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FIXED POINT INDEX AND EJECTIVE FIXED POINTS OF COMPACT ABSORBING CONTRACTION MULTIVALUED MAPPINGS

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Dedicated to the memory of Professor Francesco S. De Blasi

Abstract. At first we shall generalize the fixed point index from the case of compact admissible mappings to the case of compact absorbing contraction multivalued mappings defined on arbitrary absolute neighbourhood retracts (ANR-spaces). Then we formulate, in terms of the Lefschetz number and the fixed point index, necessary conditions for the existence of ejective fixed points and sufficient conditions for the existence of non-ejective fixed points, for the above class of mappings defined on ANR-spaces. Note that some results are presented for a larger class of spaces so called absolute neighbourhood multiretracts (ANMR-spaces).

1. Introduction

The ejective fixed point theory plays an important role in applications to population dynamics generated by both discrete as well as continuous dynamical systems (see e.g. [25, 13, 27], and the references therein). Along these lines, a particular interest concerns the periodicity problem to autonomous functional differential equations (see e.g. [5, 8, 13, 16, 17, 19, 20, 21, 22, 27]).

More precisely, desired periodic solutions are determined by fixed points of Poincaré return operators along the trajectories of such systems. However, since given differential equations often possess trivial (constant) solutions, domains of Poincaré return operators, or their closures, contain known stationary fixed points, which brings some obstructions in order to distinguish them from nonconstant periodic solutions. For a coexistence of both types of solutions a certain sort of a weak local instability is suitable, called ejectivity.

This notion was introduced in 1965 by F. E. Browder, who also proved in [6, 7] the first fixed point theorems of this type, i.e. about ejective, non-ejective and, in particular, repulsive and non-repulsive fixed points for single-valued maps.

In application to differential equations without uniqueness or if the right-hand sides have discontinuities in state variables, the associated Poincaré return operators become naturally multivalued. It might be expected (cf. [1, 2]) that, under natural

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regularity assumptions imposed on the right-hand sides, these operators can be locally compact, admissible in the sense of the second coauthor and with a compact attractor (shortly, CA-maps).

Thus, a natural question arises whether also here, in the presence of trivial constant solutions, periodic solutions can be again considered in the same way, i.e. via fixed points of multivalued CA-maps? In order to answer this question, we need an adequate multivalued ejective fixed point theory. Let us note that until now, as far as we know, the multivalued case was considered only in two papers [9, 10] by Fenske and Peitgen. For some further papers in the single-valued case, see e.g. [6, 7, 15, 18, 20, 21, 22, 26].

An important step for this aim is to develop a sufficiently general Lefschetz fixed point theorem and the fixed point index (cf. [2, 4, 6, 7, 9, 10, 14, 15, 18, 23]). For compact admissible maps on ANR-spaces, the fixed point index is well known (cf. [2, 14]). In the present paper, we extend it to the case of compact absorbing contraction maps (CAC-maps), at first. Then we formulate one of the most general versions of the Lefschetz fixed point theorem, namely the one for CAC-mappings on absolute neighbourhood multiretract spaces (ANMR-spaces). Finally, we employ it for two main theorems about the existence of ejective, non-ejective, repulsive and non-repulsive fixed points.

2. Some auxiliary definitions

In the entire text, all topological spaces are metric and all single-valued mappings are continuous. Let X be a metric space and let x be a point of X. By $U(x)$ we shall denote the family of all open neighbourhoods of x in X.

Let Top_2 be the category of pairs of topological spaces and continuous mappings of such pairs. By a pair (X, A) in Top_2, we understand a space X and its subset A; a pair $(X, \emptyset)$ will be denoted for short by X. By a map $f\colon (X, A) \to (Y, B)$, we shall understand a continuous map from X to Y such that $f(A) \subset B$.

We shall use the following notations: if $f\colon (X, A) \to (Y, B)$ is a map of pairs, then by $f_X\colon X \to Y$ and $f_A\colon A \to B$, we shall understand the respective induced mappings. Let us also denote by Vect_G the category of graded vector spaces over the field of rational numbers $\mathbb{Q}$ and linear maps of degree zero between such spaces. By $H\colon \mathrm{Top}_2 \to \mathrm{Vect}_G$, we shall denote the Čech homology functor with compact carriers and coefficients in $\mathbb{Q}$.

