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MULTIPLICITY RESULTS FOR SYSTEMS OF FIRST ORDER DIFFERENTIAL INCLUSIONS

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Dedicated to the memory of Professor Francesco S. De Blasi

Abstract. Multiplicity results are obtained for systems of first order differential inclusions with periodic boundary or initial value conditions. To this aim, we introduce notions of strict solution-tubes. The cases where the nonlinearity satisfies an upper or lower semi-continuity condition are considered. Our results are new even in the particular cases where the nonlinearity is single-valued or it has real values.

1. Introduction

In this paper, we establish multiplicity results for the following system of first order differential inclusions:

$$\begin{aligned} &x'(t) \in F(t, x(t)) \quad \text{a.e. } t \in [0,1], \\ &x(0) = x(1); \end{aligned} \tag{1.1}$$

where $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a set-valued mapping with compact values which satisfies an upper or lower semi-continuity condition. In this first case, the values are also convex.

The theory of differential inclusions is well developed in the literature, see for example [4, 6, 11, 12] and the references therein. One approach to obtain existence results for problem (1.1) is to establish the existence of fixed points to the associated Poincaré operator [2, 3, 8, 9, 10, 19, 22]. Here, we use an other approach which is to obtain a solution as a fixed point of a compact operator defined on the space of periodic, continuous functions on $[0,1]$.

In order to establish the existence of a solution for (1.1), we use the notion of solution-tube for problem (1.1) similar to the notion of solution-tube introduced in [13], (see also [15] and [16] for systems of first order differential equations). In the particular case where F has real values ($n = 1$), the notion of solution-tube coincides with the notions of lower and upper solutions $\alpha \leq \beta$, see [1, 5].

To our knowledge, here, the first multiplicity results for problem (1.1) are established. Indeed, there are no multiplicity results in this generality but there are some in the particular case where F has real single values, see for example [20, 21]. In order to obtain the existence of at least three solutions of (1.1), we introduce the notions of strict solution-tubes of (1.1). It is inspired by a notion introduced in [14] for second order system of differential equations. It is worth to mention that our

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results are new even in the particular cases where the nonlinearity is single-valued or it has real values.

This paper is divided in five sections. Section 2 contains preliminaries on set-valued mappings. In section 3, we obtain existence and multiplicity results in the case where F satisfies an upper semi-continuity condition and has convex, compact values. In section 4, the convexity assumption on the values of F is removed and the upper semi-continuity condition is replaced by a lower semi-continuity condition. In the last section, we present multiplicity results for the following system of first order differential inclusions with initial condition:

$$\begin{aligned} &x'(t) \in F(t, x(t)) \quad \text{a.e. } t \in [0,1],\\ &x(0) = x_0. \end{aligned}$$

2. Preliminaries

In what follows, we will use the following notations: $I = [0,1]$, $L^1(I, \mathbb{R}^n)$ is the space of integrable functions; $C(I, \mathbb{R}^n)$ is the space of continuous functions endowed with the usual norm $\|\cdot\|_0$; $W^{1,1}(I, \mathbb{R}^n)$ is the Sobolev space $\{x \in C(I, \mathbb{R}^n) : x$ is absolutely continuous and $x' \in L^1(I, \mathbb{R}^n)\}$. We denote $W_P^{1,1}(I, \mathbb{R}^n) = \{x \in W^{1,1}(I, \mathbb{R}^n) : x(0) = x(1)\}$. Let $L : W^{1,1}(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ be defined by

$$L(x) = x'. \tag{2.1}$$

It is well known that the continuous linear operator $L + \text{id} : W_P^{1,1}(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ is invertible.

For sake of completeness, we recall some definitions. Let X and Y be topological spaces, and Z a measurable space. We say that a set-valued mapping $F : X \to Y$ is *compact* if $F(X) = \cup_{x \in X} F(x)$ is relatively compact, and F is *upper semi-continuous* (u.s.c.) (resp. *lower semi-continuous* (l.s.c.)) if $\{x \in X : F(x) \cap B \neq \emptyset\}$ is closed (resp. open) for every closed (resp. open) set $B \subset Y$. We say that a set-valued mapping $F : Z \to Y$ is *measurable* if $\{z \in Z : F(z) \cap B \neq \emptyset\}$ is measurable for each closed set $B \subset Y$. The reader is referred to [7, 11, 17, 18] for more details on set-valued mappings.

Here are some conditions which permit to obtain more precision on the location of the solutions of (1.1).

Lemma 2.1. *Let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued mapping. Assume there exist $v \in W^{1,1}(I, \mathbb{R}^n)$ and $\rho \in W^{1,1}(I, [0, \infty))$ such that $\|v(0) - v(1)\| \leq \rho(0) - \rho(1)$ and*

$$F(t,x) \subset \{y \in \mathbb{R}^n : \langle x - v(t), y - v'(t)\rangle < \|x - v(t)\|\rho'(t)\} \quad \text{a.e. } t \in I \text{ and for all } x \in \mathbb{R}^n \text{ such that } \|x - v(t)\| > \rho(t). \tag{2.2}$$

Then any solution $x \in W^{1,1}(I, \mathbb{R}^n)$ of (1.1) satisfies $\|x(t) - v(t)\| \leq \rho(t)$ for all $t \in I$.

Proof. Let assume that x is a solution of (1.1) such that

$$J = \{t \in I : \|x(t) - v(t)\| > \rho(t)\} \neq \emptyset.$$

Let t_1 be the largest $t \in J$ such that

$$\|x(t_1) - v(t_1)\| - \rho(t_1) = \sup_{t \in J} \|x(t) - v(t)\| - \rho(t).$$

Since x satisfies the periodic boundary condition,

$$\|x(0) - v(0)\| - \rho(0) \leq \|x(1) - v(1)\| + \|v(1) - v(0)\| - \rho(0) \leq \|x(1) - v(1)\| - \rho(1).$$

So, $t_1 > 0$. Let $t_0 < t_1$ be such that $[t_0, t_1] \subset J$. Then

$$\begin{aligned}
0 &\leq \|x(t_1) - v(t_1)\| - \rho(t_1) - \|x(t_0) - v(t_0)\| - \rho(t_0) \\
&= \int_{t_0}^{t_1} \frac{d}{dt}\Big(\|x(t) - v(t)\| - \rho(t)\Big)\, dt \\
&= \int_{t_0}^{t_1} \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle}{\|x(t) - v(t)\|} - \rho'(t)\, dt \\
&< 0.
\end{aligned}$$

Contradiction. □

3. Upper semi-continuity condition

3.1. **Existence result.** In this section, we consider the case where the mapping F is upper semi-continuous with respect to the second variable.

Definition 3.1. A set-valued mapping $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ with nonempty closed, convex values is said to be *Carathéodory* if the following conditions hold:

(i) $t \mapsto F(t, x)$ is measurable for every $x \in \mathbb{R}^n$;
(ii) $x \mapsto F(t, x)$ is upper semi-continuous for a.e. $t \in I$;
(iii) for every $r > 0$, there exists $h_r \in L^1(I, \mathbb{R})$ such that for almost every $t \in I$ and every $x \in \mathbb{R}^n$ satisfying $\|x\| \leq r$, one has $\|y\| \leq h_r(t)$ for all $y \in F(t, x)$.

Remark 3.2. A single-valued mapping $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is *Carathéodory* if and only if $F = \{f\}$ is Carathéodory in the sense of Definition 3.1.

Definition 3.3. A set-valued mapping $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ is *integrably bounded* if there exists $h \in L^1(I, \mathbb{R})$ such that for almost every $t \in I$ and every $x \in \mathbb{R}^n$, one has $\|y\| \leq h(t)$ for all $y \in F(t, x)$.

