$ converging to x in the norm implies that $\{Tx_n\}_{n\in\mathbb{N}}$ converges weakly to Tx.

In recent years, several types of iterative schemes have been constructed and proved in order to get strong convergence results for nonexpansive mappings in various settings. One of the most important iterative algorithms for approximating fixed points of a nonexpansive mapping $T : C \to C$ is Halpern iteration, where C is a closed and convex subset of a Banach space E. Recall that the Halpern iteration is given by the following formula

$$(1.2) \qquad \begin{cases} u \in C,\ x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. The construction of fixed points of nonexpansive mappings via Halpern's algorithm [9] has been extensively investigated recently in the current literature (see, for example, [20] and the references therein). Numerous results have been proved on Halpern's iterations for nonexpansive mappings in Hilbert and Banach spaces (see, e.g., [9, 2]). Because of a simple construction, Halpern's iterations are widely used to approximate a solution of fixed points for nonexpansive mappings and other classes of nonlinear mappings by many authors in different styles.

Let E be a strictly convex and reflexive Banach space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2,\ \|x\| = \|f\|\},\ \forall x \in E.$$

Let C be a nonempty, closed and convex subset of E. The generalized projection Π_C from E onto C is defined and denoted by

$$\Pi_C(x) = \mathrm{argmin}_{y\in C} \phi(y, x),\ \forall x \in E,$$

where $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$; see [1, 12]. It is obvious from the definition of the function ϕ that

$$(\|x\| + \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2, \ \forall x,y \in E. \tag{1.3}$$

It is also clear that

$$\phi(x,y) = 0 \Longleftrightarrow x = y.$$

We have from the definition of ϕ that

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \ \forall x,y,z \in E. \tag{1.4}$$

In particular, it can be easily seen that

$$\phi(x,y) = -\phi(y,x) + 2\langle y - x, Jy - Jx\rangle, \ \forall x,y \in E. \tag{1.5}$$

Indeed, by letting $z = x$ in (1.4) and taking into account that $\phi(x,x) = 0$, we get the desired result. Let $\phi_* : E^* \times E^* \to \mathbb{R}$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2 \tag{1.6}$$

for $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi_*(Jy, Jx) = \phi(x,y), \ \forall x,y \in E. \tag{1.7}$$

We have from the definition of ϕ_* that

$$\phi_*(x^*, y^*) = \phi_*(x^*, z^*) + \phi_*(z^*, y^*) + 2\langle J^{-1}x^* - J^{-1}z^*, z^* - y^*\rangle, \ \forall x^*, y^*, z^* \in E^*. \tag{1.8}$$

In particular,

$$\phi_*(x^*, y^*) = -\phi_*(y^*, x^*) + 2\langle J^{-1}y^* - J^{-1}x^*, y^* - x^*\rangle, \ \forall x^*, y^* \in E^*. \tag{1.9}$$

Indeed, there exist $x, y, z \in E$ such that $J(x) = x^*$, $J(y) = y^*$ and $J(z) = z^*$. Therefore,

$$\begin{aligned} \phi_*(x^*, y^*) &= \phi_*(Jx, Jy) = \phi(y,x) = \phi(y,z) + \phi(z,x) + 2\langle y - z, Jz - Jx\rangle \\ &= \phi_*(Jz, Jy) + \phi_*(Jx, Jz) + 2\langle J^{-1}y^* - J^{-1}z^*, z^* - x^*\rangle \\ &= \phi_*(z^*, y^*) + \phi_*(x^*, z^*) + 2\langle J^{-1}x^* - J^{-1}z^*, z^* - y^*\rangle. \end{aligned}$$

Let C be a nonempty, closed and convex subset of a smooth Banach space E, and let T be a mapping from C into itself. A point $p \in C$ is said to be an *asymptotic fixed point* [17] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Let E be a real Banach space and let $g : E \to (-\infty, +\infty]$ be a convex function. The domain of g is denoted by dom $g = \{x \in E : g(x) < \infty\}$. Let $x \in$ int dom g and $y \in E$. The *right-hand derivative* of g at x in the direction y is defined and denoted by

$$g^o(x,y) = \lim_{t\downarrow 0} \frac{g(x+ty) - g(x)}{t}. \tag{1.10}$$

The function g is called be *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case $g^o(x,y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x. The function g is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function g is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. The function g is said to be *Fréchet* *differentiable* if it is Fréchet differentiable everywhere. It is well-known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [5, Proposition 1.1.10]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [14, p. 508]). The function g is said to be *bounded on bounded subsets of* E if $g(U)$ is bounded for each bounded subset U of E. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit (1.10) is attained uniformly for all $x \in X$ and $\|y\| = 1$. In that case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$ for all $x \in E$.

Let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by dom $A = \{x \in E : Ax \neq \emptyset\}$ and ran $A = \cup_{x \in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [21] if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* [22] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex.

Let C be a nonempty, closed and convex subset of a Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem [4]: Find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{1.12}$$

For solving the equilibrium problem, let us assume that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:
(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $y \in C$, the function $x \longmapsto f(x, y)$ is upper semicontinuous;
(A4) for each $x \in C$, the function $y \longmapsto f(x, y)$ is convex and lower semicontinuous.
The set of solutions of problem (1.10) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle y - x, Tx \rangle$ for all $x, y \in C$. Then $z \in EP(f)$ if and only if $\langle y - z, Tx \rangle \geq 0$ for all $y \in C$, *i.e.*, z is a solution of the variational inequality.

Following Matsushita and Takahashi [17], a mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* if $F(T)$ is nonempty and $\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T),\ x \in C$. The mapping T is called *relatively nonexpansive* if the following conditions are satisfied:
(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T),\ x \in C$;
(3) $\hat{F}(T) = F(T)$.

Recently, Takahashi and Zembayashi [25] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let T be a relatively nonexpansive mapping from C into* *itself such that* $F(T) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by*

(1.13)
$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for every $n \in \mathbb{N}$, *where* J *is the normalized duality mapping on* E, $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ *satisfies* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\}_{n\in\mathbb{N}} \subset [a,\infty)$ *for some* $a > 0$. *Then,* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\Pi_{F(T)\cap EP(f)}x$ *as* $n \to \infty$.

In 2010, Plubtieng and Ungchiterakool [18] proved the following strong convergence theorem for equilibrium problems in a Banach space.

Theorem 1.2. *Let* C *be a nonempty, closed and convex subset of uniformly smooth and uniformly convex Banach space* E. *Let* f *be a bifunction from* $C \times C \to \mathbb{R}$ *satisfying (A1)-(A4) and* $EP(f) \neq \emptyset$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{u_n\}_{n\in\mathbb{N}}$ *be sequences generated by*

(1.14)
$$\begin{cases} x_1 = x \in E, \\ u_n \in C = C_1 \quad \textit{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Ju_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ *satisfies either*

(a) $0 \leq \alpha_n < 1$ *for all* $n \in \mathbb{N}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$ *or,*
(b) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

Let $\{r_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0,\infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. *Then* $\{x_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$, *and* $\{y_n\}_{n\in\mathbb{N}}$ *converge strongly to* $\Pi_{EP(f)}x$ *as* $n \to \infty$.