Thus, for any pair (X, A), we have $H(X, A) = \{H_q(X, A)\}_{q \geq 0}$, a graded vector space in Vect_G and, for any map $f\colon (X, A) \to (Y, B)$, we have the induced linear map $f_* = \{f_{*q}\}\colon H(X, A) \to H(Y, B)$, where $f_{*q}\colon H_q(X, A) \to H_q(Y, B)$ is a linear map from the q-dimensional homology $H_q(X, A)$ of the pair (X, A) into the q-dimensional homology $H_q(Y, B)$ of the pair (Y, B).

For the properties of H, we recommend [14].

A non-empty space X is called *acyclic* provided:

(i) $H_q(X) = 0$, for every $q \geq 1$, and
(ii) $H_0(X) = \mathbb{Q}$.

Definition 2.1. A map $p\colon \Gamma \to X$ is called a *Vietoris map* if the following conditions are safisfied:

(i) p is onto and closed,
(ii) for every $x \in X$, the set $p^{-1}(x)$ is compact and acyclic.

Theorem 2.2 ((**Vietoris**) see e.g. [14])**.** *If $p\colon \Gamma \to X$ is a Vietoris map, then the induced linear map $p_*\colon H(\Gamma) \xrightarrow{\sim} H(X)$ is an isomorphism, i.e., for every $q \geq 0$ the linear map $p_{*q}\colon H_q(\Gamma) \xrightarrow{\sim} H_q(X)$ is a linear isomorphism.*

For further properties of Vietoris mappings, see e.g. [14].

The following notions will play a crucial role. At first, by $\varphi\colon X \multimap Y$, we shall denote a multivalued map, i.e., a map which assigns to every point $x \in X$ a compact nonempty set $\varphi(x) \subset Y$.

A multivalued map $\varphi\colon X \multimap Y$ is called *admissible* (see [2, 14]) provided there exists a diagram

$$X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y$$

in which p is a Vietoris map, such that $\varphi(x) = q(p^{-1}(x))$. The pair (p,q) is called a *selected pair* of φ (write $(p,q) \subset \varphi$). In what follows, we shall use the following notation:

$$\Gamma \overset{p}{\Rightarrow} X$$

for Vietoris mappings.

Note that the superposition $\psi \circ \varphi\colon X \multimap Z$ of two admissible maps $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ is again an admissible map.

For a map $\varphi\colon X \multimap X$, we shall consider the set $\mathrm{Fix}(\varphi)$ of fixed points φ, i.e.,

$$\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}.$$

More information about admissible mappings will be presented in the next section.

Recall that the space X is an *absolute neighbourhood retract* ($X \in$ ANR), provided there exists an open set U in a normed space E and two maps:

$$r\colon U \to X \quad \text{and} \quad s\colon X \to U$$

such that $r \circ s = \mathrm{id}_X$.

We shall also use the notion of a multiretraction.

Definition 2.3 ([23, 3, 4])**.** A map $r\colon Y \to X$ is said to be a *multiretraction* if there exists an admissible map $\varphi\colon X \multimap Y$ such that $r \circ \varphi = \mathrm{id}_X$.

Definition 2.4 ([23])**.** A space X is called an *absolute neighbourhood multiretract* ($X \in$ ANMR) if there exists an open set U of a normed space E and a multiretraction $r\colon U \to X$.

Evidently, we have:

$$\mathrm{ANR} \subset \mathrm{ANMR},$$

i.e. that the class of ANMR-spaces is obviously larger than the one of ANR-spaces (see [23, 3, 4]).

For some nontrivial examples and more details concerning AMR-spaces, we recommend [23].

3. Compact absorbing contraction mappings

Let $\varphi\colon X \multimap Y$ be an admissible mapping and $(p, q) \subset \varphi$ be a selected pair of φ. Using the Vietoris Theorem 2.2, we are able to define the induced by (p, q) linear map by putting:

$$q_* \circ p_*^{-1}\colon H_*(X) \to H_*(Y).$$

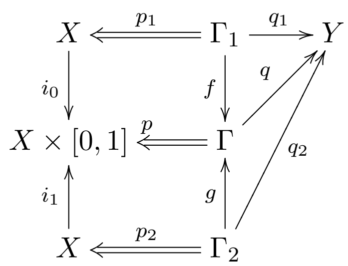
We let: $\varphi_* = \{q_* \circ p_*^{-1} \mid (p, q) \subset \varphi\}$.