For a set-valued mapping $F : I \times \mathbb{R}^n \to \mathbb{R}^n$, we define $\mathbf{F} : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ by

$$\mathbf{F}(x) = \{y \in L^1(I, \mathbb{R}^n) : y(t) \in F(t, x(t)) \text{ a.e. } t \in I\}.$$

Arguing as in the proof of Proposition 4.1 in [13], we obtain the following result.

Proposition 3.4. *Let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be an integrably bounded, Carathéodory set-valued mapping with nonempty, closed, convex values. If $L^1(I, \mathbb{R}^n)$ is endowed with the weak topology, then the associated operator $\mathbf{F} : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ is u.s.c. and has nonempty, compact, convex values.*

To a set-valued mapping $F : I \times \mathbb{R}^n \to \mathbb{R}^n$, we associate the set-valued operator

$$\mathcal{F} : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n) \quad \text{defined by} \quad \mathcal{F} = i \circ (L + \mathrm{id})^{-1} \circ \mathbf{F}, \tag{3.1}$$

where L is defined in (2.1) and $i : W^{1,1}(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ is the continuous embedding. Using Proposition 3.4, it is easy to prove the following result.

Proposition 3.5. *Let $F : I\times\mathbb{R}^n \to \mathbb{R}^n$ be an integrably bounded, Carathéodory set-valued mapping with nonempty, closed, convex values. Then the associated operator $\mathcal{F} : C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ is u.s.c., compact and has nonempty, compact, convex values.*

We consider the problem

$$x'(t) + x(t) \in F(t, x(t)) \quad \text{a.e. } t \in [0,1], \qquad x(0) = x(1). \tag{3.2}$$

It is well known that if F is Carathéodory and integrably bounded then (3.2) has a solution. We present the proof for sake of completeness.

Proposition 3.6. *Let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be an integrably bounded, Carathéodory set-valued mapping with nonempty, closed, convex values. Then* (3.2) *has a solution $x \in W^{1,1}(I,\mathbb{R}^n)$. Moreover, there exists a bounded open set $\Omega \subset C(I,\mathbb{R}^n)$ such that the topological degree $deg(id - \mathcal{F},\Omega) = 1$.*

Proof. A fixed point of $\mathcal{F}$ is a solution of (3.2). Let $H : [0,1]\times C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ be defined by $H(\lambda, x) = \lambda\mathcal{F}(x)$. Proposition 3.5 insures that H is an u.s.c., compact set-valued mapping with nonempty, compact, convex values. Hence, there exists a bounded open set $\Omega \subset C(I,\mathbb{R}^n)$ such that $H\big([0,1]\times C(I,\mathbb{R}^n)\big) \subset \Omega$. The topological degree theory implies that

$$1 = \deg(\mathrm{id},\Omega) = \deg(\mathrm{id} - H(0,\cdot),\Omega) = \deg(\mathrm{id} - H(1,\cdot),\Omega) = \deg(\mathrm{id} - \mathcal{F},\Omega).$$

Thus, (3.2) has a solution. □

Here is the notion of solution-tube of (1.1). It generalizes the notion of upper and lower solutions in the case where F has real values. It will play a crucial role in what follows.

Definition 3.7. Let $(v, \rho) \in W^{1,1}(I,\mathbb{R}^n) \times W^{1,1}(I, [0,\infty))$. We say that (v, ρ) is a *solution-tube of* (1.1) if the following conditions hold:

(i) for a.e. $t \in I$ and for any $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = \rho(t)$, there exists $y \in F(t, x)$ such that
$$\langle x - v(t), y - v'(t)\rangle \le \rho(t)\rho'(t);$$
(ii) $v'(t) \in F(t, v(t))$ for a.e. on $\{t \in I : \rho(t) = 0\}$;
(iii) $\|v(0) - v(1)\| \le \rho(0) - \rho(1)$.

We denote
$$T(v, \rho) = \{x \in C(I,\mathbb{R}^n) : \|x(t) - v(t)\| \le \rho(t) \text{ for all } t \in I\}.$$

Notice that when the problem has only one equation (i.e. $n = 1$), then for $\alpha \le \beta$ respectively lower and upper solutions of (1.1), one has $(\frac{\beta+\alpha}{2}, \frac{\beta-\alpha}{2})$ is a solution-tube of (1.1). The reader is referred to [1] for the definition of upper and lower solutions.

Remark 3.8. In the case where the nonlinearity is single-valued, we consider the following system of first order differential equations:

$$x'(t) = f(t, x(t)) \quad \text{a.e. } t \in [0,1], \qquad x(0) = x(1). \tag{3.3}$$

We say that $(v,\rho)\in W^{1,1}(I,\mathbb{R}^n)\times W^{1,1}(I,[0,\infty))$ is a *solution-tube of* (3.3) if it is a solution-tube of (1.1) with $F=\{f\}$.

The existence of a solution-tube insures the existence of a solution to (1.1).

Theorem 3.9. *Let $F: I\times\mathbb{R}^n\to\mathbb{R}^n$ be a Carathéodory set-valued mapping with nonempty, closed, convex values. Assume there exists $(v,\rho)\in W^{1,1}(I,\mathbb{R}^n)\times W^{1,1}(I,[0,\infty))$ a solution-tube of* (1.1). *Then, problem* (1.1) *has a solution $x\in W^{1,1}(I,\mathbb{R}^n)$ such that $\|x(t)-v(t)\|\le\rho(t)$ for every $t\in I$.*

Proof. We denote the projection of x on the closed ball centered in $v(t)$ of radius $\rho(t)$ by

$$\bar{x}_{(t,v,\rho)}=\begin{cases}x & \text{if } \|x-v(t)\|\le\rho(t),\\ v(t)+\frac{\rho(t)}{\|x-v(t)\|}\big(x-v(t)\big) & \text{if } \|x-v(t)\|>\rho(t).\end{cases}\tag{3.4}$$

Let us define $G_{(v,\rho)}: I\times\mathbb{R}^n\to\mathbb{R}^n$ and $F_{(v,\rho)}: I\times\mathbb{R}^n\to\mathbb{R}^n$ respectively by

$$G_{(v,\rho)}(t,x)=\begin{cases}\{z\in\mathbb{R}^n:\langle x-v(t),z-v'(t)\rangle\\ \qquad\le\rho'(t)\|x-v(t)\|\} & \text{if } \|x-v(t)\|>\rho(t)>0,\\ v'(t) & \text{if } \rho(t)=0,\\ \mathbb{R}^n & \text{otherwise.}\end{cases}\tag{3.5}$$

and

$$F_{(v,\rho)}(t,x)=\bar{x}_{(t,v,\rho)}+F\big(t,\bar{x}_{(t,v,\rho)}\big)\cap G_{(v,\rho)}(t,x).\tag{3.6}$$

It follows from the definition of solution-tube that $F_{(v,\rho)}$ has nonempty values. It is easy to verify that $F_{(v,\rho)}$ is Carathéodory and has convex, compact values. Also, the fact that $\big\|\bar{x}_{(t,v,\rho)}\big\|\le\|v\|_0+\|\rho\|_0$ and the condition (iii) of Definition 3.1 imply that $F_{(v,\rho)}$ is integrably bounded. It follows from Proposition 3.6 that the problem

$$\begin{aligned}&x'(t)+x(t)\in F_{(v,\rho)}(t,x(t)) \quad \text{a.e. } t\in[0,1],\\ &x(0)=x(1);\end{aligned}$$

has a solution $\hat{x}\in W^{1,1}(I,\mathbb{R}^n)$. Observe that for almost every $t\in I$ and every $x\in\mathbb{R}^n$ such that $\|x-v(t)\|>\rho(t)$, one has for every $u\in F_{(v,\rho)}(t,x)$, there exists $z\in G_{(v,\rho)}(t,x)$ such that $u=\bar{x}_{(t,v,\rho)}+z$, and hence,

$$\begin{aligned}\langle x-v(t),u-x-v'(t)\rangle&=\langle x-v(t),\bar{x}_{(t,v,\rho)}-x\rangle+\langle x-v(t),z-v'(t)\rangle\\ &\le\big(\rho(t)-\|x-v(t)\|\big)\|x-v(t)\|\\ &\qquad+\begin{cases}\rho'(t)\|x-v(t)\| & \text{if } \rho(t)>0,\\ 0 & \text{if } \rho(t)=0,\end{cases}\\ &<\rho'(t)\|x-v(t)\|,\end{aligned}\tag{3.7}$$

since $\rho'(t)=0$ almost everywhere on $\{t\in I:\rho(t)=0\}$.

