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GENERALIZED NONEXPANSIVE MAPPINGS AND A KRASNOSEL'SKIĬ THEOREM FOR THE DE BLASI NONCOMPACTNESS MEASURE

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Abstract. Some fixed point theorems for a broad class of mappings which contains the nonexpansive mappings as well as many of their generalizations will be given. Moreover, we will prove a Krasnosel'skiĭ type fixed point result for operators splitting into a sum of a generalized nonexpansive mapping and a mapping whose compactness depends on the measure of noncompactness introduced in 1977 by de Blasi.

1. Introduction

A celebrated result due to Krasnosel'skiĭ [10] established that the sum $A+B$ of two operators has a fixed point in a nonempty closed convex subset C of a real Banach space $(X, \|\cdot\|)$ whenever A and B satisfy.

(1) $A(C)+B(C) \subseteq C$,
(2) A is continuous on C and $A(C)$ is a relatively compact subset of X,
(3) B is a strict contraction.

It is not hard to see that this theorem is a particular case of the Darbo's well known theorem [4], which, in turn is an extension of Schauder's celebrated fixed point theorem to the setting of some noncompact operators. Darbo's result relies upon the notion of Kuratowski measure of noncompactness, and it was in turn generalized by Sadovskii in 1967 (see [18]). Such early extensions of Krasnosel'skii's theorem can be found, for example, in the paper by S. Reich [16] and in references therein.

In the last decades, different authors have obtained Krasnosel'skiĭ type fixed point theorems by relaxing either the compactness, or the contractivity, of the mappings under consideration. Recently, several papers give generalizations of Krasnosel'skiĭ theorem using the weak topology, in particular, the extensions obtained in [7, 13] rely on the concept of measure of weak noncompactness given by de Blasi in [5].

The aim of this paper is twofold. First, we will give some fixed point theorems for a broad class of mappings which contains the so called (L)-type mappings. The class of such (L)-type mappings was introduced in 2011 (see [12]), and it contains all the nonexpansive mappings as well as many of their generalizations.

Second, we will give a Krasnosel'skiĭ type fixed point result for operators splitting into a sum of an (L)-type mapping and a mapping whose compactness depends on the de Blasi's measure of noncompactness.

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2. Preliminaries

Throughout this paper we suppose that $(X, \|\cdot\|)$ is a real Banach space. For any $r > 0$, B_r denotes the closed ball in X centered in 0_X and with radius r.

We denote by $\mathcal{B}(X)$ the collection of all nonempty bounded subsets of X, and by $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. The notion of the measure of weak noncompactness was introduced by de Blasi [5] and it is the map $\omega : \mathcal{B}(X) \to [0, \infty[$ defined for every $M \in \mathcal{B}(X)$ by

$$\omega(M) := \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\}.$$

Now, we are going to recall some basic properties of $\omega(\cdot)$ needed later. Let M_1, M_2 be two elements of $\mathcal{B}(X)$. The following properties hold (for instance see [1, 5]):

(1) If $M_1 \subseteq M_2$, then $\omega(M_1) \leq \omega(M_2)$,
(2) $\omega(M_1) = 0$ if and only if, $\overline{M_1}^w \in \mathcal{W}(X)$ ($\overline{M_1}^w$ means the weak closure of M_1),
(3) $\omega(\overline{M_1}^w) = \omega(M_1)$,
(4) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$,
(5) $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ for all $\lambda \in \mathbb{R}$,
(6) $\omega(conv(M_1)) = \omega(M_1)$,
(7) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.

A mapping $T : C \subseteq X \to X$ is said to be ω-condensing if T is continuous and $\omega(T(A)) < \omega(A)$ for every bounded set $A \subseteq C$ with $\omega(A) > 0$. Since $\omega(A) = 0$ for every $A \in \mathcal{B}(X)$ if X is a reflexive space, this concept has a real sense provided that X is a nonreflexive space.

Let X be Banach space and $T : D(T) \subseteq X \to X$ a mapping. We will use the following condition for this mapping T:

($\mathcal{A}1$) If (x_n) is a sequence in $D(T)$ which is weakly convergent in X, then (Tx_n) has a strongly convergent subsequence in X.

Condition ($\mathcal{A}1$) was already considered, among others, in the papers [7, 13, 14]. Recall that this condition does not imply the compactness of T even if T is a linear mapping. Nevertheless, condition ($\mathcal{A}1$) reduces to compactness in the reflexive case.