Now, let us consider two admissible mappings $\varphi, \psi\colon X \multimap Y$. We shall say that φ is *homotopic* to ψ (written: $\varphi \sim \psi$), provided there exists an admissible mapping $\chi\colon X \times [0, 1] \to Y$ such that $\chi(x, 0) = \varphi(x)$ and $\chi(x, 1) = \psi(x)$, for every $x \in X$.

We have the following proposition (for its proof, see [14]):

Proposition 3.1. *If $\varphi \sim \psi$, then $\varphi_* \cap \psi_* \neq \emptyset$.*

Let $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$. We shall say that the above selected pairs are *homotopic* (written $(p_1, q_1) \sim (p_2, q_2)$), provided there exists the following commutative diagram:



where $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$, Γ is a given space and f, g are also given.

Evidently, we have:

Proposition 3.2. *If $(p_1, q_1) \sim (p_2, q_2)$, then $q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}$.*

We say that an admissible map $\varphi\colon X \multimap X$ is a *Lefschetz map* provided, for every selected pair $(p, q) \subset \varphi$, the *generalized Lefschetz number* $\Lambda(p, q) = \Lambda(q_* \circ p_*^{-1})$ is well defined (for details, see [14]).

For a Lefschetz map $\varphi\colon X \multimap X$, we define the *Lefschetz set* $\Lambda(\varphi)$ of φ by putting:

$$\Lambda(\varphi) = \{\Lambda(p, q) \mid (p, q) \subset \varphi\}. \tag{3.1}$$

We have (see [14]):

(1) If $\varphi \sim \psi$, then $\Lambda(\varphi) \cap \Lambda(\psi) \neq \emptyset$.
(2) If $(p_1, q_1) \sim (p_2, q_2)$, then $\Lambda((p_1, q_1)) = \Lambda((p_2, q_2))$.

Definition 3.3 ([2, 11, 14])**.** An admissible map $\varphi\colon X \multimap X$ is called a *compact absorbing contraction* ($\varphi \in \mathrm{CAC}(X)$) if there exists an open set $U \subset X$ such that:

(i) $\varphi(U) \subset U$,
(ii) the closure $\overline{\varphi(U)}$ of $\varphi(U)$ is contained in a compact subset of U,
(iii) for every $x \in X$, there exists a natural number n_x such that

$$\varphi^{n_x}(x) \subset U.$$

We say that $\varphi\colon X \multimap X$ is a *locally compact map* provided, for every $x \in X$, there exists $V \in U(x)$ such that $\varphi|_V\colon V \multimap X$ is a compact map, i.e. $\overline{\varphi|_V(V)}$ is compact.

We let:

$\mathrm{K}(X) = \{\varphi\colon X \multimap X \mid \varphi$ is admissible and compact$\}$.

$\mathrm{EC}(X) = \{\varphi\colon X \multimap X \mid \varphi$ is admissible locally compact and there exists a natural number n such that the n-th iteration $\varphi^n\colon X \multimap X$ of φ is a compact map$\}$.

$\mathrm{ASC}(X) = \{\varphi\colon X \multimap X \mid \varphi$ is admissible locally compact, the orbit $O(x) = \bigcup_{n=1}^{\infty} \varphi^n(x)$ is, for every $x \in X$, relatively compact and the core $C(\varphi) = \bigcap_{n=1}^{\infty} \varphi^n(x)$ is nonempty and relatively compact$\}$.

$\mathrm{CA}(X) = \{\varphi\colon X \multimap X \mid \varphi$ is admissible locally compact and has a compact attractor, i.e., then exists a compact set $A \subset X$ such that, for every open set $W \subset X$ containing A and for every point $x \in X$, there is n_x such that $\varphi^{n_x}(x) \subset W\}$.

The following hierarchy holds ([2, 14]):

$$\mathrm{K}(X) \subset \mathrm{EC}(X) \subset \mathrm{ASC}(X) \subset \mathrm{CA}(X) \subset \mathrm{CAC}(X). \tag{3.2}$$

Moreover, each of the above inclusions is proper.

Let $\varphi \in \mathrm{CAC}(X)$ and let U be chosen according to Definition 3.3. Then

(3.3) $\varphi_U\colon U \multimap U$, defined by the formula $\varphi_U(x) = \varphi(x)$, for every $x \in U$, is a compact admissible map.

Recall that if $\psi\colon Y \multimap Y$ is a compact admissible map and $Y \in \mathrm{ANMR}$, then ψ is a Lefschetz map and $\Lambda(\psi) \neq \{0\}$ implies that ψ has a fixed point (see [23, 4]).

