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IMPLICIT AND EXPLICIT ALGORITHMS FOR A SYSTEM OF NONLINEAR VARIATIONAL INEQUALITIES IN BANACH SPACES

LU-CHUAN CENG, HIMANSHU GUPTA, AND QAMRUL HASAN ANSARI

Abstract. In this paper, we consider a general system of nonlinear variational inequalities (in short, GSVI) in the setting of Banach spaces. We first establish the equivalence between GSVI and a system of fixed point problems. By utilizing this equivalence, we construct an implicit algorithm of Mann's type for solving GSNVI. We also propose another explicit algorithm of Mann's type for solving GSNVI. Finally, under very mild conditions, we prove the strong convergence of the sequences generated by the proposed algorithms.

1. Introduction and formulations

Let X be a real Banach space with its topological dual X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined as

$$J(x) := \left\{\varphi \in X^* : \langle \varphi, x\rangle = \|x\|^2 = \|\varphi\|^2\right\}, \quad \forall x \in X, \tag{1.1}$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. For further details on normalized duality, we refer to [1] and the references therein.

Let $C \subseteq X$ be a nonempty, closed and convex set, $A, B : C \to X$ be two nonlinear mappings and λ, μ be two positive real numbers. The general system of nonlinear variational inequalities (in short, GSNVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{1.2}$$

It is considered and studied by Yao et al. [29]. They proposed and analyzed implicit and explicit iterative algorithms for solving the GSNVI (1.2). The equivalence between GSNVI (1.2) and the fixed point problem of some nonexpansive mapping defined on a Banach space is also established. By using this equivalence, they constructed an implicit iterative algorithm and another one explicit iterative algorithm for solving the GSNVI (1.2), and proved the strong convergence of the sequences generated by the proposed algorithms. It is worth to mention that the system of variational inequalities plays an important role in game theory and economics.

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Namely, the Nash equilibrium problem can be formulated in the form of a system of variational inequalities; See for example [2, 3, 9, 14] and the references therein.

If X is a real Hilbert space, then the GSNVI (1.2) is introduced and studied by Ceng et al. [6]. In this case, for $A \equiv B$, it is considered by Verma [25]. Further, in this case, when $x^* = y^*$, problem (1.2) reduces to the following classical variational inequality (VI) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This problem is a fundamental problem in the variational analysis; in particular, in the optimization theory and mechanics; See for example [10, 16, 17, 18, 19] and the references therein. A large number of algorithms for solving this problem are essentially projection algorithms that employ projections onto the feasible set C of the VI, or onto some related set, so as to iteratively reach a solution. In particular, Korpelevich [20] proposed an algorithm for solving the VI in Euclidean space, known as the extragradient method (see also [9]). This method further has been improved by several researchers; See for example [7, 13, 22] and the references therein.

In case of Banach space setting, that is, if $A \equiv B$ and $x^* = y^*$, the VI is defined as

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{1.4}$$

Aoyama et al. [4] proposed an iterative scheme to find the approximate solution of (1.4) and they proved the weak convergence of the sequences generated by the proposed scheme. Note that this problem is connected with the fixed point problem for nonlinear mapping, the problem of finding a zero point of a nonlinear operator and so on.

It is an interesting problem how to construct some algorithms with strong convergence for solving the GSNVI (1.2) which contains problem (1.4) as a special case.

Our purpose in this paper is to continue the study of the iterative methods for finding the solutions of GSNVI (1.2). By utilizing the equivalence between GSNVI (1.2) and fixed point problem as mentioned as, we construct an implicit algorithm of Mann's type for solving GSNVI (1.2). We also propose another explicit algorithm of Mann's type for solving GSNVI (1.2). Finally, under very mild conditions, we prove the strong convergence of the sequences generated by the proposed algorithms.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Banach space X. We write $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x.

A mapping F with domain $D(F)$ and range $R(F)$ in X is called

(a) accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0,$$

where J is the normalized duality mapping;

(b) δ-strongly accretive if for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \geq \delta \|x-y\|^2 \quad \text{for some } \delta \in (0,1).$$

(c) α-inverse-strongly accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha \|Ax - Ay\|^2, \quad \text{for some } \alpha \in (0,1).$$

(d) λ-strictly pseudocontractive [5] if for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \leq \|x-y\|^2 - \lambda \|x-y-(Fx-Fy)\|^2 \quad \text{for some } \lambda \in (0,1). \tag{2.1}$$

It is easy to see that (2.1) can be written as [30]

$$\langle (I-F)x - (I-F)y, j(x-y)\rangle \geq \lambda \|(I-F)x - (I-F)y\|^2. \tag{2.2}$$

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x-y\| \geq \varepsilon \;\Rightarrow\; \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that an uniformly convex Banach space is reflexive and strictly convex. Also, it is known that if a Banach space X is reflexive, then X is strictly convex if and only if X^* is smooth as well as X is smooth if and only if X^* is strictly convex. Here we define a function $\rho : [0,\infty) \to [0,\infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \tau\right\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau\to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For further detail on geometry of Banach spaces, we refer to [1, 12] and the references therein.

Remark 2.1. Takahashi, Hashimoto and Kato [24] reminded us of the fact that no Banach space is q-uniformly smooth for $q > 2$. So, in this paper, we focus on only a 2-uniformly smooth Banach space as in [29].

In the sequel, we use the following lemmas to establish the main results of this paper.

Lemma 2.2 ([27])**.** *Let q be a given real number with $1 < q \leq 2$ and let X be a q-uniformly smooth Banach space. Then*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|\kappa y\|^q, \quad \forall x, y \in X,$$

where κ is the q-uniformly smooth constant of X and J_q is the generalized duality mapping from X into 2^{X^*} defined by

$$J_q(x) = \{\varphi \in X^* : \langle \varphi, x\rangle = \|x\|^q,\ \|\varphi\| = \|x\|^{q-1}\},$$

for all $x \in X$.

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Lemma 2.3 ([21]). *Let C be a closed convex subset of a smooth Banach space X, let D be a nonempty subset of C and Π be a retraction from C onto D. Then Π is sunny and nonexpansive if and only if*

$$\langle u - \Pi(u), j(y - \Pi(u)) \rangle \leq 0,$$

for all $u \in C$ and $y \in D$.

Remark 2.4. (a) It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Π_C coincides with the metric projection from X onto C.