Moreover, such condition holds also true for the class of weakly compact operators acting on Banach spaces with the Dunford-Pettis property. (A Banach space X has the Dunford-Pettis property if every weakly compact operator defined on X takes weakly compact sets into norm compact sets.) Finally, notice that operators satisfying ($\mathcal{A}1$) are not necessarily weakly continuous.

We recall also some geometric properties of normed spaces that will appear at some points of the remainder of this paper.

A normed space $(X, \|.\|)$ is said to satisfy *the Opial condition* if for any sequence (x_n) in X which converges weakly to $x_0 \in X$ it happens that for all $y \in X$, $y \neq x_0$,

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - y\| .$$

It can be readily established, on the extraction of appropriate subsequences, that the lower limit can be replaced with upper limit in the above definition. For example the classical ℓ_p spaces satisfy the Opial condition whenever $1 \leq p < \infty$.

A geometric property which plays an important role in metric fixed point theory is the so called normal structure, which was introduced in 1948 by Brodskii and Milman. Recall that a Banach space $(X, \|\cdot\|)$ is said to have *normal structure* if for each bounded, convex, subset C of X with $\mathrm{diam}(C) > 0$ there exists a *nondiametral* point $p \in C$, that is a point $p \in C$ such that

$$\sup\{\|p - x\| : x \in C\} < \text{ diam } (C).$$

It is well known that every uniformly convex Banach space, as well as every reflexive Banach space satisfying the Opial condition, enjoys normal structure.

3. Generalized nonexpansive mappings

Let C be a nonempty closed convex subset of $(X, \|\cdot\|)$.

In 2010 García Falset, Llorens Fuster and Suzuki introduced a class of generalized nonexpansive mappings in [9]. For $\lambda \in (0,1)$ a mapping $T : C \to X$ is said to satisfy condition (C_λ) on C if for any $x, y \in C$ such that $\lambda\|x - Tx\| \leq \|x - y\|$ it holds that $\|Tx - Ty\| \leq \|x - y\|$.

In the particular case $\lambda = \frac{1}{2}$ we recover the class of type (C) mappings defined by Suzuki in 2008 ([19]). Mappings satisfying condition (C) satisfy condition (L) in turn, but this is not true in general for mappings satisfying condition (C_λ) ([12]).

Definition 3.1. Kaewkao, Sokhuma [11]

A bounded sequence (x_n) in X is said to be an *asymptotic center sequence* for the mapping T if for every $x \in C$

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

According with [8], if, in particular, $x_n \equiv x_0 \in X$, we say that x_0 is a *center* for the mapping T in C.

Definition 3.2. We say that the mapping $T : C \to X$ is a (DL)-type mapping (or that T satisfies condition (DL)) provided that there exists a bounded sequence (x_n) in C such that it is an asymptotic center sequence for T in C.

Let us point out that a continuous mapping T which admits an asymptotic center sequence in X was called a (D)-type mapping by A. Kaewkao and K. Sokhuma in [11, Definition 3.1]. However, it is easy to check that the class of (DL)-type mappings neither is contained nor contains the class of (D)-type mappings.

The class of (DL)-type mappings contains the so called (L)-type mappings. If C is also a bounded set, the class of the (L)-type self-mappings of C in turn contains the class of the nonexpansive (i.e. 1-Lipschitzian) mappings $T : C \to C$.

Recall that an almost fixed point sequence for $T : C \to X$ (a.f.p.s. from now on) is a sequence (x_n) on C such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Definition 3.3. Llorens-Fuster, Moreno-Gálvez [12]

We say that the mapping $T : C \to C$ is an (L)-type mapping (or that T satisfies condition (L)) on C provided that

(a) If $D \subset C$ is nonempty, closed, convex and T invariant, then there exists an a.f.p.s. (x_n) for T in D.

(b) For every a.f.p.s. (x_n) for T in C, and for each $x \in C$,

$$\limsup_{n\to\infty} \|x_n - Tx\| \le \limsup_{n\to\infty} \|x_n - x\|.$$

Of course, if T satisfies condition (L) on C then it satisfies condition (DL) on C. As we will see below, the converse is untrue.