Very recently, Zegeye and Shahzad [27] proved the following strong convergence theorem for two monotone mappings in Banach spaces.

Theorem 1.3. *Let* C *be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space* E *which also enjoys Kadec-Klee property. Let* A_1, $A_2 : C \to E^*$ *be two continuous monotone mappings. Let* $F := \cap_{i=1}^{2} VI(C, A_i) \neq \emptyset$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in C_1 = C, \\ u_n = T_{1,\gamma_n} x_n; \ v_n = T_{2,\gamma_n} x_n, \\ w_n = J^{-1}(\beta Ju_n + (1-\beta)Jv_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.15}$$

where $\beta \in (0,1)$, $\{\gamma_n\}_{n\in\mathbb{N}} \subset [c_1,\infty)$ *for some* $c_1 > 0$ *and* $T_{i,\gamma_n}(x) = \{z \in C : \langle y-z, A_i z\rangle + \frac{1}{\gamma_n}\langle y-z, Jz-Jx\rangle \geq 0,\ \forall y \in C\}$ *for all* $x \in E$, $i = 1,2$. *Then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by (1.15) converges strongly to* $\Pi_F x$ *as* $n \to \infty$.

The following open question was raised by Zegeye and Shahzad in their final remark of [27].

Open question 1.1. *Is it possible to obtain a strongly convergent sequence* $\{x_n\}_{n\in\mathbb{N}}$ *to a common solution of a variational inequality problem for two monotone operators without using the generalized projection of a point* x_0 *on the closed and convex sets* C_{n+1} *in more general Banach spaces?*

Remark 1.4. Though the iteration processes (1.13)-(1.15) as introduced by the authors mentioned above worked, it is easy to see that these processes seem cumbersome and complicated in the sense that at each stage of iteration, two different sets C_n and Q_n are computed and the next iterate taken as the generalized projection of x_0 on the intersection of C_n and Q_n. This seems difficult to do in application. But it is worth mentioning that, in all the above results for nonexpansive type mappings, the computation of closed and convex sets C_n and Q_n for each $n \in \mathbb{N}$ are required.

In this paper, we deal with a system of equilibrium problems in a uniformly smooth and strictly convex Banach space. First, we consider disadvantages of the iterative sequences in known results. Namely, generalized projections are not always available in a practical calculation. We attempt to improve these schemes and, by combining them with iterative method of the Halpern type, we obtain a new type of strong convergence theorem, which overcomes the drawbacks of the previous results. Next, we study Halpern type iterative schemes for finding common solutions of a system of equilibrium problems in a uniformly smooth and strictly convex Banach space. Then, we apply our results to the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space under suitable assumptions. The computation of closed and convex sets C_n and Q_n for each $n \in \mathbb{N}$ are not required. Consequently, the above question is answered in the affirmative in a reflexive Banach space setting. Our results improve and generalize many known results in the current literature; see, for example, [18, 25, 27].

2. Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be used in the sequel. The following lemma which is a generalization of Lemma 3.2 in [2] plays a key role in our results.

Lemma 2.1. *Let* E *be a reflexive, smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let* C *be a nonempty subset of* E *and* $\{T_n\}_{n\in\mathbb{N}}$ *be a family of mappings from* C *into* E. *Suppose that for any bounded subset* B *of* C *there exists a continuous increasing function* $h_B : [0,\infty) \to [0,\infty)$ *such that* $h_B(0) = 0$ *and*

$$\lim_{k,l\to\infty} \theta_l^k = 0, \tag{2.1}$$

where $\theta_l^k := \sup\{h_B(\|JT_kz - JT_lz\|) : z \in B\} < \infty$*, for all* $k, l \in \mathbb{N}$*. Then, for each* $x \in C$*,* $\{JT_nx\}_{n\in\mathbb{N}}$ *converges strongly to some point of* E^**. Moreover, let the mapping* T *be defined by*

$$Tx = \lim_{n\to\infty} T_nx, \quad \forall x \in C.$$

Then, $\limsup_{n\to\infty}\{h_B(\|JT_nz - JTz\|) : z \in B\} = 0$.

Proof. We first show that $\{JT_nx\}_{n\in\mathbb{N}}$ is a Cauchy sequence for each $x \in C$. To this end, let $k, l \in \mathbb{N}$ such that $k > l$. For any $x \in C$ let B be a bounded subset of C such that $x \in B$. Then we obtain

$$h_B(\|JT_kx - JT_lx\|) \leq \theta_l^k,$$

which implies that

$$\lim_{l\to\infty} h_B(\|JT_kx - JT_lx\|) = 0. \tag{2.2}$$

From the properties of the mapping h_B, we conclude that $\lim_{l\to\infty}\|JT_kx - JT_lx\| = 0$. This implies that $\{JT_nx\}_{n\in\mathbb{N}}$ is a Cauchy sequence in E^*. Since E^* is a Banach space, then there exists $w^* \in E^*$ such that $\lim_{n\to\infty}\|JT_nx - w^*\| = 0$. From $J(E) = E^*$, it follows that there exists $w \in E$ such that $Jw = w^*$. Thus we have $\lim_{n\to\infty}\|JT_nx - Jw\| = 0$. Since J^{-1} is demi-continuous, we conclude that $T_nx \rightharpoonup w$ as $n \to \infty$. In view of (1.6) and (1.7), we obtain

$$\lim_{n\to\infty}\phi(T_nx, w) = \lim_{n\to\infty}\phi_*(Jw, JT_nx) = 0.$$

This, together with (1.3), implies that $\|T_nx\| \to \|w\|$ as $n \to \infty$. By the Kadec-Klee property of E, we deduce that $\lim_{n\to\infty}\|T_nx - w\| = 0$. Now, we define the function $T : C \to C$ by

$$Tx = \lim_{n\to\infty} T_nx, \quad \forall x \in C.$$

Let $\epsilon > 0$ be fixed. It follows from (2.1) that there exists $n_0 \in \mathbb{N}$ such that for all $k, l > n_0$

$$h_B(\|JT_ky - JT_ly\|) < \frac{\epsilon}{2}, \quad \forall y \in B.$$

Let l be fixed and let $k \to \infty$. By the continuity of h_B and $\|.\|$, we deduce that

$$\begin{aligned} h_B(\|JTy - JT_ly\|) &= h_B(\|\lim_{k\to\infty} JT_ky - JT_ly\|) \\ &= \lim_{k\to\infty} h_B(\|JT_ky - JT_ly\|) \leq \frac{\epsilon}{2}, \quad \forall y \in B. \end{aligned} \tag{2.3}$$

In view of (2.3), we conclude that $\sup\{h_B(\|JTy - JT_ly\|) : y \in B\} \leq \frac{\epsilon}{2}$ and hence

$$\limsup_{l\to\infty}\sup\{h_B(\|JTy - JT_ly\|) : y \in B\} \leq \frac{\epsilon}{2} < \epsilon.$$

Since ϵ is arbitrary, we obtain that

$$\limsup_{l\to\infty}\sup\{h_B(\|JTy - JT_ly\|) : y \in B\} = 0,$$

which completes the proof. □

The following two lemmas have been proved in [18].