(b) Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X and let T be a nonexpansive mapping of C into itself with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$. Then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C; See for example [29].

Lemma 2.5 ([11]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T, that is, $x \in \mathrm{Fix}(T)$.*

Lemma 2.6 ([29]). *Let C be a nonempty closed convex subset of a real Banach space X. Assume that the mapping $F : C \to X$ is accretive and weakly continuous along segments (that is, $F(x + ty) \rightharpoonup F(x)$ as $t \to 0$). Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C$$

is equivalent to the following Minty type variational inequality:

$$x^* \in C, \quad \langle Fx, j(x - x^*) \rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.7 ([23]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Suppose that $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)z_n$, $\forall n \geq 0$ and

$$\limsup_{n\to\infty} \Big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \Big) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.8 ([28]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, We study the iterative methods for computing the approximate solutions of GSNVI (1.2). We introduce the implicit and explicit algorithms of Mann's type for solving the GSNVI (1.2). We show the strong converge theorems for the sequences generated by the proposed algorithms.

The following proposition will be used frequently throughout the paper. For the sake of completeness, we include its proof.

Proposition 3.1. *Let* X *be a real smooth Banach space and* $F : C \to X$ *be a mapping.*

(a) *If* F *is* ζ*-strictly pseudocontractive, then* F *is Lipschitz continuous with constant* $(1+\frac{1}{\zeta})$.

(b) *If* F *is* δ*-strongly accretive and* ζ*-strictly pseudocontractive with* $\delta+\zeta>1$, *then* $I-F$ *is contractive with constant* $\sqrt{\frac{1-\delta}{\zeta}} \in (0,1)$.

(c) *If* F *is* δ*-strongly accretive and* ζ*-strictly pseudocontractive with* $\delta+\zeta>1$, *then for any fixed number* $\tau \in (0,1)$, $I-\tau F$ *is contractive with constant* $1-\tau\Big(1-\sqrt{\frac{1-\delta}{\zeta}}\Big) \in (0,1)$.

Proof. (a) Utilizing the definition of the ζ-strict pseudocontraction F, we derive for all $x, y \in C$,

$$\begin{aligned} \zeta\|(I-F)x-(I-F)y\|^2 &\leq \langle (I-F)x-(I-F)y, j(x-y)\rangle \\ &\leq \|(I-F)x-(I-F)y\|\|x-y\|, \end{aligned}$$

which implies that

$$\|(I-F)x-(I-F)y\| \leq \frac{1}{\zeta}\|x-y\|.$$

Thus

$$\begin{aligned} \|Fx-Fy\| &\leq \|(I-F)x-(I-F)y\| + \|x-y\| \\ &\leq \Big(1+\frac{1}{\zeta}\Big)\|x-y\|, \end{aligned}$$

and so F is Lipschitz continuous with constant $(1+\frac{1}{\zeta})$.

(b) Since F is δ-strongly accretive and ζ-strictly pseudocontractive, we have

$$\begin{aligned}\zeta\|(I-F)x-(I-F)y\|^2 &\le \|x-y\|^2-\langle Fx-Fy, j(x-y)\rangle\\ &\le (1-\delta)\|x-y\|^2.\end{aligned}$$

Note that $\delta+\zeta>1 \iff \sqrt{\frac{1-\delta}{\zeta}}\in(0,1)$. Hence we obtain

$$\|(I-F)x-(I-F)y\| \le \Big(\sqrt{\frac{1-\delta}{\zeta}}\Big)\|x-y\|.$$

This implies that $I-F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}}\in(0,1)$.

(c) Since $I-F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}}$, for each fixed number $\tau\in(0,1)$, we have

$$\begin{aligned}\|(x-y)-\tau(Fx-Fy)\| &= \Big\|(1-\tau)(x-y)+\tau[(I-F)x-(I-F)y]\Big\|\\ &\le (1-\tau)\|x-y\|+\tau\|(I-F)x-(I-F)y\|\\ &\le (1-\tau)\|x-y\|+\tau\Big(\sqrt{\frac{1-\delta}{\zeta}}\Big)\|x-y\|\\ &= \Big(1-\tau\Big(1-\sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|x-y\|.\end{aligned}$$

This shows that $I-\tau F$ is contractive with constant $1-\tau\Big(1-\sqrt{\frac{1-\delta}{\zeta}}\Big)\in(0,1)$. $\square$

We recall several useful lemmas.

Lemma 3.2 ([29])**.** *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mappings $A,B: C\to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Then,*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \le \|x-y\|^2+2\lambda(\kappa^2\lambda-\alpha)\|Ax-Ay\|^2,$$

and

$$\|(I-\mu B)x-(I-\mu B)y\|^2 \le \|x-y\|^2+2\mu(\kappa^2\mu-\beta)\|Bx-By\|^2.$$

In particular, if $0\le\lambda\le\frac{\alpha}{\kappa^2}$ and $0\le\mu\le\frac{\beta}{\kappa^2}$, then $I-\lambda A$ and $I-\mu B$ are nonexpansive.

Lemma 3.3 ([29])**.** *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A,B: C\to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $G: C\to C$ be a mapping defined by*

$$G(x)=\Pi_C[\Pi_C(x-\mu Bx)-\lambda A\Pi_C(x-\mu Bx)],\quad \forall x\in C.$$

If $0\le\lambda\le\frac{\alpha}{\kappa^2}$ and $0\le\mu\le\frac{\beta}{\kappa^2}$, then $G: C\to C$ is nonexpansive.

Lemma 3.4 ([29]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the GSNVI (1.2) if and only if $x^* = \Pi_C(y^* - \lambda A y^*)$ where $y^* = \Pi_C(x^* - \mu B x^*)$.*

Remark 3.5. From Lemma 3.4, we have

$$x^* = \Pi_C[\Pi_C(x^* - \mu B x^*) - \lambda A \Pi_C(x^* - \mu B x^*)],$$

which implies that x^* is a fixed point of the mapping G.

Throughout the paper, the set of fixed points of the mapping G is denoted by Ω.