The above assumption (a) was called Condition (A) by Dhompongsa and Nanan in [6]. In other words, the mapping $T : C \to C$ is an (L)-type mapping whenever it satisfies condition (A) on C and any a.f.p.s. on C is an asymptotic center sequence for T. From now on, if not specified, a mapping is said to satisfy condition (L), whenever it satisfies this condition on its domain. Assumption (a) of this definition (i.e. Condition (A)) is automatically satisfied by several classes of nonlinear mappings. For instance, it is a well-known property of nonexpansive mappings. Indeed, if C is also a bounded set and $T : C \to C$ is a nonexpansive mapping with respect to some equivalent renorming of X, then T satisfies (a). Asymptotically regular mappings automatically satisfy (a) too. The existence of a.f.p.s. for nonexpansive mappings is discussed, for example, in the papers by S. Reich [17] and by E. Matoušková and S. Reich [15].

Example 3.4. Let $T : [0,1] \to [0,1]$ given by $T(x) = \sqrt{x}$. For every $x \in [0,1]$ one has

$$1 - \sqrt{x} = \frac{1-x}{1+\sqrt{x}} \le 1 - x.$$

Therefore, $|1 - T(x)| \le |1 - x|$. Thus, the sequence (x_n) defined as $x_n \equiv 1$ is an asymptotic center sequence for T on $[0,1]$, and T is a (non Lipschitzian) (DL)-type mapping. On the other hand, in [12, Example 3.11.] it is shown that T fails to satisfy the condition (L) on $[0,1]$.

Since every nonexpansive mappings is an (L) type mapping, and hence a (DL)-type mapping, provided that there exists well known fixed-point free nonexpansive mappings, condition (DL) does not guarantee fixed points. In other words, to get a fixed point theorem for (DL)-type mappings some additional requirement is necessary.

Although (L)-type mapping need not be continuous, regarding fixed point properties, the main features of the (L)-type mappings are the following.

Theorem 3.5 ([12, Theorem 4.2.]). *Let C be a nonempty compact convex subset of a Banach space X and $T : C \to C$ be a mapping satisfying condition (L). Then, T has a fixed point.*

Theorem 3.6 ([12, Theorem 4.4.]). *Let K be a nonempty weakly compact convex subset of a Banach space X with normal structure. Let $T : K \to K$ be a mapping satisfying condition (L). Then, T has a fixed point.*

Theorem 3.7 ([12, Theorem 4.6.]). *Let X be a Banach space which satisfies the Opial condition. Let C be a nonempty, closed, convex, bounded subset of X. Let $T : C \to C$ be a mapping satisfying condition (L). Then, if (x_n) is an a.f.p.s. for T such that it converges weakly to $z \in C$, then z is a fixed point of T.*

In the remaining of this section, we will show that, in some sense, the above results also hold for (DL)-type mappings.

Theorem 3.8. *Let C be a nonempty compact convex subset of a Banach space X and $T : C \to C$ be a mapping satisfying condition (DL). Then, T has a fixed point.*

Proof. From the definition of (DL)-type mappings, we know that there exists in C an asymptotic center sequence for T, say (x_n). Since K is a compact set, after passing to a subsequence it necessary, we may assume that $x_n \to x_0$. Again by definition of (DL) mapping,

$$\limsup_{n\to\infty} \|x_n - T(x_0)\| \le \limsup_{n\to\infty} \|x_n - x_0\| = 0,$$

which implies that $x_n \to T(x_0)$. By uniqueness of limits, then $x_0 = T(x_0)$. □

Theorem 3.9. *Let X be a Banach space which satisfies the Opial condition. Let $T : C \to C$ be a mapping satisfying condition (DL). If (x_n) is an asymptotic center sequence for T such that it converges weakly to $z \in C$, then z is a fixed point of T.*

Proof. Since (x_n) is an asymptotic center sequence for T,

$$\limsup_{n\to\infty} \|x_n - T(z)\| \le \limsup_{n\to\infty} \|x_n - z\|.$$

If $T(z) \neq z$, from the Opial condition it follows that

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - T(z)\|,$$

and this is a contradiction. Thus, $T(z) = z$. □

Theorem 3.10. *Let K be a nonempty weakly compact convex subset of a Banach space X with normal structure. Let $T : K \to K$ be a mapping which admits asymptotic center sequences in each nonempty closed convex T invariant subset of K. Then, T has a fixed point.*