Lemma 2.2. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Let $T : C \to C$ be a quasi-ϕ-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.3. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and $\{T_n\}_{n\in\mathbb{N}}$ an infinite family of quasi-ϕ-nonexpansive mappings from C into itself such that $F := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let the mapping $T : C \to C$ be defined by*

$$Tx = \lim_{n\to\infty} T_n x.$$

Then, T is a quasi-ϕ-nonexpansive mapping.

The following result was first proved in [1] (see also [14]).

Lemma 2.4. *Let E be a smooth, strictly convex and reflexive Banach space and let V be the function defined by*

$$V(x, x^*) = \|x\|^2 - \langle x, x^*\rangle + \|x^*\|^2, \qquad \forall x \in E,\ \forall x^* \in E^*.$$

Then the following assertions hold:

(1) *$\phi(x, J^{-1}x^*) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.*
(2) *$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.*

The following result has been proved in [6].

Lemma 2.5. *Let E be a uniformly convex Banach space and $r > 0$ be a constant. Then there exists a continuous, strictly increasing and convex function $h : [0,\infty) \to [0,\infty)$ such that*

$$\left\| \sum_{k=0}^{\infty} \alpha_k x_k \right\|^2 \leq \sum_{k=0}^{\infty} \alpha_k \|x_k\|^2 - \alpha_i \alpha_j h(\|x_i - x_j\|)$$

for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r := \{z \in E : \|z\| \leq r\}$, $\alpha_k \in (0,1)$ and $k \in \mathbb{N} \cup \{0\}$ with $\sum_{k=0}^{\infty} \alpha_k = 1$.

Lemma 2.6. *Let E be a uniformly smooth and strictly convex Banach space. Let $s > 0$ be a constant. Then there exists a continuous, strictly increasing and convex function $\rho_s : [0,\infty) \to [0,\infty)$ such that*

$$\rho_s(\|x^* - y^*\|) \leq \phi_*(x^*, y^*)$$

for any $x^*, y^* \in B_s := \{z^* \in E^* : \|z^*\| \leq s\}$.

Proof. Since E is a uniformly smooth Banach space, E^* is a uniformly convex Banach space. Then, in view of Lemma 2.5, there exists a continuous, strictly increasing and convex function $\rho_s : [0,\infty) \to [0,\infty)$ such that

$$\|\alpha x^* + (1-\alpha)y^*\|^2 \leq \alpha\|x^*\|^2 + (1-\alpha)\|y^*\|^2 - \alpha(1-\alpha)\rho_s(\|x^* - y^*\|)$$

for all $x^*, y^* \in B_s = \{z^* \in E^* : \|z^*\| \leq s\}$ and all $\alpha \in (0,1)$. If $x^*, y^* \in B_s$, then we obtain

$$\frac{\|\alpha x^* + (1-\alpha)y^*\|^2 - \|y^*\|^2}{\alpha} \leq \|x^*\|^2 - \|y^*\|^2 - (1-\alpha)\rho_s(\|x^* - y^*\|).$$

Letting $\alpha \to 0$ in the above inequality, we arrive at

$$\langle 2J^{-1}y^*, x^* - y^* \rangle \leq \|x^*\|^2 - \|y^*\|^2 - \rho_s(\|x^* - y^*\|).$$

This implies that

$$\rho_s(\|x^* - y^*\|) \leq \phi_*(x^*, y^*),$$

which completes the proof. □

Lemma 2.7. *Let E be a uniformly smooth and strictly convex Banach space. Let $\{x_n^*\}_{n\in\mathbb{N}}$ and $\{y_n^*\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent:*

(1) $\lim_{n\to\infty} \phi_*(x_n^*, y_n^*) = 0$.
(2) $\lim_{n\to\infty} \|x_n^* - y_n^*\| = 0$.

Proof. The implication (1) $\Longrightarrow$ (2) is an immediate consequence of Lemma 2.6. For the converse implication, we assume that $\lim_{n\to\infty} \|x_n^* - y_n^*\| = 0$. Then, in view of (1.9), we have

$$\begin{aligned} \phi_*(x_n^*, y_n^*) &= -\phi_*(y_n^*, x_n^*) + 2\langle J^{-1}y_n^* - J^{-1}x_n^*, y_n^* - x_n^* \rangle \\ &\leq 2\|x_n^* - y_n^*\|\|J^{-1}x_n^* - J^{-1}y_n^*\|, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.4}$$

Since J^{-1} is bounded on bounded subsets of E^* (see, for example, [23] for more details). This, together with (2.4), implies that $\lim_{n\to\infty} \phi_*(x_n^*, y_n^*) = 0$, which completes the proof. □

Lemma 2.8 ([16])**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.9 ([26])**.** *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the inequality:*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy the conditions:

(i) *$\{\gamma_n\}_{n\in\mathbb{N}} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$, or equivalently, $\Pi_{n=1}^{\infty}(1 - \gamma_n) = 0$;*
(ii) *$\limsup_{n\to\infty} \delta_n \leq 1$, or*
(ii)′ *$\sum_{n=1}^{\infty} \gamma_n \delta_n < \infty$.*

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.10 ([12])**.** *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E, $x \in W$ and $z \in C$. Then*

(i) *$z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$;*
(ii) *$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$ for all $y \in C$.*

3. Equilibrium problems

In this section, we prove strong convergence theorems in a reflexive Banach space. Let E be a Banach space and C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $EP(f) \neq \emptyset$. For $r > 0$, we define a mapping $T_r : E \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C \right\} \tag{3.1}$$

for all $x \in E$.

Lemma 3.1 ([25]). *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For $r > 0$, let $T_r : E \to C$ be the mapping defined by (3.1). Then, dom $(T_r) = E$.*

Lemma 3.2 ([25]). *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) such that $EP(f) \neq \emptyset$. For $r > 0$, let $T_r : E \to C$ be the mapping defined by (3.1). Then, the following statements hold:*

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive mapping* [25]*, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex;*
(5) *T_r is a quasi-ϕ-nonexpansive mapping;*
(6) *$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x), \ \forall q \in F(T_r)$.*

Using ideas in [7], we can prove the following result.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E which also enjoys Kadec-Klee property. For any $j \in \mathbb{N}$, let $f_j : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For $r > 0$ and $j \in \mathbb{N}$, let $T_{r,j} : E \to C$ be the mapping defined by (3.1). Suppose that $F := \cap_{j=1}^{\infty} EP(f_j)$ is a nonempty subset of C, where $EP(f_j)$ is the set of solutions to the equilibrium problem (1.12). Let $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_{n,j}\}_{n\in\mathbb{N}, j\in\mathbb{N}\cup\{0\}}$ be sequences in $[0, 1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $\beta_{n,0} + \sum_{j=1}^{\infty} \beta_{n,j} = 1, \ \forall n \in \mathbb{N}$;
(d) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,j} > 0, \ \forall j \in \mathbb{N}$.

Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in C, \; x_1 \in C \quad \text{chosen arbitrarily}, \\ u_{n,j} \in C \quad \text{such that} \\ \quad f_j(u_{n,j}, y) + \frac{1}{r_n}\langle y - u_{n,j}, Ju_{n,j} - Jx_n\rangle \geq 0, \quad \forall j \in \mathbb{N}, \; y \in C, \\ y_n = J^{-1}[\beta_{n,0}Jx_n + \sum_{j=1}^{\infty}\beta_{n,j}Ju_{n,j}], \\ x_{n+1} = \Pi_C(J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n]) \; \text{and} \; n \in \mathbb{N}. \end{cases} \tag{3.2}$$

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.2) converges strongly to $\Pi_F u$ as $n \to \infty$.

Proof. We divide the proof into several steps. In view of Lemma 3.2, we conclude that F is closed and convex. Set

$$z = \Pi_F u.$$

Step 1. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{u_{n,j}\}_{n,j\in\mathbb{N}}$ are bounded sequences in E. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in \cap_{j=1}^{\infty}EP(f_j)$ be fixed. In view of Lemmas 2.4, 3.2 and (3.2), we have

$$\begin{aligned} \phi(p, y_n) &= \phi(p, J^{-1}[\beta_{n,0}Jx_n + \sum_{j=1}^{\infty}\beta_{n,j}JT_{r_n,j}x_n]) \\ &= V(p, \beta_{n,0}Jx_n + \sum_{j=1}^{\infty}\beta_{n,j}JT_{r_n,j}x_n) \\ &\leq \beta_{n,0}V(p, Jx_n) + \sum_{j=1}^{\infty}\beta_{n,j}V(p, JT_{r_n,j}x_n) \\ &= \beta_{n,0}\phi(p, x_n) + \sum_{j=1}^{\infty}\beta_{n,j}\phi(p, T_{r_n,j}x_n) \\ &\leq \beta_{n,0}\phi(p, x_n) + \sum_{j=1}^{\infty}\beta_{n,j}\phi(p, x_n) \\ &= \phi(p, x_n). \end{aligned} \tag{3.3}$$

This implies that

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, \Pi_C(J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n])) \\ &\leq \phi(p, J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n]) \\ &= V(p, \alpha_n Ju + (1-\alpha_n)Jy_n) \\ &\leq \alpha_n V(p, Ju) + (1-\alpha_n)V(p, Jy_n) \\ &= \alpha_n\phi(p, u) + (1-\alpha_n)\phi(p, y_n) \\ &\leq \alpha_n\phi(p, u) + (1-\alpha_n)\phi(p, y_n) \\ &\leq \alpha_n\phi(p, u) + (1-\alpha_n)\phi(p, x_n) \\ &\leq \max\{\phi(p, u), \phi(p, x_n)\}. \end{aligned} \tag{3.4}$$

By induction, we obtain

$$\phi(p, x_{n+1}) \leq \max\{\phi(p, u), \phi(p, x_1)\} \tag{3.5}$$

for all $n \in \mathbb{N}$. It follows from (3.5) that the sequence $\{\phi(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_0 > 0$ such that

$$\phi(x_n, x) \leq M_0, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

In view of (1.3), we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{T_{r_n,j}\}_{n,j\in\mathbb{N}}$ is an infinite family of relatively nonexpansive mappings from C into itself, we deduce that

$$\phi(p, u_{m,j}) = \phi(p, T_{r_n,j}x_m) \leq \phi(p, x_m), \quad \forall n, m, j \in \mathbb{N}. \tag{3.7}$$

This, together with (1.3) and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{T_{r_n,j}x_n\}_{n,j\in\mathbb{N}}$ is bounded. Since J is also bounded on bounded subsets of E, the sequences $\{Jx_n\}_{n\in\mathbb{N}}$, $\{Jy_n\}_{n\in\mathbb{N}}$ and $\{JT_{r_n,j}x_n\}_{n\in\mathbb{N}}$ are bounded in E^*.

Step 2. We show that, for any $j \in \mathbb{N}$, there exists a mapping $T_j : C \to C$ such that

$$T_j x = \lim_{n\to\infty} T_{r_n,j}x, \quad \forall x \in C$$

and

$$F(T_j) = \cap_{n=1}^{\infty} F(T_{r_n,j}) = \cap_{n=1}^{\infty} \hat{F}(T_{r_n,j}) = \hat{F}(T_j).$$

Since $T_{r_n,j}$ is a quasi-ϕ-nonexpansive mapping, we have

$$\phi(z, T_{r_n,j}v) \leq \phi(z, v), \quad \forall v \in E, \ n, j \in \mathbb{N}.$$

This, together with (1.3), implies that for any bounded subset B of E with $\{x_n\}_{n\in\mathbb{N}} \subset B$, $\{T_{r_n,j}v : v \in B\}$ is bounded. For any $v \in E$, we set $v_{n,j} = T_{r_n,j}v$. Then we get

$$f_j(v_{l,j}, y) + \frac{1}{r_l}\langle y - v_{l,j}, Jv_{l,j} - Jv\rangle \geq 0, \quad \forall y \in C \tag{3.8}$$

and

$$f_j(v_{k,j}, y) + \frac{1}{r_k}\langle y - v_{k,j}, Jv_{k,j} - Jv\rangle \geq 0, \quad \forall y \in C. \tag{3.9}$$

Letting $y = v_{k,j}$ in (3.8) and $y = v_{l,j}$ in (3.9), we conclude that

$$f_j(v_{l,j}, v_{k,j}) + \frac{1}{r_l}\langle v_{k,j} - v_{l,j}, Jv_{l,j} - Jv\rangle \geq 0$$

and

$$f_j(v_{k,j}, v_{l,j}) + \frac{1}{r_k}\langle v_{l,j} - v_{k,j}, Jv_{k,j} - Jv\rangle \geq 0.$$

Now, in view of (A2) we obtain

$$\Big\langle v_{k,j} - v_{l,j}, \frac{Jv_{l,j} - Jv}{r_l} - \frac{Jv_{k,j} - Jv}{r_k}\Big\rangle \geq 0$$

and hence

$$\Big\langle v_{k,j} - v_{l,j}, Jv_{l,j} - Jv - \frac{r_l}{r_k}(Jv_{k,j} - Jv)\Big\rangle \geq 0.$$

Therefore,

$$\langle v_{k,j} - v_{l,j}, Jv_{k,j} - Jv_{l,j}\rangle + \Big\langle v_{k,j} - v_{l,j}, \Big(1 - \frac{r_l}{r_k}\Big)(Jv_{k,j} - Jv)\Big\rangle \geq 0.$$

Without loss of generality, we may assume that there exists a real number a such that $r_n > a$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\langle v_{k,j} - v_{l,j}, Jv_{k,j} - Jv_{l,j}\rangle &\le \Big\langle v_{k,j} - v_{l,j}, \Big(1 - \frac{r_l}{r_k}\Big)(Jv_{l,j} - Jv)\Big\rangle \\ &\le \frac{1}{a}\|v_{k,j} - v_{l,j}\||r_k - r_l|\|Jv_{k,j} - Jv\| \\ &= \frac{1}{a}\|T_{r_k,j}v - T_{r_l,j}v\|\|JT_{r_k,j}v - JT_{r_l,j}v\||r_k - r_l|.\end{aligned}$$