It follows from Lemma 2.1 that the solution $\hat{x}$ verifies $\|\hat{x}(t)-v(t)\|\le\rho(t)$ for all $t\in I$. Therefore,

$$\overline{\hat{x}(t)}_{(t,v,\rho)}=\hat{x}(t)\quad\text{and}\quad F_{(v,\rho)}(t,\hat{x}(t))\subset\hat{x}(t)+F(t,\hat{x}(t))\quad\text{a.e. } t\in I.$$

So, $\hat{x}$ is a solution of (1.1). □

3.2. **Multiplicity results.** In order to establish our multiplicity results, we introduce the notion of strict solution-tube of (1.1).

Definition 3.10. Let $(v,\rho) \in W^{1,1}(I,\mathbb{R}^n) \times W^{1,1}(I,(0,\infty))$. We say that (v,ρ) is a *strict solution-tube of* (1.1) if the following conditions hold:

(i) there exists a l.s.c. mapping $\epsilon : I \to (0,\infty)$ such that for a.e. $t \in I$ and all $x \in \mathbb{R}^n$ such that $\rho(t) - \epsilon(t) < \|x - v(t)\| \le \rho(t)$, there exists $y \in F(t,x)$ such that
$$\langle x - v(t), y - v'(t)\rangle \le \rho(t)\rho'(t);$$
(ii) $\|v(0) - v(1)\| < \rho(0) - \rho(1)$.

Obviously, a strict solution-tube is a solution-tube of (1.1).

Remark 3.11. In the case where the nonlinearity is single-valued, we say that $(v,\rho) \in W^{1,1}(I,\mathbb{R}^n) \times W^{1,1}(I,(0,\infty))$ is a *strict solution-tube of* (3.3) if it is a strict solution-tube of (1.1) with $F = \{f\}$.

Definition 3.12. Let (v_1,ρ_1) and (v_2,ρ_2) be two strict solution-tubes of (1.1). They are said *compatible* if the l.s.c. functions ϵ_1 and ϵ_2 in Definition 3.10(i) can be chosen such that for a.e. $t \in I$ and all $x \in \mathbb{R}^n$ such that
$$\rho_i(t) - \epsilon_i(t) < \|x - v_i(t)\| < \rho_i(t) \quad \text{for } i = 1,2,$$
there exists $y \in F(t,x)$ such that
$$\langle x - v_i(t), y - v_i'(t)\rangle \le \rho_i(t)\rho_i'(t) \quad \text{for } i = 1,2.$$

Remark 3.13. Any two strict solution-tubes of (3.3) are compatible.

Here is our main theorem for Carathéodory set-valued mappings.

Theorem 3.14. *Let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory set-valued mapping with nonempty, closed, convex values. Assume the following conditions hold:*

(i) *there exists (v_0,ρ_0) a solution-tube of* (1.1)*;*
(ii) *there exist (v_1,ρ_1) and (v_2,ρ_2) two compatible strict solution-tubes of* (1.1) *such that*
 (a) *$T(v_i,\rho_i) \subset T(v_0,\rho_0)$ for $i = 1,2$;*
 (b) *$T(v_1,\rho_1) \cap T(v_2,\rho_2) = \emptyset$.*

Then problem (1.1) *has at least three distinct solutions $x_0, x_1, x_2 \in W^{1,1}(I,\mathbb{R}^n)$ such that $x_j \in T(v_j,\rho_j)$ and $x_0 \notin T(v_i,\rho_i)$ for $i = 1,2$ and $j = 0,1,2$.*

Proof. For $i = 1,2$ define $K_i : I \times \mathbb{R}^n \to \mathbb{R}^n$ by
$$K_i(t,x) = \begin{cases} \{z \in \mathbb{R}^n : \langle x - v_i(t), z - v_i'(t)\rangle \\ \qquad \le \rho_i(t)\rho_i'(t)\} & \text{if } \rho_i(t) - \epsilon_i(t) < \|x - v_i(t)\| < \rho_i(t), \\ \mathbb{R}^n & \text{otherwise.} \end{cases}$$
The set-valued mapping K_i is measurable in t, u.s.c. in x, and has nonempty closed, convex values.

For $j = 0, 1, 2$, Condition (ii)(a) implies that $\rho_j(t) > 0$ for all $t \in I$. Let $\bar{x}_{(t,v_j,\rho_j)}$ and $G_{(v_j,\rho_j)}(t,x)$ be defined in (3.4) and (3.5) respectively. We define $F_j : I \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned}
F_0(t,x) &= \bar{x}_{(t,v_0,\rho_0)} + F\big(t, \bar{x}_{(t,v_0,\rho_0)}\big) \cap G_{(v_0,\rho_0)}(t,x) \cap K_1(t,x) \cap K_2(t,x),\\
F_1(t,x) &= \bar{x}_{(t,v_1,\rho_1)} + F\big(t, \bar{x}_{(t,v_1,\rho_1)}\big) \cap G_{(v_1,\rho_1)}(t,x) \cap K_1(t,x) \cap K_2(t, \bar{x}_{(t,v_1,\rho_1)}),\\
F_2(t,x) &= \bar{x}_{(t,v_2,\rho_2)} + F\big(t, \bar{x}_{(t,v_2,\rho_2)}\big) \cap G_{(v_2,\rho_2)}(t,x) \cap K_1(t, \bar{x}_{(t,v_2,\rho_2)}) \cap K_2(t,x).
\end{aligned}$$

In view of assumption (ii)(a), F_j is integrably bounded by the function $h(t) = r + h_r(t)$ for $r = \|v_0\|_0 + \|\rho_0\|_0$ and h_r the function given in Definition 3.1(iii).

Assumptions (i) and (ii) imply that for almost every $t \in I$,

$$\begin{aligned}
&\emptyset \neq F\big(t, \bar{x}_{(t,v_0,\rho_0)}\big) \cap G_{(v_0,\rho_0)}(t,x) \subset K_1(t,x) \cap K_2(t,x) \quad \text{if } \|x - v_0(t)\| > \rho_0(t),\\
&\emptyset \neq F\big(t, \bar{x}_{(t,v_0,\rho_0)}\big) \cap K_1(t,x) \cap K_2(t,x) \subset G_{(v_0,\rho_0)}(t,x) \quad \text{if } \|x - v_0(t)\| \leq \rho_0(t).
\end{aligned}$$

Also,

$$\begin{aligned}
&\emptyset \neq F\big(t, \bar{x}_{(t,v_1,\rho_1)}\big) \cap K_1(t,x) \cap K_2(t, \bar{x}_{(t,v_1,\rho_1)}) \subset G_{(v_1,\rho_1)}(t,x)\\
&\qquad\qquad \text{if } \|x - v_1(t)\| \leq \rho_1(t),\\
&\emptyset \neq F\big(t, \bar{x}_{(t,v_1,\rho_1)}\big) \cap G_{(v_1,\rho_1)}(t,x) \subset K_1(t,x) \cap K_2(t, \bar{x}_{(t,v_1,\rho_1)})\\
&\qquad\qquad \text{if } \|x - v_1(t)\| > \rho_1(t) \text{ and } \|\bar{x}_{(t,v_1,\rho_1)} - v_2(t)\| \notin \big(\rho_2(t) - \epsilon_2(t), \rho_2(t)\big).
\end{aligned}$$