In order to solve GSNVI (1.2), we first introduce an implicit algorithm of Mann's type. Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $\lambda \in \Big(0, \frac{\alpha}{\kappa^2}\Big)$ and $\mu \in (0, \frac{\beta}{\kappa^2})$ where κ is the 2-uniformly smooth constant of X (see Lemma 2.2). For each $t \in (0,1)$, choose a number $\theta_t \in (0,1)$ arbitrarily. For any $x \in C$, we consider the following mapping

$$\begin{aligned} W_t x \;&:= \{t\Pi_C(I - \lambda A)\Pi_C(I - \mu B) \\ &\quad + (1-t)\Pi_C(I - \theta_t F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)\}x. \end{aligned} \tag{3.1}$$

We note that $\Pi_C(I - \lambda A)$ and $\Pi_C(I - \mu B)$ are nonexpansive (by Lemma 3.2), $G = \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$ is also nonexpansive (by Lemma 3.3), and $I - \theta_t F$ is contractive with coefficient $1 - \theta_t\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big) \in (0,1)$ (by Proposition 3.1 (c)). Hence for all $x, y \in C$,

$$\begin{aligned} \|W_t x - W_t y\| &= \|\{t\Pi_C(I - \lambda A)\Pi_C(I - \mu B) \\ &\quad + (1-t)\Pi_C(I - \theta_t F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)\}x \\ &\quad - \{t\Pi_C(I - \lambda A)\Pi_C(I - \mu B) \\ &\quad + (1-t)\Pi_C(I - \theta_t F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)\}y\| \\ &= \|t(G(x) - G(y)) + (1-t)[\Pi_C(I - \theta_t F)G(x) - \Pi_C(I - \theta_t F)G(y)]\| \\ &\le t\|G(x) - G(y)\| + (1-t)\|\Pi_C(I - \theta_t F)G(x) - \Pi_C(I - \theta_t F)G(y)\| \\ &\le t\|x - y\| + (1-t)\|(I - \theta_t F)G(x) - (I - \theta_t F)G(y)\| \\ &\le t\|x - y\| + (1-t)\Big(1 - \theta_t\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|G(x) - G(y)\| \\ &\le t\|x - y\| + (1-t)\Big(1 - \theta_t\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|x - y\| \\ &= \Big[1 - (1-t)\theta_t\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big]\|x - y\|. \end{aligned}$$

Since $\theta_t \in (0,1)$, $\forall t \in (0,1)$ and $\delta + \zeta > 1$ with $\delta, \zeta \in (0,1)$, we obtain

$$0 < \theta_t \Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big) < 1,$$

and so,

$$0 < 1 - (1-t)\theta_t \Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big) < 1.$$

This means that the mapping W_t is a contraction. Therefore, the following implicit algorithm of Mann's type for solving GSNVI (1.2) is well defined.

Algorithm 3.6. For each $t \in (0,1)$, choose a number $\theta_t \in (0,1)$ arbitrarily. The net $\{x_t\}$ is generated by the implicit method

(3.2)

$$\begin{aligned} x_t \;&= \{t\Pi_C(I-\lambda A)\Pi_C(I-\mu B) \\ &\quad + (1-t)\Pi_C(I-\theta_t F)\Pi_C(I-\lambda A)\Pi_C(I-\mu B)\}x_t, \quad \forall t \in (0,1), \end{aligned}$$

where x_t is a unique fixed point of the contraction

$$W_t = t\Pi_C(I-\lambda A)\Pi_C(I-\mu B) + (1-t)\Pi_C(I-\theta_t F)\Pi_C(I-\lambda A)\Pi_C(I-\mu B).$$

We prove that the sequences generated by the Algorithm 3.6 converge strongly to a solution of a VI.

Theorem 3.7. *The net $\{x_t\}$ generated by Algorithm 3.6 converges in norm, as $t \to 0^+$, to the unique solution $\tilde{x}$ of the following VI:*

$$\tilde{x} \in \Omega, \quad \langle F(\tilde{x}), j(\tilde{x} - z)\rangle \le 0, \quad \forall z \in \Omega, \tag{3.3}$$

provided $\lim_{t\to 0^+} \theta_t = 0$.

Proof. Set $z_t = \Pi_C(I-\mu B)x_t$ and $y_t = \Pi_C(I-\lambda A)z_t$ for all $t \in (0,1)$. Then we have $x_t = ty_t + (1-t)\Pi_C(I-\theta_t F)y_t$. Let $x^* \in \Omega$, then from Lemma 3.4, we have

$$x^* = \Pi_C[\Pi_C(x^* - \mu Bx^*) - \lambda A\Pi_C(x^* - \mu Bx^*)].$$

Set $y^* = \Pi_C(x^* - \mu Bx^*)$. Then $x^* = \Pi_C(y^* - \lambda Ay^*)$.

From Lemma 3.2, we know that $\Pi_C(I-\lambda A)$ and $\Pi_C(I-\mu B)$ are nonexpansive. Hence, we have

$$\begin{aligned} \|y_t - x^*\| \;&= \|\Pi_C(I-\lambda A)z_t - \Pi_C(I-\lambda A)y^*\| \\ &\le \|z_t - y^*\| = \|\Pi_C(I-\mu B)x_t - \Pi_C(I-\mu B)x^*\| \\ &\le \|x_t - x^*\|. \end{aligned}$$

So, by Proposition 3.1 (c), we get

$$\begin{aligned} \|x_t - x^*\| &= \|ty_t + (1-t)\Pi_C(I-\theta_t F)y_t - (tx^* + (1-t)\Pi_C(x^*))\| \\ &= \|t(y_t - x^*) + (1-t)(\Pi_C(I-\theta_t F)y_t - \Pi_C(x^*))\| \\ &\le t\|y_t - x^*\| + (1-t)\|\Pi_C(I-\theta_t F)y_t - \Pi_C(x^*)\| \\ &\le t\|x_t - x^*\| + (1-t)\|(I-\theta_t F)y_t - x^*\| \\ &= t\|x_t - x^*\| + (1-t)\|(I-\theta_t F)y_t - (I-\theta_t F)x^* - \theta_t F(x^*)\| \end{aligned}$$

$$\begin{aligned}
&\le t\|x_t - x^*\| + (1-t)\Big(\|(I-\theta_t F)y_t - (I-\theta_t F)x^*\| + \theta_t\|F(x^*)\|\Big)\\
&\le t\|x_t - x^*\| + (1-t)\Big[\Big(1-\theta_t\Big(1-\sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|y_t - x^*\| + \theta_t\|F(x^*)\|\Big]\\
&\le t\|x_t - x^*\| + (1-t)\Big[\Big(1-\theta_t\Big(1-\sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|x_t - x^*\| + \theta_t\|F(x^*)\|\Big]\\
&= \Big[1-(1-t)\theta_t\Big(1-\sqrt{\frac{1-\delta}{\zeta}}\Big)\Big]\|x_t - x^*\| + (1-t)\theta_t\|F(x^*)\|.
\end{aligned}$$