In view of Lemma 3.2, we have $EP(f_j) = \cap_{n=1}^{\infty} F(T_{r_n,j})$. Let

$$M_1 = \sup\Big\{\frac{1}{a}\|T_{r_k,j}v - T_{r_l,j}v\|\|JT_{r_k,j}v - JT_{r_l,j}v\| : v \in B,\ j,k,l \in \mathbb{N}\Big\}.$$

Putting $s_1 = \sup\{\|T_{r_k,j}v\|, \|T_{r_l,j}v\|, \|JT_{r_k,j}v\|, \|JT_{r_l,j}v\|, \|v\|, \|Jv\| : k,l,j \in \mathbb{N},\ v \in B\}$, in view of Lemma 2.6, there exists a strictly increasing, continuous and convex function $\rho_{s_1} : [0,\infty) \to [0,\infty)$ such that for all $v \in B$,

$$\begin{aligned}\rho_{s_1}(\|JT_{r_k,j}v - JT_{r_l,j}v\|) &= \rho_{s_1}(\|Jv_{k,j} - Jv_{l,j}\|) \le \phi_*(Jv_{k,j}, Jv_{l,j}) \\ &= -\phi_*(Jv_l, Jv_k) + 2\langle v_k - v_l, Jv_k - Jv_l\rangle \\ &\le 2\|v_{k,j} - v_{l,j}\|\|Jv_{k,j} - Jv_{l,j}\| \le M_1|r_k - r_l| \\ &\le 2M_1\sum_{n=l}^{k-1}|r_{n+1} - r_n| \le 2M_1\sum_{n=l}^{\infty}|r_{n+1} - r_n| < \infty.\end{aligned}$$

Let

$$\theta_l^k := \sup\{\rho_{s_1}(\|JT_{r_k,j}v - JT_{r_l,j}v\|) : j \in \mathbb{N}, v \in B\} \le 2M_1\sum_{n=l}^{\infty}|r_{n+1} - r_n| < \infty.$$

Letting $l \to \infty$ in the above inequality, we get $\lim_{k,l\to\infty}\theta_l^k = 0$. This implies that, for any $x \in E$, $\lim_{k,l\to\infty}\|JT_{r_k,j}x - JT_{r_l,j}x\| = 0$. Since E^* is a Banach space, then there exists $w_j^* \in E^*$ such that $\lim_{n\to\infty}\|JT_{r_n,j}x - w_j^*\| = 0$. From $J(E) = E^*$, it follows that there exists $w_j \in E$ such that $Jw_j = w_j^*$. Thus we have $\lim_{n\to\infty}\|JT_{r_n,j}x - Jw_j\| = 0$. Since J^{-1} is demicontinuous, we conclude that $T_{r_n,j}x \rightharpoonup w_j$ as $n \to \infty$. In view of (1.6) and (1.7), we obtain

$$\lim_{n\to\infty}\phi(T_{r_n,j}x, w_j) = \lim_{n\to\infty}\phi_*(Jw_j, JT_{r_n,j}x) = 0.$$

This, together with (1.3), implies that $\|T_{r_n,j}x\| \to \|w_j\|$ as $n \to \infty$. By the Kadec-Klee property of E, we deduce that $\lim_{n\to\infty}\|T_{r_n,j}x - w_j\| = 0$. Now, for any $j \in \mathbb{N}$, we define the mapping $T_j : C \to C$ by

$$T_jx = \lim_{n\to\infty}T_{r_n,j}x, \quad \forall x \in C.$$

We prove that

$$F(T_j) = \cap_{n=1}^{\infty}F(T_{r_n,j}) = \cap_{n=1}^{\infty}\hat{F}(T_{r_n,j}) = \hat{F}(T_j). \tag{3.10}$$

We first note that the following assertions are obvious:

(1) $\cap_{n=1}^{\infty}F(T_{r_n,j}) = \cap_{n=1}^{\infty}\hat{F}(T_{r_n,j}) = \hat{F}(T_j)$.

(2) $\cap_{n=1}^{\infty}F(T_{r_n,j}) \subset F(T_j)$ and $\cap_{n=1}^{\infty}\hat{F}(T_{r_n,j}) \subset \hat{F}(T_j)$.

It remains to prove that (3) $F(T_j) \subset \cap_{n=1}^{\infty} F(T_{r_n,j})$ and (4) $\hat{F}(T) \subset \cap_{n=1}^{\infty} \hat{F}(T_{r_n,j})$.

(3) Let $p \in F(T_j)$ be fixed. In view of the definition of T_r, we have

$$f_j(T_{r_n,j}p, y) + \frac{1}{r_n}\langle y - T_{r_n,j}p, JT_{r_n,j}p - Jp\rangle \geq 0, \quad \forall y \in C.$$

In view of (A2), we obtain

$$\frac{1}{r_n}\langle y - T_{r_n,j}p, JT_{r_n,j}p - Jp\rangle \geq f_j(y, T_{r_n,j}p), \quad \forall y \in C.$$

Since $T_{r_n,j}p \to T_j p = p$ as $n \to \infty$, J is uniformly continuous on bounded subsets of E and $f_j(y, .)$ is lower semicontinuous, we conclude that $f_j(y, p) \leq 0$ for all $y \in C$. Take any $y \in C$ and set $x_t = ty + (1-t)p$, for $t \in (0, 1]$. Then, we obtain

$$0 \leq f_j(x_t, x_t) \leq tf_j(x_t, y) + (1-t)f_j(x_t, p) \leq tf_j(x_t, y).$$

This implies that $f_j(x_t, y) \geq 0$. Letting $t \downarrow 0$ and taking into account (A3), we get $f_j(p, y) \geq 0$ for all $y \in C$ and hence $p \in EP(f_j) = \cap_{n=1}^{\infty} F(T_{r_n,j})$.

(4) Let $q \in \hat{F}(T_j)$. Then, there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset E$ such that $v_n \rightharpoonup q$ as $n \to \infty$ and $\lim_{n\to\infty} \|v_n - T_j v_n\| = 0$. This implies that $T_j v_n \rightharpoonup q$ as $n \to \infty$. Hence $q \in C$. Since J is uniformly continuous on bounded subsets of E, we conclude that $\lim_{n\to\infty} \|Jv_n - JT_j v_n\| = 0$. For any $m \in \mathbb{N}$, it follows from the definition of $T_{r_m,j}$ that

$$f_j(T_{r_m,j}v_n, y) + \frac{1}{r_m}\langle y - T_{r_m,j}v_n, JT_{r_m,j}v_n - Jv_n\rangle \geq 0, \quad \forall y \in C.$$

In view of (A2) and taking into account $\frac{1}{r_m} \leq \frac{1}{a}$, we obtain

$$\begin{aligned} f_j(y, T_{r_m,j}v_n) &\leq \frac{1}{r_m}\langle y - T_{r_m,j}v_n, JT_{r_m,j}z_n - Jz_n\rangle \\ &\leq \frac{1}{a}\|y - T_{r_m,j}v_n\|\|JT_{r_m,j}v_n - Jv_n\|, \quad \forall y \in C. \end{aligned}$$