For almost every $t \in I$ and for $x \in \mathbb{R}^n$ such that

$$\|x - v_1(t)\| > \rho_1(t) \quad \text{and} \quad \|\bar{x}_{(t,v_1,\rho_1)} - v_2(t)\| \in \big(\rho_2(t) - \epsilon_2(t), \rho_2(t)\big),$$

there exists a sequence $\{x_n\}$ such that

$$x_n \to \bar{x}_{(t,v_1,\rho_1)}, \quad \text{and} \quad \|x_n - v_k(t)\| \in \big(\rho_k(t) - \epsilon_k(t), \rho_k(t)\big) \quad \text{for } k = 1, 2.$$

Since (v_1, ρ_1) and (v_2, ρ_2) are compatible strict solution-tubes, there exists $z_n \in F(t, x_n)$ such that

$$\langle x_n - v_k(t), z_n - v_k'(t)\rangle \leq \rho_k(t)\rho_k'(t) \quad \text{for } k = 1, 2.$$

The sequence $\{z_n\}$ being bounded, it has a subsequence converging to some $z \in \mathbb{R}^n$. Since $y \mapsto F(t,y)$ is u.s.c., we deduce that

$$z \in F\big(t, \bar{x}_{(t,v_1,\rho_1)}\big),$$

and

$$\langle \bar{x}_{(t,v_1,\rho_1)} - v_k(t), z - v_k'(t)\rangle \leq \rho_k(t)\rho_k'(t) \quad \text{for } k = 1, 2.$$

In particular, by definition of $\bar{x}_{(t,v_1,\rho_1)}$, one has

$$\langle x - v_1(t), z - v_1'(t)\rangle \leq \|x - v_1(t)\|\rho_1'(t).$$

So,

$$z \in F\big(t, \bar{x}_{(t,v_1,\rho_1)}\big) \cap G_{(v_1,\rho_1)}(t,x) \cap K_2(t, \bar{x}_{(t,v_1,\rho_1)}) \subset K_1(t,x).$$

We get similar relations by interchanging the role of 1 and 2. Therefore,

$$F_j(t,x) \text{ has nonempty values for a.e. } t \in I, \text{ for all } x \in \mathbb{R}^n \text{ and for } j = 0, 1, 2.$$

It is easy to verify that F_j is a Carathéodory set-valued mapping with closed, convex values.

Let $\mathcal{F}_j : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ be the operator associated to F_j defined in (3.1). It follows from Proposition 3.6 that there exists an open bounded set $\Omega \subset C(I, \mathbb{R}^n)$ such that

$$\deg(\mathrm{id} - \mathcal{F}_j, \Omega) = 1 \quad \text{for } j = 0, 1, 2. \tag{3.8}$$

Thus, there exists $x_j \in W^{1,1}(I, \mathbb{R}^n)$ a solution to

$$\begin{aligned} &x'(t) + x(t) \in F_j(t, x(t)) \quad \text{a.e. } t \in I, \\ &x(0) = x(1). \end{aligned} \tag{3.9$_j$}$$

We define the open set

$$U_j = \{x \in C(I, \mathbb{R}^n) : \|x(t) - v_j(t)\| < \rho_j(t) \text{ for all } t \in I\}.$$

Observe that $F_j(t, x) \subset F_{(v_j, \rho_j)}(t, x)$ the function defined in (3.6). So, by (3.7),

$$\begin{aligned} -x + F_j(t, x) \subset \{y \in \mathbb{R}^n : \langle x - v_j(t), y - v_j'(t)\rangle < \|x - v_j(t)\|\rho_j'(t)\} \\ \text{a.e. } t \in I \text{ and for all } x \in \mathbb{R}^n \text{ such that } \|x - v_j(t)\| > \rho_j(t). \end{aligned}$$

Lemma 2.1 implies that all solutions of (3.9$_j$) are in $\overline{U}_j$. So

$$\{x \in \Omega : 0 \in x - \mathcal{F}_j(x)\} \subset \overline{U}_j \quad \text{for } j = 0, 1, 2. \tag{3.10}$$

Observe that for every $x \in \overline{U}_j$,

$$x(t) = \overline{x(t)}_{(t, v_j, \rho_j)} \quad \text{for all } t \in I.$$

Hence, for $j = 0, 1, 2$,

$$x(t) + F(t, x(t)) \supset F_j(t, x(t)) \quad \text{a.e. } t \in I \quad \text{for all } x \in \overline{U}_j. \tag{3.11}$$

Now, we show that

$$S_i = \{x \in \Omega : 0 \in x - \mathcal{F}_i(x)\} \subset U_i \quad \text{for } i = 1, 2. \tag{3.12}$$

Without loss of generality, assume this is false for $i = 1$. Then, using (3.10), there exists $x \in S_1$ such that

$$B = \{t \in I : \|x(t) - v_1(t)\| = \rho_1(t)\} \neq \emptyset.$$

Observe that $0 \notin B$. Indeed, the periodic boundary condition, and Definition 3.10 imply that

$$\|x(0) - v_1(0)\| \leq \|x(1) - v_1(1)\| + \|v_1(1) - v_1(0)\| < \rho_1(0).$$

Let $t_1 = \inf B > 0$. Since $\rho_1(t_1) - \epsilon_1(t_1) < \|x(t_1) - \nu_1(t_1)\| = \rho_1(t_1)$, the lower semi-continuity of ϵ_1 and the continuity of x, v_1 and ρ_1 imply that there exists $0 \leq t_0 < t_1$ such that

$$\rho_1(t) - \epsilon_1(t) < \|x(t) - \nu_1(t)\| < \rho_1(t) \quad \text{for all } t \in [t_0, t_1).$$

By definition of the mappings K_1, F_1 and $\bar{x}_{(t, v_1, \rho_1)}$,

$$x'(t) + x(t) \in F_1(t, x(t)) \subset x(t) + K_1(t, x(t)) \quad \text{a.e. } t \in [t_0, t_1).$$

Thus,

$$\langle x(t) - v_1(t), x'(t) - v_1'(t)\rangle \leq \rho_1(t)\rho_1'(t) \quad \text{a.e. } t \in [t_0, t_1).$$

Consequently,

$$\begin{aligned}0 &< \rho_1^2(t_0) - \|x(t_0) - v_1(t_0)\|^2 \\ &= \int_{t_0}^{t_1} \frac{d}{dt}\Big(\|x(t) - v_1(t)\|^2 - \rho_1^2(t)\Big)\,dt \\ &= 2\int_{t_0}^{t_1} \langle x(t) - v_1(t), x'(t) - v_1'(t)\rangle - \rho_1(t)\rho_1'(t)\,dt \\ &\le 0;\end{aligned}$$

which is a contradiction.

For $i = 1, 2$, by (ii)(a), for every $x \in \overline{U}_i$,

$$x(t) = \overline{x(t)}_{(t,v_i,\rho_i)} = \overline{x(t)}_{(t,v_0,\rho_0)} \quad \text{for all } t \in I,$$

and

$$\mathbb{R}^n = G_{(v_i,\rho_i)}(t, x(t)) = G_{(v_0,\rho_0)}(t, x(t)) \quad \text{for all } t \in I.$$

So,

$$\mathcal{F}_0(x) = \mathcal{F}_i(x) \quad \text{for all } x \in \overline{U}_i. \tag{3.13}$$

The topological degree theory, combined with (3.8), (3.12), and (3.13) imply that

$$\begin{aligned}&\deg\Big(\mathrm{id} - \mathcal{F}_0, \Omega\backslash\big(\overline{U_1 \cup U_2}\big)\Big) \\ &\quad = \deg\big(\mathrm{id} - \mathcal{F}_0, \Omega\big) - \Big(\deg\big(\mathrm{id} - \mathcal{F}_0, U_1\big) + \deg\big(\mathrm{id} - \mathcal{F}_0, U_2\big)\Big) \\ &\quad = \deg\big(\mathrm{id} - \mathcal{F}_0, \Omega\big) - \Big(\deg\big(\mathrm{id} - \mathcal{F}_1, U_1\big) + \deg\big(\mathrm{id} - \mathcal{F}_2, U_2\big)\Big) \\ &\quad = \deg\big(\mathrm{id} - \mathcal{F}_0, \Omega\big) - \Big(\deg\big(\mathrm{id} - \mathcal{F}_1, \Omega\big) + \deg\big(\mathrm{id} - \mathcal{F}_2, \Omega\big)\Big) \\ &\quad = 1 - (1 + 1).\end{aligned}$$

So, for $j = 0, 1, 2$, problem (3.9_j) has a solution $x_j \in T(v_j, \rho_j)$. Moreover, $x_0 \notin T(v_1, \rho_1) \cup T(v_2, \rho_2)$. Assumption (ii)(b) implies that x_1 and x_2 are distinct. Finally, using (3.11), we conclude that x_0, x_1 and x_2 are three distinct solutions of (1.1). □

In the particular case where the nonlinearity is single-valued, we consider the system of first order differential equations (3.3) and we obtain the following corollary.