It follows that

$$\|x_t - x^*\| \le \Big(1-\sqrt{\frac{1-\delta}{\zeta}}\Big)^{-1}\|F(x^*)\|.$$

Therefore, $\{x_t\}$ is bounded. Hence $\{y_t\}$, $\{z_t\}$, $\{Ay_t\}$, $\{Bx_t\}$ and $\{F(y_t)\}$ are also bounded. We observe that

$$\begin{aligned}
\|x_t - y_t\| &= \|ty_t + (1-t)\Pi_C(I-\theta_t F)y_t - (ty_t + (1-t)\Pi_C y_t)\|\\
&= (1-t)\|\Pi_C(I-\theta_t F)y_t - \Pi_C y_t\|\\
&\le \|(I-\theta_t F)y_t - y_t\| \qquad (3.4)\\
&= \theta_t\|F(y_t)\| \to 0 \text{ as } t\to 0^+.
\end{aligned}$$

From Lemma 3.3, it is known that $G : C \to C$ is nonexpansive. Thus, we have

$$\begin{aligned}
\|y_t - G(y_t)\| &= \|\Pi_C[\Pi_C(x_t - \mu Bx_t) - \lambda A\Pi_C(x_t - \mu Bx_t)] - G(y_t)\|\\
&= \|G(x_t) - G(y_t)\|\\
&\le \|x_t - y_t\| \to 0 \text{ as } t\to 0^+.
\end{aligned}$$

Therefore,

$$\lim_{t\to 0^+}\|x_t - G(x_t)\| = 0. \qquad (3.5)$$

Next, we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Assume that $\{t_n\} \subset (0,1)$ is such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$ and $\theta_n := \theta_{t_n}$. It follows from (3.5) that

$$\|x_n - G(x_n)\| \to 0 \quad \text{as } n\to\infty. \qquad (3.6)$$

We can rewrite (3.2) as

$$x_t = ty_t + (1-t)\Big[\Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t + (I-\theta_t F)y_t\Big].$$

For any $x^* \in \Omega \subset C$, by Lemma 2.3, we have

$$\begin{aligned}
&\langle x_t - (I-\theta_t F)y_t, j(x_t - x^*)\rangle\\
&= t\langle y_t - (I-\theta_t F)y_t, j(x_t - x^*)\rangle\\
&\quad + (1-t)\langle \Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t, j(x_t - x^*)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= t\theta_t\langle F(y_t), j(x_t - x^*)\rangle \\
&\quad + (1-t)[\langle \Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t, j(\Pi_C(I-\theta_t F)y_t - x^*)\rangle \\
&\quad + \langle \Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t, j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\rangle] \\
&\le t\theta_t\langle F(y_t), j(x_t - x^*)\rangle + (1-t)\langle \Pi_C(I-\theta_t F)y_t \\
&\quad - (I-\theta_t F)y_t, j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\rangle \\
&\le t\theta_t\langle F(y_t), j(x_t - x^*)\rangle + (1-t)\|\Pi_C(I-\theta_t F)y_t \\
&\quad - (I-\theta_t F)y_t\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\le t\theta_t\langle F(y_t), j(x_t - x^*)\rangle + (1-t)(\|\Pi_C(I-\theta_t F)y_t - \Pi_C y_t\| \\
&\quad + \theta_t\|F(y_t)\|)\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\le t\theta_t\langle F(y_t), j(x_t - x^*)\rangle + (1-t)(\|(I-\theta_t F)y_t - y_t\| \\
&\quad + \theta_t\|F(y_t)\|)\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\le t\theta_t\|F(y_t)\|\|x_t - x^*\| + 2\theta_t\|F(y_t)\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\|.
\end{aligned}
$$

With this fact, we deduce that

$$
\begin{aligned}
&\|x_t - x^*\|^2 = \langle x_t - x^*, j(x_t - x^*)\rangle \\
&= \langle x_t - (I-\theta_t F)y_t, j(x_t - x^*)\rangle + \langle (I-\theta_t F)y_t - x^*, j(x_t - x^*)\rangle \\
&\le t\theta_t\|F(y_t)\|\|x_t - x^*\| + 2\theta_t\|F(y_t)\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\quad + \langle (I-\theta_t F)y_t - x^*, j(x_t - x^*)\rangle \\
&= t\theta_t\|F(y_t)\|\|x_t - x^*\| + 2\theta_t\|F(y_t)\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\quad + \langle (I-\theta_t F)y_t - (I-\theta_t F)x^*, j(x_t - x^*)\rangle - \theta_t\langle F(x^*), j(x_t - x^*)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le t\theta_t\|F(y_t)\|\|x_t - x^*\| + 2\theta_t\|F(y_t)\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\quad + \|(I-\theta_t F)y_t - (I-\theta_t F)x^*\|\|x_t - x^*\| - \theta_t\langle F(x^*), j(x_t - x^*)\rangle \\
&\le t\theta_t\|F(y_t)\|\|x_t - x^*\| + 2\theta_t\|F(y_t)\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\quad + \Big(1 - \theta_t\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|y_t - x^*\|\|x_t - x^*\| - \theta_t\langle F(x^*), j(x_t - x^*)\rangle \\
&\le t\theta_t\|F(y_t)\|\|x_t - x^*\| + 2\theta_t\|F(y_t)\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\| \\
&\quad + \Big(1 - \theta_t\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|x_t - x^*\|^2 - \theta_t\langle F(x^*), j(x_t - x^*)\rangle.
\end{aligned} \tag{3.7}
$$

It turns out that

$$
\begin{aligned}
\|x_t - x^*\|^2 \le \Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)^{-1}\big[&\langle F(x^*), j(x^* - x_t)\rangle + t\|F(y_t)\|\|x_t - x^*\| \\
&+ 2\|F(y_t)\|\|j(x_t - x^*) - j(\Pi_C(I-\theta_t F)y_t - x^*)\|\big], \quad \forall x^* \in \Omega.
\end{aligned}
$$