We prove the following theorem.

Theorem 3.4. *Let $\varphi \in \mathrm{CAC}(X)$, where $X \in \mathrm{ANMR}$. Assume further that U is chosen according to Definition 3.3 and $\varphi_U\colon U \multimap U$ be a map defined in (3.3). Then φ is a Lefschetz map and*

$$\Lambda(\varphi) \subset \Lambda(\varphi_U).$$

Proof. Let (p, q) be a selected pair of φ, i.e., we have a diagram:

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$$

such that $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$.

Consider still the following diagram:

$$U \overset{p_1}{\Longleftarrow} p^{-1}(U) \overset{q_1}{\longrightarrow} U$$

in which p_1 and q_1 are respective contractions of p and q.

We have also the following diagram:

$$(X, U) \overset{\overline{p}}{\Longleftarrow} (\Gamma, p^{-1}(U)) \overset{\overline{q}}{\longrightarrow} (X, U)$$

in which $\overline{p}(y) = p(y)$ and $\overline{q}(y) = q(y)$, for every $y \in \Gamma$.

Now, we shall use the following formula proved in [14]. If two Lefschetz numbers from the following three numbers $\Lambda(\overline{p}, \overline{q})$, $\Lambda(p, q)$ and $\Lambda(p_1, q_1)$ are well defined, then the third one is well defined too, and we have:

$$\Lambda(p, q) = \Lambda(\overline{p}, \overline{q}) + \Lambda(p_1, q_1).$$

Since an open subset of an ANMR-space is an ANMR-space, too, we infer from above that $\Lambda(p_1, q_1)$ is well defined.

Now, since we consider the homology with compact carriers from (2), it follows that $\Lambda(\overline{p},\overline{q})=0$. Consequently, we get that $\Lambda(p,q)$ is well defined, and

$$\Lambda(p,q)=\Lambda(p_1,q_1).$$

The proof is completed. □

Corollary 3.5. *If* $\varphi\in \mathrm{CAC}(X)$ *and* $X\in \mathrm{ANMR}$, *then* φ *is a Lefschetz map and* $\Lambda(\varphi)\neq\{0\}$ *implies that* φ *has a fixed point.*

4. The fixed point index

Firstly, let us assume that $\varphi\colon X\multimap X$ is a compact admissible map, where $X\in\mathrm{ANR}$.

Let $(p,q)\subset\varphi$ and $V\subset X$ be an open set such that $\{x\in V\mid x\in\varphi(x)\}$ is compact. Then the fixed point index $\mathrm{ind}((p,q),V)$ of the pair (p,q) with respect to V is well defined (see [14] and also [9, 10]). Note that $\mathrm{ind}((p,q),V)$ is an integer.

We define the fixed point index of φ as the following set:

$$\mathrm{Ind}(\varphi,V)=\{\mathrm{ind}((p,q),V)\mid (p,q)\subset\varphi\}. \tag{4.1}$$

Below, we shall list the important properties of the fixed index which we shall need in the next section.

(a) (Existence). If $\mathrm{ind}((p,q),V)\neq 0$ ($\mathrm{Ind}((\varphi,V)\neq\{0\}$), then

$$\mathrm{Fix}(p,q)\cap V\neq\emptyset.$$

(b) (Excision). If $\mathrm{Fix}(\varphi)\cap W\subset V\subset W$ is compact, then

$$\mathrm{ind}((p,q),V)=\mathrm{ind}((p,q),W)\quad (\mathrm{Ind}(\varphi,V)=\mathrm{Ind}(\varphi,W)).$$

(c) (Additivity). If V_1,V_2 are open subsets of X such that $V_1\cap V_2=\emptyset$ and $\mathrm{Fix}(\varphi)\cap V_1$, $\mathrm{Fix}(\varphi)\cap V_2$ are compact sets, then

$$\mathrm{ind}((p,q),V_1\cup V_2)=\mathrm{ind}((p,q),V_1)+\mathrm{ind}((p,q),V_2).$$

(d) (Homotopy). If $(p_1,q_1)\sim(p_2,q_2)$ ($\varphi\sim\psi$), then

$$\mathrm{ind}((p_1,q_1),V)=\mathrm{ind}((p_2,q_2),V)\quad (\mathrm{Ind}(\varphi,V)\cap\mathrm{Ind}(\psi,V)\neq\emptyset),$$

where $(p_1,q_1)\subset\varphi$ and $(p_2,q_2)\subset\psi$.