Corollary 3.15. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory single-valued function. Assume the following assumptions are satisfied:*

- (i) *there exists (v_0, ρ_0) a solution-tube of* (3.3)*;*
- (ii) *there exist (v_1, ρ_1) and (v_2, ρ_2) two strict solution-tubes of* (3.3) *such that*
  - (a) *$T(v_i, \rho_i) \subset T(v_0, \rho_0)$ for $i = 1, 2$;*
  - (b) *$T(v_1, \rho_1) \cap T(v_2, \rho_2) = \emptyset$.*

Then problem (3.3) *has at least three distinct solutions $x_0, x_1, x_2 \in W^{1,1}(I, \mathbb{R}^n)$ such that $x_j \in T(v_j, \rho_j)$ and $x_0 \notin T(v_i, \rho_i)$ for $i = 1, 2$ and $j = 0, 1, 2$.*

We obtain the following corollary for real valued nonlinearity.

Corollary 3.16. *For $i = 1, 2$, let $\alpha_i, \beta_i \in W^{1,1}(I, \mathbb{R})$, and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory set-valued mapping with nonempty, closed, convex values. Assume the following conditions hold:*

(i) $\alpha_1(t) < \beta_1(t) \leq \beta_2(t)$ *and* $\alpha_1(t) \leq \alpha_2(t) < \beta_2(t)$ *for all* $t \in I$*;*
(ii) *there exists* $t \in I$ *such that* $\beta_1(t) < \alpha_2(t)$*;*
(iii) $\alpha_i(0) < \alpha_i(1)$ *and* $\beta_i(0) > \beta_i(1)$ *for* $i = 1, 2$*;*
(iv) *for* $i = 1, 2$*, there exists a l.s.c. mapping* $\epsilon_i : I \to (0, \infty)$ *such that,*
- $F(t, x) \cap [\alpha_i'(t), \infty) \neq \emptyset$ *for a.e.* $t \in I$ *and all* $x \in \mathbb{R}$ *such that* $\alpha_i(t) \leq x < \alpha_i(t) + \epsilon_i(t)$*;*
- $F(t, x) \cap (-\infty, \beta_i'(t)] \neq \emptyset$ *for a.e.* $t \in I$ *and all* $x \in \mathbb{R}$ *such that* $\beta_i(t) \geq x > \beta_i(t) - \epsilon_i(t)$*;*

(v)
- *there exits* $y \in F(t, x)$ *such that* $y \geq \max\{\alpha_1'(t), \alpha_2'(t)\}$ *for a.e.* $t \in I$ *and all* $x \in \big(\alpha_1(t), \alpha_1(t) + \epsilon_1(t)\big) \cap \big(\alpha_2(t), \alpha_2(t) + \epsilon_2(t)\big)$*;*
- *there exits* $y \in F(t, x)$ *such that* $y \leq \min\{\beta_1'(t), \beta_2'(t)\}$ *for a.e.* $t \in I$ *and all* $x \in \big(\beta_1(t) - \epsilon_1(t), \beta_1(t)\big) \cap \big(\beta_2(t) - \epsilon_2(t), \beta_2(t)\big)$*;*
- *for* $i, j \in \{1, 2\}$ *with* $i \neq j$*, there exits* $y \in F(t, x)$ *such that* $\alpha_j'(t) \leq y \leq \beta_i'(t)$ *for a.e.* $t \in I$ *and all* $x \in \big(\beta_i(t) - \epsilon_i(t), \beta_i(t)\big) \cap \big(\alpha_j(t), \alpha_j(t) + \epsilon_j(t)\big)$.

Then problem (1.1) *has at least three distinct solutions* $x_0, x_1, x_2 \in W^{1,1}(I, \mathbb{R})$ *such that* $x_j \in T(v_j, \rho_j)$ *and* $x_0 \notin T(v_i, \rho_i)$ *for* $i = 1, 2$ *and* $j = 0, 1, 2$.

Proof. Let

$$v_0 = \frac{\alpha_1 + \beta_2}{2} \quad \text{and} \quad \rho_0 = \frac{\beta_2 - \alpha_1}{2},$$

and for $i = 1, 2$,

$$v_i = \frac{\alpha_i + \beta_i}{2} \quad \text{and} \quad \rho_i = \frac{\beta_i - \alpha_i}{2},$$

Assumptions (i), (iii) and (iv) imply that (v_0, ρ_0) is a solution-tube of (1.1) and (v_i, ρ_i) are strict solution-tubes of (1.1) for $i = 1, 2$. It follows from (v) that (v_1, ρ_1) and (v_2, ρ_2) are compatible strict solution-tubes. Theorem 3.14 gives the conclusion. □

Remark 3.17. In the particular case where F is single-valued, condition (v) of the previous corollary can be omitted. Indeed, it follows directly from (iv).

Assuming the existence of more strict solution-tubes leads to the existence of more solutions of (1.1). The proof is left to the reader.

Theorem 3.18. *Let* $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be a Carathéodory set-valued mapping with nonempty, closed, convex values and let* $m \geq 2$*. Assume the following conditions hold:*

(i) *there exists* (v_0, ρ_0) *a solution-tube of* (1.1)*;*
(ii) *there exist* $(v_1, \rho_1), \ldots, (v_m, \rho_m)$ *strict solution-tubes of* (1.1) *such that*
  (a) $T(v_i, \rho_i) \subset T(v_0, \rho_0)$ *for* $i = 1, \ldots, m$*;*
  (b) $T(v_i, \rho_i) \cap T(v_j, \rho_j) = \emptyset$ *for all* $i, j \in \{1, \ldots, m\}$ *such that* $i \neq j$*;*
  (c) *the l.s.c. functions* $\epsilon_1, \ldots, \epsilon_m$ *in Definition 3.10(i) can be chosen such that for a.e.* $t \in I$ *and all* $x \in \mathbb{R}^n$ *such that card*$\big(I(t, x)\big) \geq 2$*, there exists* $y \in F(t, x)$ *such that*

$$\langle x - v_i(t), y - v_i'(t)\rangle \leq \rho_i(t)\rho_i'(t) \quad \textit{for all } i \in I(t, x),$$

where

$$I(t,x) = \big\{i \in \{1,\dots,m\} : \rho_i(t) - \epsilon_i(t) < \|x - v_i(t)\| < \rho_i(t)\big\}.$$

Then problem (1.1) *has at least* $m+1$ *distinct solutions* $x_0,\dots,x_m \in W^{1,1}(I,\mathbb{R}^n)$ *such that* $x_j \in T(v_j,\rho_j)$ *and* $x_0 \notin T(v_i,\rho_i)$ *for* $i = 1,\dots,m$ *and* $j = 0,\dots,m$.

4. Lower semi-continuity condition

4.1. **Existence result.** In this section, we consider a mapping F lower semi-continuous with respect to the second variable. In this case, F may have non-convex values.