Proof. Since K is a weakly compact set there exists a nonempty, closed, convex, T invariant subset C of K with no proper subsets enjoying these characteristics. Let (x_n) be an asymptotic center sequence for T in C. If $x_n \to x_0 \in C$ then x_0 is a fixed point of T. Indeed,

$$\|x_0 - T(x_0)\| = \limsup_{n\to\infty} \|x_n - T(x_0)\| \le \limsup_{n\to\infty} \|x_n - x_0\| = 0$$

In other case, according to [3, Corollary 1] the real function $g : C \to [0, \infty)$ defined as

$$g(x) := \limsup_{n\to\infty} \|x - x_n\|$$

is not constant on $\operatorname{conv}\{x_n : n = 1, 2, ...\}$. Thus, g takes at least two different real values. Let r be the average of these values. Consider the set

$$M := \{x \in C : g(x) \le r\}.$$

It is straightforward to check that M is nonempty, closed and convex, with $M \neq C$.

Moreover, for every $x \in M$, since (x_n) is an asymptotic center sequence for T in C, and $M \subset C$,

$$g(T(x)) = \limsup_{n\to\infty} \|T(x) - x_n\| \le \limsup_{n\to\infty} \|x - x_n\| = g(x) \le r.$$

Thus, M is a nonempty, closed, convex, T invariant subset of C with $M \neq C$, which is a contradiction with the minimality of C. □

Notice that if $T : K \to K$ is nonexpansive then it admits asymptotic center sequences in each nonempty closed convex T-invariant subset C of K. Indeed, for every $x \in C$, the sequence $(T^n(x))$ has the range contained in C and it is an asymptotic center sequence for T in C.

In the following example we give a non Lipchitzian mapping falling into the scope of the above theorem.

Example 3.11. In the standard Hilbert space ℓ_2 consider the set

$$K = \{x = (x_n) \in \ell_2 : \|x\|_2 \le 1, x_n \ge 0 \ \ n = 1, 2, \dots\},$$

where $\|x\|_2$ stands for the ordinary Euclidean norm of the vector $x \in \ell_2$.

Consider the mapping $T : K \to K$ defined by

$$T(x) = \sqrt{1 - \|x\|^2} \ \ e_1,$$

where $e_1 = (1, 0, \dots)$. It is obvious that T is not Lipschitzian on K. On the other hand, it is straightforward to check that for every positive integer k and every $x \in K$,

$$\begin{aligned} &T^2(x) = \|x\|e_1 \\ &T^3(x) = T(x) \\ &\vdots \\ &T^{2k}(x) = \|x\|e_1 \\ &T^{2k+1}(x) = T(x). \end{aligned}$$

If C is a nonempty closed convex T-invariant subset of K, and $x \in C$ we will show that the orbit $(T^k(x))$ is an asymptotic center sequence for T in C. Indeed,

$$\begin{aligned} \limsup_{n\to\infty} \|T^k(x) - T(x)\| &= \|\|x\|e_1 - T(x)\| = \left|\|x\| - \sqrt{1 - \|x\|^2}\right| \\ &= \sqrt{1 - 2\sqrt{1 - \|x\|^2} \ \|x\|}. \end{aligned}$$

On the other hand

$$\begin{aligned} \limsup_{n\to\infty} \|T^n(x) - x\| &= \max\{\|\|x\|e_1 - x\|, \|T(x) - x\|\} \\ &\ge \sqrt{\|T(x)\|^2 + \|x\|^2 - 2\langle T(x), x\rangle} \\ &= \sqrt{1 - 2\sqrt{1 - \|x\|^2} \ x_1} \\ &\ge \sqrt{1 - 2\sqrt{1 - \|x\|^2} \ \|x\|}. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \|T^k(x) - T(x)\| \le \limsup_{n\to\infty} \|T^n(x) - x\|.$$

and T admits asymptotic center sequences in every nonempty closed convex T-invariant subset C of K. Since it is well known that every Hilbert space enjoys normal structure, the above theorem can be applied to derive that the mapping T has a fixed point in K.

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a function such that

(1) $\phi(0) = 0$
(2) $\phi(r) > 0$ if $r > 0$,
(3) ϕ is continuous or increasing.