(e) (Normalization). If $V=X$, then

$$\mathrm{ind}((p,q),V)=\Lambda((p,q))\quad\text{and}\quad \Lambda(\varphi)=\mathrm{Ind}(\varphi,V).$$

Now, we shall consider the noncompact case. Assume that $\varphi\colon X\multimap X$ is an admissible compact absorbing contraction and $X\in\mathrm{ANR}$. Assume, furthermore, that V is an open set such that $\{x\in V\mid x\in\varphi(x)\}$ is compact. According to Definition 3.3, we select an open set U satisfying all its assumptions (i)-(iii). Evidently, $\mathrm{Fix}(\varphi)\subset U$. Moreover, we have that $\varphi_U: U\multimap U$ is a compact admissible map, where $\varphi_U(x)=\varphi(x)$, for every $x\in U$. Let $(p,q)\subset\varphi$. Then $(p_U,q_U)\subset\varphi_U$, where $p_U: p^{-1}(U)\Rightarrow U$ and $q_U: p^{-1}(U)\to U$ are defined as follows: $p_U(y)=p(y)$ and $q_U(y)=q(y)$, for every $y\in p^{-1}(U)$.

We let:

$$\mathrm{ind}((p,q),V)=\mathrm{ind}((p_U,q_U),V\cap U) \tag{4.2}$$

and

$$\mathrm{Ind}(\varphi, V) = \{\mathrm{ind}((p,q), V) \mid (p,q) \subset \varphi\}. \tag{4.3}$$

By means of (c),we deduce that the definitions (4.2) and (4.3) do not depend on the choice of U. Thus, all properties (a)–(e) are satisfied

For more details, we recommend [2, 9, 10, 11, 14].

Open Problem 4.1. *Is it possible to define a fixed point index for* CAC-*mappings on* ANMR-*retracts?*

5. Ejective fixed points

In this section, we shall assume that all multivalued maps are compact absorbing contractions (CAC-maps).

Definition 5.1 (cf. [6, 7, 9, 10, 12, 15])**.** Let $\varphi\colon X \multimap X$ be a given map and let $x_0 \in \mathrm{Fix}(\varphi)$.

(i) We say that x_0 is *ejective relative* to $V \in U(x_0)$ if, for any $x \in \overline{V} \setminus \{x_0\}$, there exists an integer $n \geq 1$ such that $\varphi^n(x) \subset X \setminus \overline{V}$. If there exists $V \in U(x_0)$ such that x_0 is ejective relative to V, then x_0 is called *ejective.* The set of all ejective fixed points is denoted by $\mathrm{Fix}_e(\varphi)$.

(ii) A fixed point $x_0 \in \mathrm{Fix}(\varphi)$ is called *repulsive relative* to $V \in U(x_0)$ if, for any $W \in U(x_0)$, there exists an integer $n(W) \geq 1$ such that $\varphi^n(X \setminus W) \subset X \setminus \overline{V}$, for all $n \geq n(W)$. If there exists $V \in U(x_0)$ such that x_0 is repulsive relative to V, then x_0 is called *repulsive.* The set of all repulsive fixed points is denoted by $\mathrm{Fix}_r(\varphi)$.

As an immediate consequence of the above definitions, we have:

$$\mathrm{Fix}_r(\varphi) \subset \mathrm{Fix}_e(\varphi).$$

The following example shows that the converse is not true even for single-valued mappings.

Example 5.2. Let $f\colon [0,1] \to [0,1]$ be defined by the formula $f(x) = 2(-x^2 + x)$. Then $x_0 = 0$ is ejective relative to $V = [0, 1/4)$. However, 0 is not a repulsive point because $f(1) = 0$.

Remark 5.3. Observe that every ejective fixed point is isolated in the set $\mathrm{Fix}(\varphi)$. Therefore, if $\#\,\mathrm{Fix}_e(\varphi) < \infty$, then $\mathrm{Fix}_e(\varphi)$ is open and compact in $\mathrm{Fix}(\varphi)$.