In particular,

$$\|x_n - x^*\|^2 \le \Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)^{-1}[\langle F(x^*), j(x^* - x_n)\rangle + t_n\|F(y_n)\|\|x_n - x^*\| \\ + 2\|F(y_n)\|\|j(x_n - x^*) - j(\Pi_C(I-\theta_n F)y_n - x^*)\|], \quad \forall x^* \in \Omega. \tag{3.8}$$

Since $\{x_n\}$ is bounded, without loss of generality we may assume that $\{x_n\}$ converges weakly to a point $\tilde{x} \in C$. Noticing (3.6) we can use Lemma 2.5 to get $\tilde{x} \in \Omega$. Therefore, we can substitute $\tilde{x}$ for x^* in (3.8) to get

$$\|x_n - \tilde{x}\|^2 \le \Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)^{-1}\big[\langle F(\tilde{x}), j(\tilde{x} - x_n)\rangle + t_n\|F(y_n)\|\|x_n - \tilde{x}\| \\ + 2\|F(y_n)\|\|j(x_n - \tilde{x}) - j(\Pi_C(I-\theta_n F)y_n - \tilde{x})\|\big]. \tag{3.9}$$

Note that

$$\begin{aligned}\|(x_n - \tilde{x}) - (\Pi_C(I-\theta_n F)y_n - \tilde{x})\| &= \|x_n - \Pi_C(I-\theta_n F)y_n\| \\ &= t_n\|\Pi_C y_n - \Pi_C(I-\theta_n F)y_n\| \\ &\le t_n\|y_n - (I-\theta_n F)y_n\| \\ &= t_n\theta_n\|F(y_n)\| \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Since X is uniformly smooth, we get that

$$\|j(x_n - \tilde{x}) - j(\Pi_C(I-\theta_n F)y_n - \tilde{x})\| \to 0 \quad \text{as } n \to \infty.$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ together with (3.9), actually implies that $x_n \to \tilde{x}$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0^+$.

We next show that $\tilde{x}$ solves the variational inequality (3.3). From (3.2), we have

$$\begin{aligned} x_t &= ty_t + (1-t)[\Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t + (I-\theta_t F)y_t] \\ \Rightarrow x_t &= ty_t + (1-t)[\Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t \\ &\quad - ((I-\theta_t F)x_t - (I-\theta_t F)y_t) + x_t - \theta_t F(x_t)] \\ \Rightarrow F(x_t) &= -\frac{t(x_t - y_t)}{(1-t)\theta_t} + \frac{1}{\theta_t}[\Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t \\ &\quad - ((I-\theta_t F)x_t - (I-\theta_t F)y_t)]. \end{aligned}$$

For any $z \in \Omega$, we have

$$\begin{aligned}\langle F(x_t), j(x_t - z)\rangle &= -\frac{t}{(1-t)\theta_t}\langle x_t - y_t, j(x_t - z)\rangle \\ &\quad + \frac{1}{\theta_t}\langle \Pi_C(I-\theta_t F)y_t - (I-\theta_t F)y_t, j(x_t - z)\rangle \\ &\quad - \frac{1}{\theta_t}\langle (I-\theta_t F)x_t - (I-\theta_t F)y_t), j(x_t - z)\rangle \\ &= -\frac{t}{(1-t)\theta_t}\langle x_t - y_t, j(x_t - z)\rangle + \frac{1}{\theta_t}\langle \Pi_C(I-\theta_t F)y_t \\ &\quad - (I-\theta_t F)y_t, j(\Pi_C(I-\theta_t F)y_t - z)\rangle \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&+\frac{1}{\theta_t}\langle \Pi_C(I-\theta_t F)y_t-(I-\theta_t F)y_t, j(x_t-z)\\
&-j(\Pi_C(I-\theta_t F)y_t-z)\rangle\\
&-\frac{1}{\theta_t}\langle (I-\theta_t F)x_t-(I-\theta_t F)y_t), j(x_t-z)\rangle\\
\le &-\frac{t}{(1-t)\theta_t}\langle x_t-y_t, j(x_t-z)\rangle+\frac{1}{\theta_t}\langle \Pi_C(I-\theta_t F)y_t\\
&-(I-\theta_t F)y_t, j(\Pi_C(I-\theta_t F)y_t-z)\rangle\\
&+2\|F(y_t)\|\|j(x_t-z)-j(\Pi_C(I-\theta_t F)y_t-z)\|\\
&-\frac{1}{\theta_t}\langle x_t-y_t), j(x_t-z)\rangle\\
&+\langle F(x_t)-F(y_t), j(x_t-z)\rangle.
\end{aligned}
$$

Now we prove that $\langle x_t-y_t), j(x_t-z)\rangle \ge 0$. Indeed, we can write $y_t=G(x_t)$. At the same time, we note that $z=G(z)$. So,

$$\langle x_t-y_t, j(x_t-z)\rangle = \langle x_t-G(x_t)-(z-G(z)), j(x_t-z)\rangle.$$

Since $I-G$ is accretive (this is due to the nonexpansivity of G), we can deduce immediately that

$$\langle x_t-y_t, j(x_t-z)\rangle = \langle x_t-G(x_t)-(z-G(z)), j(x_t-z)\rangle \ge 0.$$

Furthermore, utilizing Lemma 2.3 and Proposition 3.1 (a), we have

$$\langle \Pi_C(I-\theta_t F)y_t-(I-\theta_t F)y_t, j(\Pi_C(I-\theta_t F)y_t-z)\rangle \le 0$$

and

$$\|F(x_t)-F(y_t)\| \le \Big(1+\frac{1}{\zeta}\Big)\|x_t-y_t\|.$$

It follows from (3.10) that

$$
\begin{aligned}
\langle F(x_t), j(x_t-z)\rangle \le &2\|F(y_t)\|\|j(x_t-z)-j(\Pi_C(I-\theta_t F)y_t-z)\|\\
&+\Big(1+\frac{1}{\zeta}\Big)\|x_t-y_t\|\|x_t-z\|.
\end{aligned}
\tag{3.11}
$$

Since F is δ-strongly accretive, we have

$$0 \le \delta\|x_t-z\|^2 \le \langle F(x_t)-F(z), j(x_t-z)\rangle.$$

Therefore,

$$\langle F(z), j(x_t-z)\rangle \le \langle F(x_t), j(x_t-z)\rangle. \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$
\begin{aligned}
\langle F(z), j(x_t-z)\rangle \le &2\|F(y_t)\|\|j(x_t-z)-j(\Pi_C(I-\theta_t F)y_t-z)\|\\
&+\Big(1+\frac{1}{\zeta}\Big)\|x_t-y_t\|\|x_t-z\|.
\end{aligned}
\tag{3.13}
$$

Replacing t in (3.13) with t_n, and noticing that as $n\to\infty$, $x_{t_n}-y_{t_n}\to 0$ and $j(x_{t_n}-z)-j(\Pi_C(I-\theta_{t_n}F)y_{t_n}-z)\to 0$, we obtain

$$\langle F(z), j(\tilde{x}-z)\rangle \le 0, \quad \forall z\in\Omega,$$

which is equivalent to the Minty type variational inequality (see Lemma 2.6)

$$\langle F(\tilde{x}), j(\tilde{x}-z)\rangle \le 0, \quad \forall z \in \Omega. \tag{3.14}$$

That is, $\tilde{x} \in \Omega$ is a solution of (3.3).