Definition 4.1. A set-valued mapping $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ with nonempty closed values is *lower semi-continuous type* ((lsc)-type) if the following conditions hold:

(i) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$-measurable (here $I \times \mathbb{R}^n$ is endowed with the σ-algebra generated by subsets $C \times D$ where $C \subset I$ and $D \subset \mathbb{R}^n$ are respectively Lebesgue and Borel measurable);

(ii) $x \mapsto F(t,x)$ is lower semi-continuous for a.e. $t \in I$;

(iii) for every $r > 0$, there exists $h_r \in L^1(I,\mathbb{R})$ such that for almost every $t \in I$ and every $x \in \mathbb{R}^n$ satisfying $\|x\| \le r$, one has $\|y\| \le h_r(t)$ for all $y \in F(t,x)$.

Let us recall that to a set-valued mapping $F : I \times \mathbb{R}^n \to \mathbb{R}^n$, we associated $\mathbf{F} : C(I,\mathbb{R}^n) \to L^1(I,\mathbb{R}^n)$ defined by

$$\mathbf{F}(x) = \{y \in L^1(I,\mathbb{R}^n) : y(t) \in F(t,x(t)) \text{ a.e. } t \in I\}.$$

Arguing as in the proof of Proposition 4.3 in [13], we obtain the following result.

Proposition 4.2. *Let* $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be an integrably bounded, (lsc)-type set-valued mapping with nonempty, closed, values. If* $L^1(I,\mathbb{R}^n)$ *is endowed with the usual norm topology, then there exists a continuous single-valued selection*

$$\mathbf{f} : C(I,\mathbb{R}^n) \to L^1(I,\mathbb{R}^n) \quad \textit{such that} \quad \mathbf{f}(x) \in \mathbf{F}(x) \quad \textit{for all } x \in C(I,\mathbb{R}^n).$$

It follows from the previous proposition that to a (lsc)-type set-valued mapping $F : I \times \mathbb{R}^n \to \mathbb{R}^n$, we can associate the single-valued operator

$$f : C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n) \quad \text{defined by} \quad f = i \circ (L + \mathrm{id})^{-1} \circ \mathbf{f}, \tag{4.1}$$

where L is defined in (2.1) and $i : W^{1,1}(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ is the continuous embedding. The associated operator f is continuous and compact.

It is also well known that if F is (lsc)-type and integrably bounded, then (3.2) has a solution.

Proposition 4.3. *Let* $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be an integrably bounded, (lsc)-type set-valued mapping with nonempty, closed values. Then* (3.2) *has a solution* $x \in W^{1,1}(I,\mathbb{R}^n)$. *Moreover, there exists a bounded open set* $\Omega \subset C(I,\mathbb{R}^n)$ *such that the topological degree* $deg(id - f, \Omega) = 1$.

Proof. A fixed point of f is a solution of (3.2). Let $h : [0,1] \times C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ be defined by $h(\lambda,x) = \lambda f(x)$. There exists a bounded open set $\Omega \subset C(I,\mathbb{R}^n)$ such that $h\big([0,1] \times C(I,\mathbb{R}^n)\big) \subset \Omega$. The topological degree theory implies that

$$1 = \deg(\mathrm{id},\Omega) = \deg(\mathrm{id} - h(0,\cdot),\Omega) = \deg(\mathrm{id} - h(1,\cdot),\Omega) = \deg(\mathrm{id} - f,\Omega).$$

Thus, (3.2) has a solution. □

Also in this context, the existence of a solution-tube insures the existence of a solution to (1.1).

Theorem 4.4. *Let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a (lsc)-type set-valued mapping with nonempty, closed values. Assume there exists $(v, \rho) \in W^{1,1}(I, \mathbb{R}^n) \times W^{1,1}(I, [0, \infty))$ a solution-tube of (1.1). Then, problem (1.1) has a solution $x \in W^{1,1}(I, \mathbb{R}^n)$ such that $\|x(t) - v(t)\| \leq \rho(t)$ for every $t \in I$.*

Proof. Let us define $G^l_{(v,\rho)} : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $F^l_{(v,\rho)} : I \times \mathbb{R}^n \to \mathbb{R}^n$ respectively by

$$G^l_{(v,\rho)}(t,x) = \begin{cases} \{z \in \mathbb{R}^n : \langle x - v(t), z - v'(t)\rangle \\ \qquad \leq \rho'(t)\|x - v(t)\|\} & \text{if } \|x - v(t)\| \geq \rho(t) > 0, \\ v'(t) & \text{if } \rho(t) = 0, \\ \mathbb{R}^n & \text{otherwise,} \end{cases} \tag{4.2}$$

and

$$F^l_{(v,\rho)}(t,x) = \bar{x}_{(t,v,\rho)} + F\big(t, \bar{x}_{(t,v,\rho)}\big) \cap G^l_{(v,\rho)}(t,x), \tag{4.3}$$

where $\bar{x}_{(t,v,\rho)}$ is defined in (3.4).

It follows from the definition of solution-tube that $F^l_{(v,\rho)}$ has nonempty values. It is easy to verify that $F^l_{(v,\rho)}$ is an integrably bounded, (lsc)-type set-valued mapping with compact values. Proposition 4.3 implies that the problem

$$\begin{aligned} &x'(t) + x(t) \in F^l_{(v,\rho)}(t, x(t)) \quad \text{a.e. } t \in [0,1], \\ &x(0) = x(1); \end{aligned}$$

has a solution $\hat{x} \in W^{1,1}(I, \mathbb{R}^n)$. Observe that for almost every $t \in I$ and every $x \in \mathbb{R}^n$ such that $\|x - v(t)\| > \rho(t)$, one has

$$-x + F_{(v,\rho)}(t,x) \subset \{y \in \mathbb{R}^n : \langle x - v(t), y - v'(t)\rangle < \rho'(t)\|x - v(t)\|\}. \tag{4.4}$$

It follows from Lemma 2.1 that the solution $\hat{x}$ verifies $\|\hat{x}(t) - v(t)\| \leq \rho(t)$ for all $t \in I$. Therefore, $\hat{x}$ is a solution of (1.1). □

4.2. **Multiplicity results.** In the case where the set-valued mapping is lower semi-continuous with respect to the second variable a stronger compatibility condition is needed in order to establish multiplicity results.

Definition 4.5. Let (v_1, ρ_1) and (v_2, ρ_2) be two strict solution-tubes of (1.1). They are *strongly compatible* if the l.s.c. functions ϵ_1 and ϵ_2 in Definition 3.10(i) can be chosen such that for a.e. $t \in I$ and all $x \in \mathbb{R}^n$ such that

$$\rho_i(t) - \epsilon_i(t) < \|x - v_i(t)\| \leq \rho_i(t) \quad \text{for } i = 1, 2,$$

there exists $y \in F(t,x)$ such that

$$\langle x - v_i(t), y - v_i'(t)\rangle \leq \rho_i(t)\rho_i'(t) \quad \text{for } i = 1, 2.$$

Remark 4.6. Obviously, strongly compatible strict solution-tubes are compatible.

Here is our main theorem for (lsc)-type set-valued mappings. In this case, an extra condition is needed.