Since $\lim_{m\to\infty} T_{r_m,j}v_n = T_j v_n$ and $f_j(y, .)$ is lower semicontinuous, we arrive at

$$f_j(y, Tv_n) \leq \frac{1}{a}\|y - T_j v_n\|\|JT_j v_n - Jv_n\| \quad \forall y \in C.$$

Since $T_j v_n \rightharpoonup q$ as $n \to \infty$, $\lim_{n\to\infty} \|v_n - T_j v_n\| = 0$ and $f_j(y, .)$ is lower semicontinuous, we deduce that $f_j(y, q) \leq 0$ for all $y \in C$. By the same manner as above we conclude that $f_j(q, y) \geq 0$ for all $y \in C$. Therefore, $q \in EP(f_j) = \cap_{n=1}^{\infty} \hat{F}(T_{r_n,j})$.

Step 3. We prove that there exists a continuous, strictly increasing and convex function $h : [0, \infty) \to [0, \infty)$ such that for any $n \in \mathbb{N}$

$$\phi(z, y_n) \leq \phi(z, x_n) - \beta_{n,0}\beta_{n,j}h(\|Jx_n - JT_{r_n,j}x_n\|), \quad \forall j \in \mathbb{N}. \tag{3.11}$$

Let us show (3.11). Let $n, j \in \mathbb{N}$ be fixed. In view of the Lemma 2.5 there exists a continuous, strictly increasing and convex function $h : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}[\beta_{n,0}Jx_n + \sum_{j=1}^{\infty}\beta_{n,j}JT_{r_n,j}x_n]) \\
&= \|p\|^2 - 2\langle p, \beta_{n,0}Jx_n + \sum_{j=1}^{\infty}\beta_{n,j}JT_{r_n,j}x_n)\rangle \\
&\quad + \|\beta_{n,0}Jx_n + \sum_{j=1}^{\infty}\beta_{n,j}JT_{r_n,j}x_n)\|^2 \\
&\le \|p\|^2 - 2\langle p, \beta_{n,0}Jx_n\rangle - 2\sum_{j=1}^{\infty}\beta_{n,j}\langle p, JT_{r_n,j}x_n)\rangle \\
&\quad + \|\beta_{n,0}Jx_n + \sum_{j=1}^{\infty}\beta_{n,j}JT_{r_n,j}x_n)\|^2 \\
&\le \|p\|^2 - 2\beta_{n,0}\langle p, Jx_n\rangle - 2\sum_{j=1}^{\infty}\beta_{n,j}\langle p, JT_{r_n,j}x_n)\rangle \\
&\quad + \beta_{n,0}\|Jx_n\|^2 + \sum_{j=1}^{\infty}\beta_{n,j}\|JT_{r_n,j}x_n)\|^2 - \beta_{n,0}\beta_{n,j}h(\|Jx_n - JT_{r_n,j}x_n\|) \\
&= \beta_{n,0}\phi(p, x_n) + \sum_{j=1}^{\infty}\beta_{n,j}\phi(p, T_{r_n,j}x_n) - \beta_{n,0}\beta_{n,j}h(\|Jx_n - JT_{r_n,j}x_n\|) \\
&\le \beta_{n,0}\phi(p, x_n) + \sum_{j=1}^{\infty}\beta_{n,j}\phi(p, x_n) - \beta_{n,0}\beta_{n,j}h(\|Jx_n - JT_{r_n,j}x_n\|) \\
&= \phi(p, x_n) - \beta_{n,0}\beta_{n,j}h(\|Jx_n - JT_{r_n,j}x_n\|).
\end{aligned}$$

In view of Lemma 2.4 and (3.11), we obtain

$$\begin{aligned}
\phi(z, x_{n+1}) &= \phi(p, \Pi_C(J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n])) \\
&\le \phi(z, J^{-1}[\alpha_n Jz + (1-\alpha_n)Jy_n]) \\
&= V(z, \alpha_n Ju + (1-\alpha_n)Jy_n) \\
&\le \alpha_n V(z, Ju) + (1-\alpha_n)V(z, Jy_n) \\
&= \alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, y_n) \\
&\le \alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, y_n) \\
&\le \alpha_n\phi(z, u) \\
&\quad + (1-\alpha_n)[\phi(z, x_n) - \beta_{n,0}\beta_{n,j}g(\|Jx_n - JT_{r_n,j}x_n\|)].
\end{aligned} \tag{3.12}$$

Let $M_2 := \sup\{|\phi(z,u) - \phi(z,x_n)| + \beta_{n,0}\beta_{n,j}h(\|Jx_n - JT_{r_n,j}x_n\|) : n, j \in \mathbb{N}\}$. It follows from (3.12) that

$$\beta_{n,0}\beta_{n,j}h(\|Jx_n - JT_{r_n,j}x_n\|) \le \phi(z, x_n) - \phi(z, x_{n+1}) + \alpha_n M_2, \ \forall j \in \mathbb{N}. \tag{3.13}$$

Let $z_n = J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n]$. Then $x_{n+1} = \Pi_C(z_n)$ for all $n \in \mathbb{N}$. In view of Lemma 2.2 and (3.11) we obtain

$$\begin{aligned}
\phi(z, x_{n+1}) &= \phi(p, \Pi_C(J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n])) \\
&\leq D_g(z, J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n]) \\
&= V(z, \alpha_n Ju + (1-\alpha_n)Jy_n) \\
&\leq V(z, \alpha_n Ju + (1-\alpha_n)Jy_n - \alpha_n(Ju - Jz)) \\
&\quad -\langle J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n] - z, -\alpha_n(Ju - Jz)\rangle \\
&= V(z, \alpha_n Jz + (1-\alpha_n)Jy_n) + \alpha_n\langle z_n - z, Ju - Jz\rangle \\
&= \phi(z, J^{-1}[\alpha_n Jz + (1-\alpha_n)Jy_n]) \\
&\quad +\alpha_n\langle z_n - z, Ju - Jz\rangle \\
&\leq \alpha_n\phi(z, z) + (1-\alpha_n)\phi(z, y_n) + \alpha_n\langle z_n - z, Ju - Jz\rangle \\
&= (1-\alpha_n)\phi(z, x_n) + \alpha_n\langle z_n - z, Ju - Jz\rangle.
\end{aligned} \tag{3.14}$$

Step 4. We show that $x_n \to z$ as $n \to \infty$.