Recall that a mapping $T : C \to X$ is said to be ϕ-expansive if there exists a function ϕ satisfying the above assumptions, such that for every $x, y \in C$,

$$\|T(x) - T(y)\| \geq \phi(\|x - y\|).$$

Proposition 3.12. *Let X be a Banach space and C a closed, bounded and convex subset of X. Suppose that $T : C \to C$ satisfies condition (L) and that $I - T$ is ϕ-expansive. Then, any a.f.p.s. for T in C converges to the unique fixed point in C.*

Proof. Since T satisfies condition (L), we can consider an almost fixed point sequence (x_n) for T on C. Provided that $I - T$ is ϕ-expansive, we have that

$$\phi(\|x_n - x_m\|) \leq \|(I - T)x_n - (I - T)x_m\| \leq \|x_n - Tx_n\| + \|x_m - Tx_m\|.$$

By taking limits first on m and then on n,

$$\lim_{n\to\infty} \lim_{m\to\infty} \phi(\|x_n - x_m\|) \leq \lim_{n\to\infty} \|x_n - Tx_n\| + \lim_{m\to\infty} \|x_m - Tx_m\| = 0.$$

We claim that $\lim_{n\to\infty} \lim_{m\to\infty} \|x_n - x_m\| = 0$. Otherwise take a subsequence of (x_n) such that

$$l := \lim_{n,m;n\neq m} \|x_n - x_m\|$$

does exist.

Suppose that $l \neq 0$. There exists k such that $\|x_n - x_m\| \geq l/2$ if $n, m > k; n \neq m$. If ϕ is increasing, then $\phi(\|x_n - x_m\|) \geq \phi(l/2)$ if $n, m > k; m \neq n$. If ϕ is continuous choose $\delta > 0$ such that $\phi(t) > \phi(l)/2$ if $|t - l| < \delta$. There exists k such that $|l - \|x_n - x_m\|| < \delta$ if $n, m > k; n \neq m$. In both cases, $\phi(\|x_n - x_m\|) \geq \phi(l)/2$ whenever $n > k$, which implies

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \phi(\|x_n - x_m\|) \geq \min(\phi(l/2), \phi(l)/2) > 0.$$

However,

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \phi(\|x_n - x_m\|) \leq \lim_{n\to\infty} \|x_n - Tx_n\| + \lim_{m\to\infty} \|x_m - Tx_m\| = 0,$$

and this is a contradiction which proves our claim. Consequently, (x_n) is a Cauchy sequence, that is, it converges to some $x_0 \in C$. Then, taking into account that T satisfies condition (L),

$$\|x_0 - Tx_0\| = \limsup_{n\to\infty} \|x_n - Tx_0\| \leq \limsup_{n\to\infty} \|x_n - x_0\| = \|x_0 - x_0\| = 0,$$

that is, x_0 is a fixed point for T.

To prove the unicity, suppose now that y_0 is another fixed point for T on C different from x_0. In such case, given that $I-T$ is a ϕ-expansive mapping, we have that

$$\phi(\|x_0 - y_0\|) \leq \|(I-T)x_0 - (I-T)y_0\| \leq \|x_0 - Tx_0\| + \|y_0 - Ty_0\| = 0.$$

Then, necessarily, $\|x_0 - y_0\| = 0$ and we reach a contradiction, which allows us to state that x_0 is the unique fixed point for T on C. □

Theorem 2 in [9] states that for a selfmapping with condition (C_λ) for some $\lambda \in (0,1)$ on a bounded and convex subset C of a Banach space, the iterative scheme given for some $r \in [\lambda, 1)$ and some $x_1 \in C$ by

$$x_{n+1} = rTx_n + (1-r)x_n$$

for $n \geq 1$ provides and a.f.p.s. for T in C. Consequently, we can affirm that:

Corollary 3.13. *Let X be a Banach space and C a closed, bounded and convex subset of X. Suppose that $T : C \to C$ satisfies condition (L) and (C_λ) for some $\lambda \in (0,1)$ and that $I-T$ is ϕ-expansive. Then, the sequence given for some $r \in [\lambda, 1)$ and some $x_1 \in C$ by*

$$x_{n+1} = rTx_n + (1-r)x_n$$

for $n \geq 1$ converges to the unique fixed point for T on C.