Let $\varphi\colon X \multimap X$ be a CAC-map and let $U \subset X$ be chosen according to the Definition 3.3. Then we have a compact admissible map $\varphi_U\colon U \to U$ defined by $\varphi_U(x) = \varphi(x)$, for every $x \in U$. Observe that, in view of Definition 3.3, we have:

$$\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_U), \tag{5.1}$$

$$\mathrm{Fix}_e(\varphi) = \mathrm{Fix}_e(\varphi_U), \tag{5.2}$$

$$\mathrm{Fix}_r(\varphi) = \mathrm{Fix}_r(\varphi_U). \tag{5.3}$$

Therefore, all results obtained in [9] and [10] can be reformulated for CAC-mappings. Note that the class of compact attraction mappings considered in [9] and [10] is involved in the class of CAC-mappings (see (3.2)). Consequently, from (5.1)–(5.3), we can deduce the same results for compact admissible mappings on ANR-s, for all classes in (3.2).

For example, we can formulate the following two most important theorems.

Theorem 5.4. *Let* $X \in \mathrm{ANR}$ *and* $\varphi\colon X \multimap X$ *be a* CAC*-map. Assume furthermore that* x_0 *is a repulsive fixed point of* φ *with respect to* $V \in U(x_0)$. *If there exists* $W \in U(x_0)$ *such that:*

(i) $\overline{V} \subset U$,
(ii) *the inclusion map* $i\colon X \setminus W \to X$ *induces the isomorphism* $i_*\colon H_*(X \setminus W) \xrightarrow{\sim} H_*(X)$,

then $\mathrm{Ind}(\varphi, V) = \{0\}$.

Corollary 5.5. *If we assume additionally that* $\mathrm{Fix}_r(\varphi)$ *is a finite set and that* $\Lambda(\varphi) \neq \{0\}$, *then there exists a non-repulsive fixed point of* φ.

Concerning ejective fixed points, we will formulate the following theorem:

Theorem 5.6. *Let* $X \in \mathrm{ANR}$ *and* $\varphi\colon X \multimap X$ *be a* CAC*-mapping. Assume that*

$$\varphi(X \setminus \mathrm{Fix}_e(\varphi)) \subset X \setminus \mathrm{Fix}_e(\varphi) \quad \text{and} \quad \#\,\mathrm{Fix}_e(\varphi) < \infty. \tag{5.4}$$

Denote by $\varphi'\colon (X \setminus \mathrm{Fix}_e(\varphi)) \multimap (X \setminus \mathrm{Fix}_e(\varphi))$ *and* $\overline{\varphi}\colon (X, X \setminus \mathrm{Fix}_e(\varphi)) \multimap (X, X \setminus \mathrm{Fix}_e(\varphi))$ *the respective maps induced by* φ. *Then we have:*

(i) $\overline{\varphi}$ *is a Lefschetz map,*
(ii) $\Lambda(\overline{\varphi}) = \{0\}$ *and if* $\Lambda(\varphi) \neq \{0\}$, *then* φ' *has a non-ejective fixed point.*

Denoting still by $\mathrm{Fix}_{et}(\varphi) \subset \mathrm{Fix}_e(\varphi)$ the subset of trivial (obvious) ejective fixed points of φ, we can immediately reformulate Theorem 5.6 in the following form which is suitable for applications to functional differential equations.

Theorem 5.7. *Let* $X \in \mathrm{ANR}$ *and* $\varphi\colon X \multimap X$ *be a* CAC*-mapping. Assume that*

$$\varphi(X \setminus \mathrm{Fix}_{et}(\varphi)) \subset X \setminus \mathrm{Fix}_{et}(\varphi) \quad \text{and} \quad \#\,\mathrm{Fix}_{et}(\varphi) < \infty. \tag{5.5}$$

Denote by $\widetilde{\varphi}\colon (X \setminus \mathrm{Fix}_{et}(\varphi)) \multimap (X \setminus \mathrm{Fix}_{et}(\varphi))$ *and* $\widehat{\varphi}\colon (X, X \setminus \mathrm{Fix}_{et}(\varphi)) \multimap (X, X \setminus \mathrm{Fix}_{et}(\varphi))$ *the respective maps induced by* φ. *Then we have:*

(i) $\widehat{\varphi}$ *is a Lefschetz map,*
(ii) $\Lambda(\widehat{\varphi}) = \{0\}$ *and if* $\Lambda(\varphi) \neq \{0\}$, *then* $\widetilde{\varphi}$ *has either a nontrivial ejective fixed point or a non-ejective fixed point.*

Let us note that some further results concerning repulsive and ejective fixed points for CA-mappings were presented in [9, 10].