The rest of the proof will be divided into two parts:

Case 1. If there exists $n_0 \in \mathbb{N}$ such that $\{\phi(z, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing, then $\{\phi(z, x_n)\}_{n\in\mathbb{N}}$ is convergent. Thus, we have $\phi(z, x_n) - \phi(z, x_{n+1}) \to 0$ as $n \to \infty$. This, together with conditions (c) and (d), implies that

$$\lim_{n\to\infty} h(\|Jx_n - JT_{r_n,j}x_n\|) = 0.$$

Therefore, from the property of h we deduce that

$$\lim_{n\to\infty} \|Jx_n - JT_{r_n,j}x_n\| = 0.$$

We notice that by Step 2, we get that

$$\lim_{k,l\to\infty} \theta_l^k := \lim_{k,l\to\infty} \sup\{\rho_{s_1}(\|JT_{r_k,j}v - JT_{r_l,j}v\|) : j \in \mathbb{N}, v \in B\} = 0.$$

Then, in view of Lemma 2.1, we conclude that

$$\limsup_{n\to\infty} \sup\{\rho_{s_1}(\|JT_{r_n,j}y - JT_jy\|) : y \in B\} = 0, \ \forall j \in \mathbb{N}.$$

On the other hand, we have

$$\frac{1}{2}\|Jx_n - JT_jx_n\| \leq \frac{1}{2}\|Jx_n - JT_{r_n,j}x_n\| + \frac{1}{2}\|JT_{r_n,j}x_n - JT_jx_n\|.$$

This implies that

$$\begin{aligned}
\rho_{s_1}\Big(\frac{1}{2}\|Jx_n - JT_jx_n\|\Big) &\leq \rho_{s_1}\Big(\frac{1}{2}\|Jx_n - JT_{r_n,j}x_n\|\Big) + \rho_{s_1}\Big(\frac{1}{2}\|JT_{r_n,j}x_n - JT_jx_n\|\Big) \\
&\leq \frac{1}{2}\rho_{s_1}(\|Jx_n - JT_{r_n,j}x_n\|) \\
&\quad +\frac{1}{2}\sup\{\rho_{s_1}(\|JT_{r_n,j}v - JT_jv\|) : v \in B\}.
\end{aligned}$$

Exploiting Lemma 2.3, we obtain

$$\lim_{n\to\infty} \rho_{s_1}(\|Jx_n - JT_j x_n\|) = 0.$$

By the properties of ρ_{s_1}, we conclude that

$$\lim_{n\to\infty} \|Jx_n - JT_j x_n\| = 0. \tag{3.15}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup y \in C$. Using the lower semi-continuity of the norm, we get that

$$0 \le \liminf_{i\to\infty} \phi(x_{n_i}, y) \le \limsup_{i\to\infty} \phi(x_{n_i}, y) \le \phi(y, y) = 0.$$

This, together with (1.3), implies that $\|x_{n_i}\| \to \|y\|$ as $i \to \infty$. By the Kadec-Klee property of E, we conclude that $x_{n_i} \to y$ as $i \to \infty$. Thus we have $\|Jx_{n_i} - Jy\| \to 0$ as $i \to \infty$. This, together with (3.15), implies that $\|JT_j x_{n_i} - Jy\| \to 0$ as $i \to \infty$. Since J^{-1} is demi-continuous, we obtain that $T_j x_{n_i} \rightharpoonup y$ as $i \to \infty$. It follows from Lemma 2.7 that $\lim_{i\to\infty} \phi_*(Jy, JT_j x_{n_i}) = 0$ and hence $\lim_{i\to\infty} \phi(T_j x_{n_i}, y) = 0$. This, together with (1.3), implies that $\|T_j x_{n_i}\| \to \|y\|$ as $i \to \infty$. Using the Kadec-Klee property of E, we deduce that $\|T_j x_{n_i} - y\| \to 0$ as $i \to \infty$. Thus we have $\|Jx_{n_i} - Jy\| \to 0$ as $i \to \infty$. Therefore, $\lim_{i\to\infty} \|T_j x_{n_i} - x_{n_i}\| = 0$, which implies that $y \in F(T_j)$ for all $j \in \mathbb{N}$. This, together with Lemma 3.2 and (3.12), implies that $y \in F(T_j) = \cap_{n=1}^{\infty} F(T_{r_n,j}) = EP(f_j)$. On the other hand, we have

$$\limsup_{n\to\infty} \langle x_n - z, Jx - Jz\rangle = \lim_{i\to\infty} \langle x_{n_i} - z, Jx - Jz\rangle.$$

This, together with Lemma 2.10, implies that

$$\limsup_{n\to\infty} \langle x_n - z, Jx - Jz\rangle = \langle y - z, Jx - Jz\rangle \le 0.$$

From Lemma 2.10, we have that

$$\limsup_{n\to\infty} \langle z_n - z, Ju - Jz\rangle = \limsup_{n\to\infty} \langle x_n - z, Ju - Jz\rangle \le 0.$$

Thus we have the desired result by Lemma 2.9.

Case 2. If there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$\phi(z, x_{n_i}) < \phi(z, x_{n_i+1})$$

for all $i \in \mathbb{N}$, then by Lemma 2.8, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\phi(z, x_{m_k}) < \phi(z, x_{m_k+1}) \quad \text{and} \quad \phi(z, x_k) \le \phi(z, x_{m_k+1})$$

for all $k \in \mathbb{N}$. This, together with (3.13), implies that

$$\beta_{m_k,0}\beta_{m_k,j} h(\|Jx_{m_k} - JT_{r_{m_k},j} x_{m_k}\|) \le \phi(z, x_{m_k}) - \phi(z, x_{m_k+1}) + \alpha_{m_k} M_2 \le \alpha_{m_k} M_2$$

for all $k \in \mathbb{N}$. Then, by conditions (a) and (c), we get

$$\lim_{k\to\infty} h(\|Jx_{m_k} - JT_{m_k,j} x_{m_k}\|) = 0.$$

By the same argument, as in Case 1, we arrive at

$$\limsup_{k\to\infty} \langle z_{m_k} - z, Ju - Jz\rangle = \limsup_{k\to\infty} \langle x_{m_k} - z, Ju - Jz\rangle \le 0.$$

It follows from (3.14) that

$$\phi(z, x_{m_k+1}) \leq (1-\alpha_{m_k})\phi(z, x_{m_k}) + \alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle. \tag{3.16}$$

Since $\phi(z, x_{m_k}) \leq \phi(z, x_{m_k+1})$, we have that

$$\begin{aligned} \alpha_{m_k}\phi(z, x_{m_k}) &\leq \phi(z, x_{m_k}) - \phi(z, x_{m_k+1}) + \alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle \\ &\leq \alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle. \end{aligned} \tag{3.17}$$