Now we show that the solution set of (3.3) is a singleton. As a matter of fact, we assume that $\bar{x} \in \Omega$ is also a solution of (3.3). Then, we have

$$\langle F(\bar{x}), j(\bar{x}-\tilde{x})\rangle \le 0.$$

From (3.14), we have

$$\langle F(\tilde{x}), j(\tilde{x}-\bar{x})\rangle \le 0.$$

So, by δ-strong accretiveness of F, we have

$$\begin{aligned} &\langle F(\bar{x}), j(\bar{x}-\tilde{x})\rangle + \langle F(\tilde{x}), j(\tilde{x}-\bar{x})\rangle \le 0 \\ &\Rightarrow \langle F(\bar{x}) - F(\tilde{x}), j(\bar{x}-\tilde{x})\rangle \le 0 \\ &\Rightarrow \delta\|\bar{x}-\tilde{x}\|^2 \le 0. \end{aligned}$$

Therefore, $\bar{x} = \tilde{x}$. In summary, we have shown that each cluster point of $\{x_t\}$ (as $t \to 0$) equals to $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$. □

We next introduce an explicit method which is the discretization of the implicit method (3.2).

Algorithm 3.8. Let C be a nonempty closed convex subset of a real smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $A, B, F : C \to X$ be three nonlinear mappings. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{aligned} x_{n+1} &= \beta_n x_n + \gamma_n \Pi_C(I-\lambda A)\Pi_C(I-\mu B)x_n \\ &\quad + (1-\beta_n-\gamma_n)\Pi_C(I-\alpha_n F)\Pi_C(I-\lambda A)\Pi_C(I-\mu B)x_n, \end{aligned} \tag{3.15}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ such that $\beta_n + \gamma_n \le 1$, $\forall n \ge 0$, and λ, μ are two real numbers.

In particular, if $A \equiv B$, then (3.15) reduces to the following iterative scheme:

$$\begin{aligned} x_{n+1} &= \beta_n x_n + \gamma_n \Pi_C(I-\lambda A)\Pi_C(I-\mu A)x_n \\ &\quad + (1-\beta_n-\gamma_n)\Pi_C(I-\alpha_n F)\Pi_C(I-\lambda A)\Pi_C(I-\mu A)x_n. \end{aligned} \tag{3.16}$$

Theorem 3.9. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X and let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by (3.15). Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$.

Then, the sequence $\{x_n\}$ *converges strongly to* $\tilde{x} \in \Omega$ *which solves the variational inequality* (3.3).

Proof. Set $z_n = \Pi_C(I - \mu B)x_n$ and $y_n = \Pi_C(I - \lambda A)z_n$ for all $n \geq 0$. Then

$$x_{n+1} = \beta_n x_n + \gamma_n y_n + (1 - \beta_n - \gamma_n)\Pi_C(I - \alpha_n F)y_n, \quad \forall n \geq 0.$$

We take a point $x^* \in \Omega$ arbitrarily.

From Lemma 3.2, we know that $\Pi_C(I - \lambda A)$ and $\Pi_C(I - \mu B)$ are nonexpansive. Hence, we have

$$\begin{aligned}
\|y_n - x^*\| &= \|\Pi_C(I - \lambda A)z_n - \Pi_C(I - \lambda A)y^*\| \\
&\leq \|z_n - y^*\| = \|\Pi_C(I - \mu B)x_n - \Pi_C(I - \mu B)x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned}$$

So, by Proposition 3.1 (c), we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\beta_n x_n + \gamma_n y_n + (1 - \beta_n - \gamma_n)\Pi_C(I - \alpha_n F)y_n - x^*\| \\
&\leq \beta_n\|x_n - x^*\| + \gamma_n\|y_n - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n)\|\Pi_C(I - \alpha_n F)y_n - \Pi_C x^*\| \\
&\leq \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n)\|(I - \alpha_n F)y_n - (I - \alpha_n F)x^* - \alpha_n F(x^*)\| \\
&\leq (\beta_n + \gamma_n)\|x_n - x^*\| + (1 - \beta_n - \gamma_n)\|(I - \alpha_n F)y_n - (I - \alpha_n F)x^*\| \\
&\quad + \alpha_n(1 - \beta_n - \gamma_n)\|F(x^*)\| \\
&\leq (\beta_n + \gamma_n)\|x_n - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n)\Big(1 - \alpha_n\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|y_n - x^*\| \\
&\quad + \alpha_n(1 - \beta_n - \gamma_n)\|F(x^*)\| \\
&\leq \Big[1 - \alpha_n(1 - \beta_n - \gamma_n)\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big]\|x_n - x^*\| \\
&\quad + \alpha_n(1 - \beta_n - \gamma_n)\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\frac{\|F(x^*)\|}{\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)}.
\end{aligned}$$

By induction, we conclude that

$$\|x_{n+1} - x^*\| \leq \max\Big\{\|x_0 - x^*\|, \Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)^{-1}\|F(x^*)\|\Big\}.$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{z_n\}$, $\{Ay_n\}$ and $\{Bx_n\}$ are also bounded. We observe that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|\varPi_C(I-\lambda A)z_{n+1}-\varPi_C(I-\lambda A)z_n\| \\ &\le \|z_{n+1}-z_n\| \\ &= \|\varPi_C(I-\mu B)x_{n+1}-\varPi_C(I-\mu B)x_n\| \\ &\le \|x_{n+1}-x_n\|.\end{aligned}$$