Corollary 3.14. *Let X be a Banach space and C a closed, bounded and convex subset of X. Suppose that $T : C \to C$ satisfies condition C and that $I-T$ is ϕ-expansive. Then, the sequence given for some $r \in [\frac{1}{2}, 1)$ and some $x_1 \in C$ by*

$$x_{n+1} = rTx_n + (1-r)x_n$$

for $n \geq 1$ converges to the unique fixed point for T on C.

4. A fixed point result of Krasnosel'skiĭ type

We will also need the following result in order to prove the main theorem of this paper.

Theorem 4.1 ([14, Theorem 2.1])**.** *Let M be a nonempty closed convex subset of a Banach space X. Assume that $T : M \to M$ is a continuous map satisfying condition $(\mathcal{A}1)$. If $T(M)$ is relatively weakly compact, then there exists $x \in M$ such that $T(x) = x$.*

As an easy consequence of Theorem 4.1 the following result holds:

Theorem 4.2. *Let M be a nonempty bounded closed and convex subset of a Banach space X. Assume that $T : M \to M$ is an ω-condensing mapping satisfying condition $(\mathcal{A}1)$. Then T has a fixed point.*

Proof. Fix $x_0 \in M$. Consider

$$\mathcal{K} := \{A \subseteq M : T(A) \subseteq A,\ x_0 \in A, A \text{ closed and convex}\}.$$

It is straightforward to see that the set

$$K := \bigcap_{A \in \mathcal{K}} A = \operatorname{conv}(T(K) \cup \{x_0\})$$

belongs to $\mathcal{K}$.

Bearing in mind that the mapping T is ω-condensing, we obtain that K is a weakly compact convex T-invariant subset of M. The result now follows from Theorem 4.1. □

We can now state a generalization of Krasnosel'skiĭ theorem as well as of [7, Theorem 3.2].

Theorem 4.3. *Let X be a Banach space. Let M be a nonempty closed convex and bounded subset of X and let $A, B : M \to X$ be two continuous mappings. If A, B satisfy the following conditions,*

(i) *A satisfies $(\mathcal{A}1)$ and it is ω-k-contraction for some $k \in [0, 1[$,*
(ii) *$A(M) + B(M) \subseteq M$,*
(iii) *B satisfies that for any $y \in A(M)$ the mapping $B_y : M \to M$ defined by $B_y(x) = y + B(x)$ enjoys condition (L),*
(iv) *$I - B$ is ϕ-expansive,*
(v) *B is ω-s-contraction for some $s \in [0, 1-k[$.*

Then, the equation $x = A(x) + B(x)$ has a solution.

Proof. It is easily checked that $x \in M$ is a solution for the equation $x = B(x)+A(x)$ if and only if x is a fixed point for the operator $(I - B)^{-1} \circ A$, whenever it is well defined. This happens since:

(1) $(I - B)$ has an inverse over $R(I - B) := (I - B)(M)$. This is a consequence of the ϕ-expansiveness of $(I - B)$.
(2) The domain of $(I - B)^{-1}$ contains the range of A.
Take $y \in M$. Defining $T : M \to M$ such that for any $x \in M$, $Tx = Ay+Bx$, we have that T satisfies condition (L) by (iii), since $Ay + Bx \in M$. From assumption (iv) it is easy to check that the mapping $I-T$ is also ϕ-expansive. From Proposition 3.12 it follows that T has a unique fixed point $x \in M$, that is, $A(y) = (I - B)(x)$ and therefore $R(A) \subseteq R(I - B) = D((I - B)^{-1})$.

Consequently the mapping $(I - B)^{-1} \circ A : M \to M$ is well defined. Let us prove now that this operator satisfies the assumptions of Theorem 4.2, that is, that $(I - B)^{-1} \circ A$ is w-condensing and that it satisfies $(\mathcal{A}1)$.