As already pointed out, all the results in [9, 10], jointly with those for single-valued maps in [6, 7, 15, 18, 20, 21, 22, 26], can be reformulated for CAC-mappings. The proofs are quite analogous to those presented in the quoted papers.

Open Problem 5.8. *Is it possible to prove some existence results about ejective or repulsive fixed points for compact admissible mappings on* ANMR*-spaces?*

6. Concluding remarks

A possible candidate for the application of our results might be the following functional inclusion:

$$x'(t) \in F_{k,l}(x(t-1)), \tag{6.1}$$

where $F_{k,l}(x) \equiv F_{k,l}(x+\omega)$, $\omega > 0$, and

$$F_{k,l}(x) := \begin{cases} \left[-\frac{k}{2}, \frac{k}{2}\right], & \text{for } x \in \left\{0, \frac{\omega}{2}\right\}, \\ \frac{k}{2} + l \sin\left(\frac{2\pi}{\omega}x\right), & \text{for } x \in \left(0, \frac{\omega}{2}\right), \\ -\frac{k}{2} + l \sin\left(\frac{2\pi}{\omega}x\right), & \text{for } x \in \left(\frac{\omega}{2}, \omega\right). \end{cases}$$

Observe that, for $k = 0$, we have $F_{0,l}(x) := l \sin\left(\frac{2\pi}{\omega}x(t-1)\right)$, by which (6.1) reduces to the delayed differential equation

$$x'(t) = l \sin\left(\frac{2\pi}{\omega}x(t-1)\right), \tag{6.2}$$

studied in a more general form e.g. in [24, 28].

In [24] (see also the references therein), it was shown that, besides other things, for suitable values of $l > 0$, (6.2) possesses hyperbolic nontrivial periodic solutions oscillating around the unstable equilibria given by $\ldots, -\omega, 0, \omega, \ldots$, and with transversal heteroclinic connections between them.

For (6.1) with $k > 0$, the situation becomes more delicate. On one side, one can readily check that again

$$F_{k,l}(x) > 0, \text{ for } x \in \left(0, \frac{\omega}{2}\right), \quad \text{and} \quad F_{k,l}(x) < 0, \text{ for } x \in \left(\frac{\omega}{2}, \omega\right).$$

Therefore, for the stationary solution $x_1(t) \equiv \frac{\omega}{2}$, there is a negative feedback on the circle S^1 with a reaction lag, i.e.

$$x(t-1) \in \left(0, \frac{\omega}{2}\right) \text{ implies } x'(t) > 0,$$

and

$$x(t-1) \in \left(\frac{\omega}{2}, \omega\right) \text{ implies } x'(t) < 0,$$

while for $x_2(t) \equiv 0$, the feedback is positive.

On the other hand, the associated Poincaré return operator φ is naturally multivalued.

Since $|F_{k,l}(x)| \le l + \frac{k}{2}$ holds, for all $x \in \mathbb{R}$, $k > 0$, $l > 0$, the locally absolutely continuous solutions $x(\cdot)$ of (6.2) are equi-continuous, because they have uniformly bounded derivatives $x'(\cdot)$ such that $|x'(\cdot)| \le l + \frac{k}{2}$, for almost all $t \in \mathbb{R}$. Therefore, the bounded domain of the Poincaré return operator φ, associated with (6.2), can be a compact subset X of the Banach space of continuous real functions, on the initial interval $[-1, 0]$, endowed with the sup-norm. If X is still a retract of this Banach space, or of its convex subset, then the Poincaré return operator φ is defined on a compact AR-space (cf. [2, 14]).

Following and matching the ideas in [1], [2, Chapter III.4] and [24, 28], one might expect that the Poincaré return operator φ, associated with (6.2), can be an admissible mapping which, in view of the above arguments, is compact. Moreover, since φ

can be defined on a compact AR-space X, we have immediately that $\# \operatorname{Fix}_e(\varphi) < \infty$ (cf. [26]), and especially that $\Lambda(\varphi) \neq \{0\}$ (cf. [1, 2]).

Hence, in order to apply Theorem 5.7, we could only check in this way condition (5.5), provided all the above arguments are satisfied. Of course, in the case of $\operatorname{Fix}_{et}(\varphi)) = \operatorname{Fix}_e(\varphi)$, Theorem 5.7 coincides with Theorem 5.6.

References

[1] J. Andres, *On the multivalued Poincaré operators*, Topol. Meth. Nonlin. Anal. **10** (1997), 171–182.