Set $x_{n+1}=\beta_n x_n+(1-\beta_n)v_n$ for all $n\ge 0$.
Then $v_n=\dfrac{\gamma_n y_n+(1-\beta_n-\gamma_n)\varPi_C(I-\alpha_n F)y_n}{1-\beta_n}$. Note that

$$\begin{aligned}&\|\varPi_C(I-\alpha_{n+1}F)y_{n+1}-\varPi_C(I-\alpha_n F)y_n\| \\ &\le \|(I-\alpha_{n+1}F)y_{n+1}-(I-\alpha_n F)y_n\| \\ &= \|y_{n+1}-y_n-\alpha_{n+1}F(y_{n+1})+\alpha_n F(y_n)\| \\ &\le \|y_{n+1}-y_n\|+\alpha_{n+1}\|F(y_{n+1})\|+\alpha_n\|F(y_n)\| \\ &\le \|x_{n+1}-x_n\|+\alpha_{n+1}\|F(y_{n+1})\|+\alpha_n\|F(y_n)\|.\end{aligned}$$

Hence

$$\begin{aligned}&\|v_{n+1}-v_n\| \\ &= \Big\|\frac{\gamma_{n+1}y_{n+1}+(1-\beta_{n+1}-\gamma_{n+1})\varPi_C(I-\alpha_{n+1}F)y_{n+1}}{1-\beta_{n+1}} \\ &\quad -\frac{\gamma_n y_n+(1-\beta_n-\gamma_n)\varPi_C(I-\alpha_n F)y_n}{1-\beta_n}\Big\| \\ &\le \Big\|\frac{\gamma_{n+1}}{1-\beta_{n+1}}y_{n+1}-\frac{\gamma_n}{1-\beta_n}y_n\Big\| \\ &\quad +\Big\|\frac{(1-\beta_{n+1}-\gamma_{n+1})}{1-\beta_{n+1}}\varPi_C(I-\alpha_{n+1}F)y_{n+1}-\frac{(1-\beta_n-\gamma_n)}{1-\beta_n}\varPi_C(I-\alpha_n F)y_n\Big\| \\ &\le \Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\|y_{n+1}\|+\frac{\gamma_n}{1-\beta_n}\|y_{n+1}-y_n\| \\ &\quad +\Big|\frac{(1-\beta_{n+1}-\gamma_{n+1})}{1-\beta_{n+1}}-\frac{(1-\beta_n-\gamma_n)}{1-\beta_n}\Big|\|\varPi_C(I-\alpha_{n+1}F)y_{n+1}\| \\ &\quad +\frac{(1-\beta_n-\gamma_n)}{1-\beta_n}\|\varPi_C(I-\alpha_{n+1}F)y_{n+1}-\varPi_C(I-\alpha_n F)y_n\| \\ &\le \Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\Big(\|y_{n+1}\|+\|\varPi_C(I-\alpha_{n+1}F)y_{n+1}\|\Big) \\ &\quad +\frac{\gamma_n}{1-\beta_n}\|x_{n+1}-x_n\| \\ &\quad +\frac{(1-\beta_n-\gamma_n)}{1-\beta_n}(\|x_{n+1}-x_n\|+\alpha_{n+1}\|F(y_{n+1})\|+\alpha_n\|F(y_n)\|) \\ &\le \Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\Big(\|y_{n+1}\|+\|\varPi_C(I-\alpha_{n+1}F)y_{n+1}\|\Big) \\ &\quad +\|x_{n+1}-x_n\|+\alpha_{n+1}\|F(y_{n+1})\|+\alpha_n\|F(y_n)\|.\end{aligned}$$

Since $\{y_n\}$ and $\{F(y_n)\}$ are bounded, we have that $\{\|y_n\| + \|\Pi_C(I - \alpha_n F)y_n\|\}$ is bounded. So it follows from conditions (i) and (ii) that

$$\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.7, we get $\|v_n - x_n\| \to 0$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|v_n - x_n\| = 0.$$

We also note that

$$\begin{aligned}\|x_n - y_n\| &= \left\|\frac{\gamma_n y_n + (1 - \beta_n - \gamma_n)\Pi_C(I - \alpha_n F)y_n}{1 - \beta_n} - y_n\right\| \\ &= \left\|\frac{\gamma_n y_n + (1 - \beta_n - \gamma_n)\Pi_C(I - \alpha_n F)y_n - (1 - \beta_n)y_n}{1 - \beta_n}\right\| \\ &= \frac{1 - \beta_n - \gamma_n}{1 - \beta_n}\|\Pi_C(I - \alpha_n F)y_n - y_n\| \\ &\leq \|\Pi_C(I - \alpha_n F)y_n - \Pi_C y_n\| \\ &\leq \alpha_n\|F(y_n)\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

From Lemma 3.3, we know that $G : C \to C$ is nonexpansive. Thus, we have

$$\begin{aligned}\|y_n - G(y_n)\| &= \|\Pi_C[\Pi_C(x_n - \mu B x_n) - \lambda A \Pi_C(x_n - \mu B x_n)] - G(y_n)\| \\ &= \|G(x_n) - G(y_n)\| \\ &\leq \|x_n - y_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|x_n - G(x_n)\| = 0$.

Set $u_n = \Pi_C(I - \alpha_n F)y_n$ for all $n \geq 0$. We note that

$$\begin{aligned}\|u_n - G(u_n)\| &\leq \|u_n - x_n\| + \|x_n - G(x_n)\| + \|G(x_n) - G(u_n)\| \\ &\leq 2\|u_n - x_n\| + \|x_n - G(x_n)\| \\ &= 2\|\Pi_C(I - \alpha_n F)y_n - \Pi_C x_n\| + \|x_n - G(x_n)\| \\ &\leq 2(\|y_n - x_n\| + \alpha_n\|F(y_n)\|) + \|x_n - G(x_n)\| \to 0 \\ &\quad \text{as } n \to \infty.\end{aligned} \tag{3.17}$$

Next we show that

$$\limsup_{n\to\infty}\langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle \leq 0,$$

where $\tilde{x} \in \Omega$ is the unique solution of the VI (3.3).