(I) $(I - B)^{-1} \circ A$ is continuous.
First, let us see that $(I - B)^{-1} : R(I - B) \to M$ is a continuous mapping. Indeed, consider a sequence (x_n) in $R(I - B)$ converging to some $x_0 \in R(I - B)$. Let $y_n := (I-B)^{-1}(x_n)$ and $y_0 := (I-B)^{-1}(x_0)$. Hence $(I-B)(y_n) = x_n$ and $(I - B)(y_0) = x_0$. Since $I - B$ is ϕ-expansive

$$\phi(\|y_n - y_0\|) \leq \|(I - B)(y_n) - (I - B)(y_0)\| = \|x_n - x_0\|.$$

Consequently

$$\lim_{n\to\infty} \phi(\|y_n - y_0\|) = \lim_{n\to\infty} \|x_n - x_0\| = 0. \tag{1}$$

If we assume that $(\|y_n - y_0\|)$ is not a null sequence, then there exists $(\|y_{n_s} - y_0\|)$, subsequence of $(\|y_n - y_0\|)$, such that $\|y_{n_s} - y_0\| \to r > 0$, (this is a consequence of the fact that M is a bounded subset). Now, if ϕ is a continuous function, we obtain that

$$\lim_{s\to\infty} \phi(\|y_{n_s} - y_0\|) = \phi(r) > 0.$$

Otherwise, ϕ will be nondecreasing and then

$$0 < \phi\Big(\frac{r}{2}\Big) \le \lim_{s\to\infty} \phi(\|y_{n_s} - y_0\|).$$

In both cases, from (1) we have that $\lim_{s\to\infty} \phi(\|y_{n_s} - y_0\|) = 0$, which is a contradiction and therefore $\|y_n - y_0\| \to 0$, that is

$$\|(I-B)^{-1}(x_n) - (I-B)^{-1}(x_0)\| \to 0.$$

This means that $(I-B)^{-1}$ is continuous as we claimed. Since A is also continuous by hypothesis, $(I-B)^{-1} \circ A$ is continuous.

(II) If $K \subseteq M$ is a non relatively weakly compact set, then $w((I-B)^{-1} \circ A(K)) < w(K)$.
It is easy to check that $(I-B)^{-1} \circ A = A + B \circ (I-B)^{-1} \circ A$. By using the properties of $\omega(\cdot)$,

$$\begin{aligned}\omega((I-B)^{-1} \circ A(K)) &= \omega(A(K) + B(((I-B)^{-1} \circ A)(K))) \\ &\le \omega(A(K)) + \omega(B(I-B)^{-1} \circ A)(K)) \\ &\le k\omega(K) + s\omega((I-B)^{-1} \circ A)(K)).\end{aligned}$$

Therefore

$$\omega((I-B)^{-1} \circ A(K)) \le \frac{k}{1-s}\omega(K).$$

which implies that $(I-B)^{-1} \circ A$ is ω-condensing.

(III) $(I-B)^{-1} \circ A$ satisfies condition $(\mathcal{A}1)$.
It is straightforward from the condition $(\mathcal{A}1)$ of A and the continuity of $(I-B)^{-1}$.

Consequently, $(I-B)^{-1} \circ A$ satisfies the hypothesis of Theorem 4.2 as we claimed, and hence such operator has a fixed point. □

Corollary 4.4. *Let X be a Banach space. Let M be a nonempty closed convex and bounded subset of X and let $A, B : M \to X$ be two continuous mappings. If A, B satisfy the following conditions,*

(i) *A satisfies $(\mathcal{A}1)$ and it is ω-k-contraction for some $k \in [0,1[$,*
(ii) *$A(M) + B(M) \subseteq M$,*
(iii) *B is nonexpansive, and it is an ω-s-contraction for some $s \in [0, 1-k[$.*
(iv) *$I - B$ is ϕ-expansive.*

Then, the equation $x = A(x) + B(x)$ has a solution.

Proof. We only need to check that Condition (ii) of the above theorem holds. Indeed, since B is nonexpansive, then $B_y : M \to M$ is also nonexpansive and therefore it enjoys property (L). □

Next, we will give an example of two mappings satisfying assumption (i), (ii) and (iii) of Theorem 4.3 and such that Condition (iii) of the above corollary fails.

Example 4.5 ([12, Example 3.7])**.** The mapping $B : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ given by $Bx = x^2$ fails to be generalized nonexpansive, and hence, nonexpansive (see [12]).