[2] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer Academic Publishers, Dordrecht, 2003.

[3] J. Andres and L. Górniewicz, *Fixed point theorems on admissible multiretracts applicable to dynamical systems*, Fixed Point Theory **12** (2011), 255–264.

[4] J. Andres and L. Górniewicz, *On the Lefschetz fixed point theorem for random multivalued mappings*, Libertas Mathematica, New Series **33** (2013), 69–78.

[5] O. Arino, K.P. Hadeler and M.L. Hbid, *Existence of periodic solutions for delay differential equations with state dependent delay*, J. Diff. Eqns **144** (1998), 263–301.

[6] F. E. Browder, *Another generalization of the Schauder fixed point theorem*, Duke Math. J. **32** (1965), 399–406.

[7] F. E. Browder, *A futher generalization of the Schauder fixed point theorem*, Duke Math. J. **32** (1965), 575–578.

[8] S.-N. Chow, *Existence of periodic solutions of autonomous functional differential equations*, J. Diff. Eqns **15** (1974), 350–378.

[9] C. C. Fenske and H. O. Peitgen, *Attractors and the fixed point index for a class of multivalued mappings, I*, Bull. Polish Acad. Sci. Math. **25** (1977), 477–482.

[10] C. C. Fenske and H. O. Peitgen, *Attractors and the fixed point index for a class of multivalued mappings II*, Bull. Polish Acad. Sci. Math. **25** (1977), 483–487.

[11] G. Fournier and L. Górniewicz, *The Lefschetz fixed point theorem for some noncompact multivalued maps*, Fund. Math. **94** (1977), 245–254.

[12] G. Gabor, *On the classification of fixed points*, Math. Japon. **40** (1994), 361–369.

[13] D.V. Giang and Y. Lenbury, *Periodicity and knots in delay models of population growth*, Mathematical and Computer Modelling **47** (2008), 259–265.

[14] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Springer, Berlin, 2009.

[15] L. Górniewicz and H. O. Peitgen, *Degeneracy, non-ejective fixed points and the fixed point index*, J. Math. Pures Appl. **58** (1979), 217–228.

[16] J. Hale, *Theory of Functional Differential Equations*, Springer, Berlin, 1977.

[17] F. Hartung, T. Krisztin, H.-O. Walther and J. Wu, *Functional differential equations with state-dependent delay: theory and applications*, in Handbook of Differential Equations: Ordinary Differential Equations, A. Cañada, P. Drábek, A. Fonda (eds), Vol. 3, Elsevier–North-Holland, Amsterdam, 2006, pp. 435–545.

[18] W.A. Horn, *Some fixed point theorems for compact maps and flows in Banach spaces*, Trans. Amer. Math. Soc. **149** (1970), 391–401.

[19] J. Mallet-Paret, R.D. Nussbaum and P. Paraskevopoulos, *Periodic solutions for functional differential equations with multiple state dependent time lags*, Topol. Meth. Nonlin. Anal. **3** (1994), 101–162.

[20] R.D. Nussbaum, *Periodic solutions of autonomous functional differential equations*, Bull. Amer. Math. Soc. **79** (1973), 811–814.

[21] R.D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations*, Ann. Mat. Pura Appl. **101** (1974), 263–306.

[22] R.D. Nussbaum, *A global bifurcation theorem with applications to functional differential equations*, J. Funct. Anal. **19** (1975), 319–339.

[23] R. Skiba and M. Ślosarski, *On a generalization of absolute neighbourhood retracts*, Topology Appl. **156** (2009), 697–709.

[24] H. Steinlein and H.-O. Walther, *Hyperbolic sets, transversal homoclinic trajectories, and symbolic dynamics for C^1-maps in Banach spaces*, J. Dynamics Diff. Eqns **2** (1990), 325–365.

[25] P. Takáč, *Discrete monotone dynamics and time-periodic competition between two species*, Differential Integral Equations **10** (1997), 401–598.

[26] S. Thompson, *An extension of Browder's non-ejective fixed point theorem*, Fixed Point Theory **11** (2010), 143–146.

[27] H.-O. Walther, *Existence of a non-constant periodic solution of a non-linear autonomous functional differential equation representing the growth of a single species population*, J. Math. Biol. **1** (1975), 227–240.

[28] H.-O. Walther, *Bifurcation from periodic solutions in functional differential equations*, Math. Z. **182** (1983), 269–289.

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