Indeed, we first take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle = \lim_{k\to\infty} \langle F(\tilde{x}), j(\tilde{x} - u_{n_k})\rangle.$$

We may also assume that $u_{n_k} \rightharpoonup z$. Note that $z \in \Omega$ by virtue of Lemma 2.5 and (3.17). It follows from the variational inequality (3.3) that

$$\limsup_{n\to\infty}\langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle = \lim_{k\to\infty} \langle F(\tilde{x}), j(\tilde{x} - u_{n_k})\rangle = \langle F(\tilde{x}), j(\tilde{x} - z)\rangle \leq 0.$$

Since $u_n = \Pi_C(I - \alpha_n F)y_n$, according to Lemma 2.3, we have

$$\langle (I - \alpha_n F)y_n - \Pi_C(I - \alpha_n F)y_n, j(\tilde{x} - u_n)\rangle \leq 0. \tag{3.18}$$

From (3.18), we have

$$\begin{aligned}
\|u_n - \tilde{x}\|^2 &= \langle \Pi_C(I - \alpha_n F)y_n - \tilde{x}, j(u_n - \tilde{x})\rangle \\
&= \langle \Pi_C(I - \alpha_n F)y_n - (I - \alpha_n F)y_n, j(u_n - \tilde{x})\rangle \\
&\quad + \langle (I - \alpha_n F)y_n - \tilde{x}, j(u_n - \tilde{x})\rangle \\
&\leq \langle (I - \alpha_n F)y_n - \tilde{x}, j(u_n - \tilde{x})\rangle \\
&= \langle (I - \alpha_n F)y_n - (I - \alpha_n F)\tilde{x}, j(u_n - \tilde{x})\rangle + \alpha_n \langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle \\
&\leq \Big(1 - \alpha_n\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|y_n - \tilde{x}\|\|u_n - \tilde{x}\| + \alpha_n \langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle \\
&\leq \frac{1}{2}\Big(1 - \alpha_n\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)^2 \|y_n - \tilde{x}\|^2 \\
&\quad + \frac{1}{2}\|u_n - \tilde{x}\|^2 + \alpha_n \langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_n - \tilde{x}\|^2 &\leq \Big(1 - \alpha_n\Big(1 - \sqrt{\tfrac{1-\delta}{\zeta}}\Big)\Big)\|y_n - \tilde{x}\|^2 + 2\alpha_n \langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle \\
&\leq \Big(1 - \alpha_n\Big(1 - \sqrt{\tfrac{1-\delta}{\zeta}}\Big)\Big)\|x_n - \tilde{x}\|^2 + 2\alpha_n \langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle.
\end{aligned} \tag{3.19}$$

Finally, we prove $x_n \to \tilde{x}$. As a matter of fact, from (3.2) and (3.19), we have

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
&\leq \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|y_n - \tilde{x}\|^2 + (1 - \beta_n - \gamma_n)\|u_n - \tilde{x}\|^2 \\
&\leq \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|x_n - \tilde{x}\|^2 \\
&\quad + (1 - \beta_n - \gamma_n)\Big[\Big(1 - \alpha_n\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big)\|x_n - \tilde{x}\|^2 + 2\alpha_n \langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle\Big]
\end{aligned}$$

(3.20)

$$\begin{aligned}
&= \Big[1 - \alpha_n(1 - \beta_n - \gamma_n)\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)\Big]\|x_n - \tilde{x}\|^2 \\
&\quad + \alpha_n(1 - \beta_n - \gamma_n)(1 - \sqrt{\frac{1-\delta}{\zeta}})\Big\{\frac{2}{\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big)}\langle F(\tilde{x}), j(\tilde{x} - u_n)\rangle\Big\}.
\end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$ and $1 - \sqrt{\frac{1-\delta}{\zeta}} \in (0, 1)$, we get

$$\sum_{n=0}^{\infty} \alpha_n\Big(1 - \beta_n - \gamma_n\Big)\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big) = \infty.$$

Taking into account $\limsup_{n\to\infty}\langle F(\tilde{x}), j(\tilde{x}-u_n)\rangle \le 0$, we can apply Lemma 2.8 to the relation (3.20) and conclude that $x_n \to \tilde{x}$. □

We use Γ to denote the solution set of the variational inequality (1.4). We can derive easily the following corollaries.

Corollary 3.10. *Let $\theta_t \in (0,1)$, $\forall t \in (0,1)$ such that $\lim_{t\to 0^+}\theta_t = 0$. The net $\{x_t\}$ generated by the implicit method*

$$x_t = \{t\Pi_C(I-\lambda A)\Pi_C(I-\mu A)+(1-t)\Pi_C(I-\theta_t F)\Pi_C(I-\lambda A)\Pi_C(I-\mu A)\}x_t, \tag{3.21}$$

for all $t \in (0,1)$, converges in norm, as $t \to 0^+$, to $\tilde{x} \in \Gamma$ which is the unique solution of the following variational inequality:

$$\tilde{x} \in \Gamma \quad : \quad \langle F(\tilde{x}), j(\tilde{x}-z)\rangle \le 0, \quad \forall z \in \Gamma.$$

Corollary 3.11. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X and let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $A : C \to X$ be α-inverse-strongly accretive. Let $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta+\zeta > 1$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by (3.16). Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\beta_n+\gamma_n) < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $\tilde{x} \in \Gamma$, which solves the following variational inequality:

$$\tilde{x} \in \Gamma \quad : \quad \langle F(\tilde{x}), j(\tilde{x}-z)\rangle \le 0, \quad \forall z \in \Gamma.$$

Conclusions.

In the present paper, we considered and studied a general system of nonlinear variational inequalities (1.2) in the setting of Banach spaces. We proposed Mann type implicit and explicit algorithms for solving the GSNVI (1.2). We studied the strong convergence of the sequences generated by the proposed algorithms to a solution of GSNVI (1.2). We extended strongly positive linear bounded operator F in the implicit and explicit algorithms of [29] to nonlinear strongly accretive and strictly pseudocontractive mapping F, and also the implicit and explicit algorithms of [29] are extended to develop our implicit and explicit algorithms of Mann's type. Our proofs contain some new techniques which are very different from those in [29].

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Lu-Chuan Ceng
Department of Mathematics, Shanghai Normal University, and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China
E-mail address: zenglc@hotmail.com

Himanshu Gupta
Department of Mathematics, Aligarh Muslim University, Aligarh, India
E-mail address: E-mail: hmnshu08@gmail.com

Qamrul Hasan Ansari
Department of Mathematics, Aligarh Muslim University, Aligarh, India;
Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia
E-mail address: qhansari@gmail.com