If we take $A : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ as $A(x) = \frac{1}{9}$, then, Condition (ii) of Theorem 4.3 is clearly satisfied. On the other hand taking $y \in A([0, \frac{2}{3}])$, that is $y = \frac{1}{9}$ then $B_y : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ is given by $B_y x = y + Bx = \frac{1}{9} + x^2$. Let us prove that B_y satisfies condition (L) on $[0, \frac{2}{3}]$. Since the mapping is continuous, then the mapping B_y has a fixed point (and hence an a.f.p.s.) in every nonempty, closed, convex and B_y-invariant subset $D \subseteq [0, \frac{2}{3}]$. Let us point out that the only fixed point of B_y on $[0, \frac{2}{3}]$ is $x_0 = \frac{3-\sqrt{5}}{6}$. Hence, $x_0 \in D$. If (x_n) is an a.f.p.s. for B_y in D, then (x_n) must converge to x_0. Therefore $\limsup_{n\to\infty} |x_n - B_y x| = |x_0 - B_y(x)|$, and $\limsup_{n\to\infty} |x_n - x| = |x_0 - x|$. Since, por $x \in [0, \frac{2}{3}]$,

$$\begin{aligned} |B_y x - x_0| &= |\tfrac{1}{9} + x^2 - \tfrac{3-\sqrt{5}}{6}| \\ &\leq |x - \tfrac{3-\sqrt{5}}{6}|, \end{aligned}$$

then, B_y satisfies condition (L) on $[0, \frac{2}{3}]$.

Notice that even in the case that X is a reflexive space, the above Theorem 4.3 seems to be new because assumptions concerning operator B are weaker than nonexpansiveness.

Finally, we give an example of application of Corollary 4.4.

Example 4.6. Consider the following equation

$$\left.\begin{aligned} y'(t) &= f(t, y(t)) \\ y(0) &= g(y) \end{aligned}\right\}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function and g is a function from $C([0, T])$ to $\mathbb{R}$ defined as

$$g(u) = \sum_{i=1}^{n} \alpha_i u(t_i)$$

with $0 < t_1 < t_2 < \ldots < t_n \leq T$ and $\sum_{i=1}^{n} |\alpha_i| < 1$.

Let us define the operator

$$\begin{aligned} S : \ C([0,T]) &\to C([0,T]) \\ u &\mapsto S(u)(t) = g(u) + \textstyle\int_0^t f(s, u(s))ds. \end{aligned}$$

If this operator has a fixed point, then it is a solution for the problem above.

Let us write the operator S as a sum of the operators S_1 and S_2 defined as

$$\begin{aligned} S_1 : \ C([0,T]) &\to C([0,T]) \\ u &\mapsto S_1(u)(t) = g(u) \end{aligned}$$

and

$$\begin{aligned} S_2 : \ C([0,T]) &\to C([0,T]) \\ u &\mapsto S_2(u)(t) = \textstyle\int_0^t f(s, u(s))ds. \end{aligned}$$

Let us prove that S_1, S_2 satisfy assumption (iv) of Theorem 4.3 for some ball. Consider $r > 0$ and $u, v \in B_r(0)$. Then $\max\{\|u\|_\infty, \|v\|_\infty\} \le r$. Since $f(\cdot,\cdot)$ is a bounded function there exists $M > 0$ such that

$$\begin{aligned} |S_1(u)(t) + S_2(v)(t)| &\le |g(u)| + \int_0^t |f(s, v(s))| ds \\ &\le \|u\|_\infty \cdot \sum_{i=1}^n |\alpha_i| + MT \\ &\le r\left(\sum_{i=1}^n |\alpha_i| + \frac{MT}{r}\right) \end{aligned}$$

Since $\lim_{r\to\infty} \frac{MT}{r} = 0$ and $\sum_{i=1}^n |\alpha_i| < 1$, it is clear that there exists $r_0 > 0$ such that if $u, v \in B_{r_0}$ then $\|S_1(u) + S_2(v)\|_\infty \le r_0$.

It is a well known result that under these assumptions S_2 is a compact operator, hence S_2 satisfies condition $(\mathcal{A}1)$ and it maps bounded subsets into relatively compact subsets.

On the other hand, since g is a linear function and $\sum_{i=1}^n |\alpha_i| < 1$, S_1 is a contractive weakly continuous mapping, thus by Lemma 3.1 in [7], we know that S_1 is ω-condensing. Moreover, since S_1 is a contraction mapping, it is clear that assumptions (ii) and (iii) in Theorem 4.3 hold.

Consequently, the operator $S = S_1 + S_2$ is under the conditions of Theorem 4.3, and hence, S has a fixed point, which is a solution for the problem above.

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