Theorem 4.7. *Let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a (lsc)-type set-valued mapping with nonempty, closed values. Assume the following conditions hold:*

(i) *there exists (v_0, ρ_0) a solution-tube of (1.1);*

(ii) *there exist (v_1, ρ_1) and (v_2, ρ_2) two strongly compatible strict solution-tubes of (1.1) such that*

  (a) *$T(v_i, \rho_i) \subset T(v_0, \rho_0)$ for $i = 1, 2$;*

  (b) *$T(v_1, \rho_1) \cap T(v_2, \rho_2) = \emptyset$.*

(iii) *for almost every $t \in I$ and all $x \in \mathbb{R}^n$ such that $\|x - v_0(t)\| = \rho_0(t)$ and $card\big(J(t, x)\big) \geq 2$, where*

$$J(t, x) = \{j \in \{0, 1, 2\} : \|x - v_j(t)\| = \rho_j(t)\},$$

there exists $y \in F(t, x)$ such that

$$\langle x - v_j(t), y - v_j'(t)\rangle \leq \rho_j(t)\rho_j'(t) \quad \text{for each } j \in J(t, x).$$

Then problem (1.1) has at least three distinct solutions $x_0, x_1, x_2 \in W^{1,1}(I, \mathbb{R}^n)$ such that $x_j \in T(v_j, \rho_j)$ and $x_0 \notin T(v_i, \rho_i)$ for $i = 1, 2$ and $j = 0, 1, 2$.

Proof. For $i = 1, 2$ let $\epsilon_i^l : I \to (0, \infty)$ be a l.s.c. single-valued mapping such that $\epsilon_i^l(t) < \epsilon_i(t)$ for every $t \in I$. We define $K_i^l : I \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$K_i^l(t, x) = \begin{cases} \{z \in \mathbb{R}^n : \langle x - v_i(t), z - v_i'(t)\rangle \\ \qquad \leq \rho_i(t)\rho_i'(t)\} & \text{if } \rho_i(t) - \epsilon_i^l(t) \leq \|x - v_i(t)\| \leq \rho_i(t), \\ \mathbb{R}^n & \text{otherwise.} \end{cases}$$

The set-valued mapping K_i^l is $\mathcal{L} \otimes \mathcal{B}$-measurable in (t, x), l.s.c. in x, and has nonempty closed values.

For $j = 0, 1, 2$, Condition (ii)(a) implies that $\rho_j(t) > 0$ for all $t \in I$. Let $\bar{x}_{(t, v_j, \rho_j)}$ and $G^l_{(v_j, \rho_j)}(t, x)$ be defined in (3.4) and (4.2) respectively. We define $F_j^l : I \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned} F_0^l(t, x) &= \bar{x}_{(t, v_0, \rho_0)} + F\big(t, \bar{x}_{(t, v_0, \rho_0)}\big) \cap G^l_{(v_0, \rho_0)}(t, x) \cap K_1^l(t, x) \cap K_2^l(t, x), \\ F_1^l(t, x) &= \bar{x}_{(t, v_1, \rho_1)} + F\big(t, \bar{x}_{(t, v_1, \rho_1)}\big) \cap G^l_{(v_1, \rho_1)}(t, x) \cap K_1^l(t, x) \cap K_2^l(t, \bar{x}_{(t, v_1, \rho_1)}), \\ F_2^l(t, x) &= \bar{x}_{(t, v_2, \rho_2)} + F\big(t, \bar{x}_{(t, v_2, \rho_2)}\big) \cap G^l_{(v_2, \rho_2)}(t, x) \cap K_1^l(t, \bar{x}_{(t, v_2, \rho_2)}) \cap K_2^l(t, x). \end{aligned}$$

Assumption (ii)(a) insures that F_j^l is integrably bounded by the function $h(t) = r + h_r(t)$ for $r = \|v_0\|_0 + \|\rho_0\|_0$ and h_r the function given in Definition 4.1(iii).

For $j = 0, 1, 2$, assumptions (i), (ii) and (iii) imply that $F_j^l(t, x)$ has nonempty values for almost every $t \in I$ and for all $x \in \mathbb{R}^n$. It is easy to verify that F_j^l is a (lsc)-type set-valued mapping with closed values.

Let $f_j : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ be the operator associated to F_j^l defined in (4.1).

Arguing as in the proof of Theorem 3.14 and using Proposition 4.3, we get the conclusion. □

In the particular case where F has real values, we get the following corollary.

Corollary 4.8. *For $i=1,2$, let $\alpha_i, \beta_i \in W^{1,1}(I,\mathbb{R})$, and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a (lsc)-type set-valued mapping with nonempty, closed values. Assume the following conditions hold:*

(i) *$\alpha_1(t) < \beta_1(t) \leq \beta_2(t)$ and $\alpha_1(t) \leq \alpha_2(t) < \beta_2(t)$ for all $t \in I$;*
(ii) *there exists $t \in I$ such that $\beta_1(t) < \alpha_2(t)$;*
(iii) *$\alpha_i(0) < \alpha_i(1)$ and $\beta_i(0) > \beta_i(1)$ for $i = 1,2$;*
(iv) *for $i = 1,2$, there exists a l.s.c. mapping $\epsilon_i : I \to (0,\infty)$ such that,*
- *$F(t,x) \cap [\alpha_i'(t), \infty) \neq \emptyset$ for a.e. $t \in I$ and all $x \in \mathbb{R}$ such that $\alpha_i(t) \leq x < \alpha_i(t) + \epsilon_i(t)$;*
- *$F(t,x) \cap (-\infty, \beta_i'(t)] \neq \emptyset$ for a.e. $t \in I$ and all $x \in \mathbb{R}$ such that $\beta_i(t) \geq x > \beta_i(t) - \epsilon_i(t)$;*

(v)
- *there exits $y \in F(t,x)$ such that $y \geq \max\{\alpha_1'(t), \alpha_2'(t)\}$ for a.e. $t \in I$ and all $x \in \big[\alpha_1(t), \alpha_1(t)+\epsilon_1(t)\big) \cap \big[\alpha_2(t), \alpha_2(t)+\epsilon_2(t)\big)$;*
- *there exits $y \in F(t,x)$ such that $y \leq \min\{\beta_1'(t), \beta_2'(t)\}$ for a.e. $t \in I$ and all $x \in \big(\beta_1(t)-\epsilon_1(t), \beta_1(t)\big] \cap \big(\beta_2(t)-\epsilon_2(t), \beta_2(t)\big]$;*
- *for $i,j \in \{1,2\}$ with $i \neq j$, there exits $y \in F(t,x)$ such that $\alpha_j'(t) \leq y \leq \beta_i'(t)$ for a.e. $t \in I$ and all $x \in \big(\beta_i(t)-\epsilon_i(t), \beta_i(t)\big] \cap \big[\alpha_j(t), \alpha_j(t)+\epsilon_j(t)\big)$.*

Then problem (1.1) *has at least three distinct solutions $x_0, x_1, x_2 \in W^{1,1}(I,\mathbb{R})$ such that $x_j \in T(v_j,\rho_j)$ and $x_0 \notin T(v_i,\rho_i)$ for $i=1,2$ and $j=0,1,2$.*

Remark 4.9. It is left to the reader to state and prove a result analogous to Theorem 3.18 for a (lsc)-type set-valued mapping.

5. Initial value problem

In this section, we present multiplicity results for the following system of differential inclusions with initial value condition:

$$\begin{aligned} x'(t) &\in F(t,x(t)) \quad \text{a.e. } t \in [0,1], \\ x(0) &= x_0, \end{aligned} \tag{5.1}$$

where $x_0 \in \mathbb{R}^n$ is given. Again, our results will rely on the notions of solution-tube and strict solution-tube of (5.1).

Definition 5.1. Let $(v,\rho) \in W^{1,1}(I,\mathbb{R}^n) \times W^{1,1}(I,[0,\infty))$. We say that (v,ρ) is a *solution-tube of* (5.1) if it satisfies (i) and (ii) of Definition 3.7 and the following condition:

(iii)' $\|x_0 - v(0)\| \leq \rho(0)$.

Definition 5.2. Let $(v,\rho) \in W^{1,1}(I,\mathbb{R}^n) \times W^{1,1}(I,(0,\infty))$. We say that (v,ρ) is a *strict solution-tube of* (5.1) if it satisfies (i) of Definition 3.10 and the following condition:

(ii)' $\|x_0 - v(0)\| < \rho(0)$.

We obtain multiplicity results analogous to Theorems 3.14 and 4.7.