In particular, since $\alpha_{m_k} > 0$, we obtain

$$\phi(z, x_{m_k}) \leq \langle z_{m_k} - z, Ju - Jz\rangle.$$

In view of (3.16), we deduce that

$$\lim_{k\to\infty} \phi(z, x_{m_k}) = 0.$$

This, together with (3.17), implies that

$$\lim_{k\to\infty} D_g(z, x_{m_k+1}) = 0.$$

On the other hand, we have $\phi(z, x_k) \leq \phi(z, x_{m_k+1})$ for all $k \in \mathbb{N}$ which implies that $\phi(z, x_n) \to 0$ as $n \to \infty$. In view of (1.3), we obtain that $\|x_n\| \to \|z\|$ as $n \to \infty$. On the other hand, in view of (1.6), we have

$$\lim_{n\to\infty} \phi_*(Jz, Jx_n) = \lim_{n\to\infty} \phi(x_n, z) = 0.$$

Applying Lemma 2.7 we obtain that

$$\lim_{n\to\infty} \|Jz - Jx_n\| = 0.$$

Since J^{-1} is demi-continuous, we get that $x_n \rightharpoonup z$, which implies that $y = z$. Hence, by the Kadec-Klee property of E, we conclude that $x_{n_i} \to z$ as $i \to \infty$. Thus, for any subsequence $\{x_{n_l}\}_{l\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{x_{n_{l_i}}\}_{i\in\mathbb{N}}$ of $\{x_{n_l}\}_{l\in\mathbb{N}}$ such that $x_{n_{l_i}} \to z$ as $i \to \infty$. Therefore, $x_n \to z$ as $n \to \infty$ which completes the proof. □

Let E be a smooth Banach space and let C be a nonempty, closed and convex subset of E. A mapping $T : C \to C$ is called *generalized nonexpansive* [10, 11] if $F(T) \neq \emptyset$ and

$$\phi(Tx, p) \leq \phi(x, p)$$

for each $x \in C$ and $p \in F(T)$. Let D be a nonempty closed subset of a real Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx$$

for each $x \in E$. A mapping $R : E \to D$ is said to be a *retraction* if $Rx = x$ for each $x \in D$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto C is uniquely determined (see [10, 11, 15]). Then, such a sunny generalized nonexpansive retraction of E onto D is denoted by R_D. A nonempty subset D of E is said to be a *sunny generalized nonexpansive retract* (resp. a *generalized nonexpansive retract*) of E if there exists a sunny generalized nonexpansive retraction (resp. a generalized nonexpansive retraction) of E onto D. The set of all fixed points of such a sunny generalized nonexpansive retraction of E onto D is, of course, D.

Theorem 3.4 ([10, 11])**.** *Let E be a reflexive, strictly convex and smooth Banach space and let D be a nonempty subset of E. Then the following statements are equivalent:*

(1) *D is a sunny generalized nonexpansive retract of E;*
(2) *D is a generalized nonexpansive retract of E;*
(3) *JC is closed and convex.*

In this case, D is closed.

Using Theorems 3.3 and 3.4, we can prove the following result.

Corollary 3.5. *Let E be a uniformly smooth and strictly convex Banach space E which also enjoys Kadec-Klee property. Let C be a nonempty, closed and convex subset of E such that JC is closed and convex. For any $j \in \mathbb{N}$, let $f_j : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For $r > 0$, let $T_{r,j} : E \to C$ be the mapping defined by (3.1). Suppose that $F := \cap_{j=1}^{\infty} EP(f_j)$ is a nonempty subset of C, where $EP(f_j)$ is the set of solutions to the equilibrium problem (1.12). Let $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_{n,j}\}_{n\in\mathbb{N}, j\in\mathbb{N}\cup\{0\}}$ be sequences in $[0,1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $\beta_{n,0} + \sum_{j=1}^{\infty} \beta_{n,j} = 1, \ \forall n \in \mathbb{N}$;
(d) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,j} > 0, \ \forall j \in \mathbb{N}$.

Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$(3.18) \qquad \begin{cases} u \in C, \ x_1 \in C \quad \textit{chosen arbitrarily}, \\ u_{n,j} \in C \quad \textit{such that} \\ \qquad f_j(u_{n,j}, y) + \frac{1}{r_n}\langle y - u_{n,j}, Ju_{n,j} - Jx_n\rangle \geq 0, \qquad \forall j \in \mathbb{N}, \ y \in C, \\ y_n = J^{-1}[\beta_{n,0}Jx_n + \sum_{j=1}^{\infty} \beta_{n,j}Ju_{n,j}], \\ x_{n+1} = J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n] \ \textit{and} \ n \in \mathbb{N}. \end{cases}$$

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.18) converges strongly to $\Pi_F u$ as $n \to \infty$.

Remark 3.6. (1) In Theorem 3.3 and Corollary 3.5, we present two strong convergence results for a system of equilibrium problems with new algorithms and new control conditions. This is complementary to Theorem 1.2. In addition, our scheme in Corollary 3.5 has an advantage that we do not use any projections which create some difficulties in a practical calculation of the iterative sequence. Indeed, we propose a different approach, based on Halpern algorithm, to solve the problem without projecting onto intersection of closed and convex sets which is not problematic in applications. Therefore, Corollary 3.5 provides a positive answer to open question 1.1.

(2) Theorem 3.3 and Corollary 3.5 improve Theorems 1.1 and 1.2 in the following aspects:

(i) For the algorithm, we remove the sets C_n and Q_n in Theorems 1.1 and 1.2.
(ii) The iterative schemes (3.2) and (3.18) in our Theorem 3.3 and Corollary 3.5 have more advantageous and are more flexible than the iterative schemes of

[27] because they both are based on Halpern iteration schemes and involve no computation of generalized projection of a point onto the closed and convex sets C_n which are huge optimization problems.

4. Application

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space. For some properties of the gradient of continuously Fréchet differentiable and convex functions we refer the reader to [3].

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E which also enjoys Kadec-Klee property. Let $\{g_j\}_{j\in\mathbb{N}}$ be an infinite family of continuously Fréchet differentiable and convex functions on E such that the gradient of g_j, ∇g_j is continuous and monotone for each $j \in \mathbb{N}$. Assume that $\Omega := \cap_{j=1}^{\infty} \arg\min_{y\in E} g_j(y) = \{z \in E : g_j(z) = \cap_{j=1}^{\infty} \min_{y\in C} g_j(y)\} \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_{n,j}\}_{n\in\mathbb{N}, j\in\mathbb{N}\cup\{0\}}$ be sequences in $[0,1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $\beta_{n,0} + \sum_{j=1}^{\infty} \beta_{n,j} = 1, \ \forall n \in \mathbb{N}$;
(d) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,j} > 0, \ \forall j \in \mathbb{N}$.

Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$
\tag{4.1}
\begin{cases}
u \in E, \ x_1 \in C \quad \text{chosen arbitrarily}, \\
u_{n,j} \in C \ \text{such that} \\
\quad \langle y - u_{n,j}, \nabla g_j(u_{n,j})\rangle + \frac{1}{r_n}\langle y - u_{n,j}, Ju_{n,j} - Jx_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = J^{-1}[\beta_{n,0}Jx_n + \sum_{j=1}^{\infty} \beta_{n,j}Ju_{n,j}], \\
x_{n+1} = \Pi_C(J^{-1}[\alpha_n Ju + (1-\alpha_n)Jy_n]) \ \text{and} \ n \in \mathbb{N}.
\end{cases}
$$

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (4.1) converges strongly to $\Pi_\Omega u$ as $n \to \infty$.

Remark 4.2. We propose a new type of iterative scheme for common solutions of an infinite family of monotone mappings in a uniformly smooth and strictly convex Banach space. A strong convergence theorem by a new Halpern-type method for the approximation of common solutions of an infinite family of monotone mappings in a uniformly smooth and strictly convex Banach space is also derived.

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