Theorem 5.3. *Let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory set-valued mapping with nonempty, closed, convex values. Assume the following conditions hold:*

(i) *there exists (v_0,ρ_0) a solution-tube of* (5.1)*;*

(ii) *there exist* (v_1, ρ_1) *and* (v_2, ρ_2) *two compatible strict solution-tubes of* (5.1) *such that*
  (a) $T(v_i, \rho_i) \subset T(v_0, \rho_0)$ *for* $i = 1, 2$*;*
  (b) $T(v_1, \rho_1) \cap T(v_2, \rho_2) = \emptyset$.

Then problem (5.1) *has at least three distinct solutions* $x_0, x_1, x_2 \in W^{1,1}(I, \mathbb{R}^n)$ *such that* $x_j \in T(v_j, \rho_j)$ *and* $x_0 \notin T(v_i, \rho_i)$ *for* $i = 1, 2$ *and* $j = 0, 1, 2$.

Proof. Let $W_0^{1,1}(I, \mathbb{R}^n) = \{x \in W^{1,1}(I, \mathbb{R}^n) : x(0) = x_0\}$. It is well known that the continuous affine operator $L + \mathrm{id} : W_0^{1,1}(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ is invertible, where L is defined in (2.1). The proof is analogous to the proof of Theorem 3.14 by replacing $W_P^{1,1}(I, \mathbb{R}^n)$ by $W_0^{1,1}(I, \mathbb{R}^n)$ and, instead of Lemma 2.1, by applying Lemma 5.6 stated at the end of this section. □

Theorem 5.4. *Let* $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be a (lsc)-type set-valued mapping with nonempty, closed values. Assume the following conditions hold:*

(i) *there exists* (v_0, ρ_0) *a solution-tube of* (5.1)*;*
(ii) *there exist* (v_1, ρ_1) *and* (v_2, ρ_2) *two strongly compatible strict solution-tubes of* (5.1) *such that*
  (a) $T(v_i, \rho_i) \subset T(v_0, \rho_0)$ *for* $i = 1, 2$*;*
  (b) $T(v_1, \rho_1) \cap T(v_2, \rho_2) = \emptyset$.
(iii) *for almost every* $t \in I$ *and all* $x \in \mathbb{R}^n$ *such that* $\|x - v_0(t)\| = \rho_0(t)$ *and* $\mathit{card}\big(J(t,x)\big) \geq 2$*, where*

$$J(t,x) = \{j \in \{0,1,2\} : \|x - v_j(t)\| = \rho_j(t)\},$$

there exists $y \in F(t,x)$ *such that*

$$\langle x - v_j(t), y - v_j'(t)\rangle \leq \rho_j(t)\rho_j'(t) \quad \text{for each } j \in J(t,x).$$

Then problem (5.1) *has at least three distinct solutions* $x_0, x_1, x_2 \in W^{1,1}(I, \mathbb{R}^n)$ *such that* $x_j \in T(v_j, \rho_j)$ *and* $x_0 \notin T(v_i, \rho_i)$ *for* $i = 1, 2$ *and* $j = 0, 1, 2$.

Remark 5.5. Results analogous to Corollaries 3.16 and 4.8 and Theorem 3.18 can be obtained for problem (5.1).

Lemma 5.6. *Let* $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be a set-valued mapping and* $x_0 \in \mathbb{R}^n$*. Assume there exist* $v \in W^{1,1}(I, \mathbb{R}^n)$ *and* $\rho \in W^{1,1}(I, [0, \infty))$ *such that* $\|x_0 - v(0)\| \leq \rho(0)$ *and*

$$(5.2) \quad F(t,x) \subset \{y \in \mathbb{R}^n : \langle x - v(t), y - v'(t)\rangle < \|x - v(t)\|\rho'(t)\}$$
$$\text{a.e. } t \in I \text{ and for all } x \in \mathbb{R}^n \text{ such that } \|x - v(t)\| > \rho(t).$$

Then any solution $x \in W^{1,1}(I, \mathbb{R}^n)$ *of* (5.1) *satisfies* $\|x(t) - v(t)\| \leq \rho(t)$ *for all* $t \in I$.

References

[1] R. P. Agarwal, M. Frigon, V. Lakshmikantham and D. O'Regan, *A note on upper and lower solutions for first order inclusions of upper semicontinuous or lower semicontinuous type*, Nonlinear Stud. **12** (2005), 159–163.

[2] J. Andres, *On the multivalued Poincaré operators*, Topol. Methods Nonlinear Anal. **10** (1997), 171–182.

[3] J. Andres, L. Górniewicz and M. Lewicka, *Partially dissipative periodic processes*, Topology in nonlinear analysis (Warsaw, 1994), Banach Center Publ., **35**, Polish Acad. Sci. Inst. Math., Warsaw, 1996, pp. 109–118.

[4] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[5] M. Benchohra and A. Boucherif, *On first order multivalued initial and periodic value problems*, Dynam. Systems Appl. **9** (2000), 559–568.

[6] V. I. Blagodat·skikh and A. F. Filippov, *Differential inclusions and optimal control*, Proc. Steklov Inst. Math. **4** (1986), 199–259; transl. from Topology, ordinary differential equations, dynamical systems, Trudy Mat. Inst. Steklov. **169** (1985), 194–255.

[7] Y. G. Borisovich, B. D. Gel'man, A. D. Myshkis and V. V. Obukhovskii, *Multivalued mappings*, J. Soviet Math. **24** (1982), 719–791; transl. from Itogi Nauki i Tekhniki, Ser. Mat. Anal. **19** (1982), 127–230.

[8] A. Cellina, *On mappings defined by differential equations*, Zeszyty Nauk. Uniw. Jagiello. Prace Mat. **15** (1971), 17–19.

[9] F. S. De Blasi, L. Górniewicz and G. Pianigiani, *Topological degree and periodic solutions of differential inclusions*, Nonlinear Anal. Ser. A: Theory Methods **37** (1999), 217–243.

[10] F. S. De Blasi and G. Pianigiani, *Non-convex-valued differential inclusions in Banach spaces*, J. Math. Anal. Appl. **157** (1991), 469–494.

[11] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.

[12] M. Frigon, *Théorèmes d'existence de solutions d'inclusions différentielles*, Topological methods in differential equations and inclusions (Montreal, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **472**, Kluwer Acad. Publ., Dordrecht, 1995, pp. 51–87.

[13] M. Frigon, *Systems of first order differential inclusions with maximal monotone terms*, Nonlinear Anal. **66** (2007), 2064–2077.

[14] M. Frigon and E. Montoki, *Multiplicity results for systems of second order differential equations*, Nonlinear Stud. **15** (2008), 71–92.

[15] M. Frigon and D. O'Regan, *Existence results for initial value problems in Banach spaces*, Differential Equations Dynam. Systems **2** (1994), 41–48.

[16] M. Frigon and D. O'Regan, *Nonlinear first order initial and periodic problems in Banach spaces*, Appl. Math. Lett. **10** (1997), 41–46.

[17] C. J. Himmelberg, *Measurable relations*, Fund. Math. **87** (1975), 53–72.

[18] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis*, Vol. 1, Kluwer, Dordrecht, 1997.

[19] S. Hu and N.S. Papageorgiou, *On the existence of periodic solutions for nonconvex-valued differential inclusions in* $\mathbb{R}^N$. Proc. Amer. Math. Soc. **123** (1995), 3043–3050.

[20] J. Mawhin, *First order ordinary differential equations with several periodic solutions*, Z. Angew. Math. Phys. **38** (1987), 257–265.

[21] M. N. Nkashama, *A generalized upper and lower solutions method and multiplicity results for nonlinear first-order ordinary differential equations*, J. Math. Anal. Appl. **140** (1989), 381–395.

[22] S. Plaskacz, *Periodic solutions of differential inclusions on compact subsets of* $\mathbb{R}^n$, J. Math. Anal. Appl. **148** (1990), 202–212.

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