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A CANTOR TYPE INTERSECTION THEOREM FOR SUPERREFLEXIVE BANACH SPACES AND FIXED POINTS OF ALMOST AFFINE MAPPINGS

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Dedicated to the memory of Professor Francesco S. De Blasi

Abstract. We obtain the following Cantor type intersection theorem which, in fact, is a partial extension of Šmulian's theorem by weakening the convexity assumption. Let (A_n) be a decreasing sequence of closed subsets of a superreflexive Banach space such that for any $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $A_k + A_k \subseteq 2A_n$. Then the intersection of all sets A_n is nonempty closed and convex if and only if there exists a bounded sequence (a_n) such that $a_n \in A_n$ for each $n \in \mathbb{N}$. We present a few applications of this result in metric fixed point theory. In particular, we get a fixed point theorem for mappings, which are 'almost affine' on sets of 'almost fixed' points, generalizing the well known Browder–Göhde–Kirk theorem. We also establish a common fixed point theorem for a family of nonexpansive mappings (not necessarily commuting).

1. Introduction

It is well known that the Banach contraction principle can be derived from Cantor's intersection theorem. Such a proof was first published by Boyd and Wong [2] (see also [12, p. 2] or [11, p. 9]), but it could also have been known to Banach. The following argument is used in that proof: Let (X, d) be a metric space, C be a nonempty subset of X and $T: C \to X$ be a mapping. For any decreasing sequence (α_n) of positive reals such that $\alpha_n \searrow 0$, define

$$A_n := \{x \in C : d(x, Tx) \le \alpha_n\} \quad \text{for } n \in \mathbb{N}. \tag{1.1}$$

Clearly, (A_n) is a decreasing sequence of sets and $\bigcap_{n\in\mathbb{N}} A_n = \operatorname{Fix} T$, the set of all fixed points of T. Thus T has a fixed point if and only if the intersection of all sets A_n is nonempty. Now Boyd and Wong [2] showed that under the assumptions of the Banach principle, Cantor's intersection theorem can be applied to (A_n), so $\bigcap_{n\in\mathbb{N}} A_n$ is then a singleton and hence T has a unique fixed point. A more comprehensive study of other applications of Cantor's intersection theorem in metric fixed point theory was done in [15]. Here we quote a selected part of [15, Proposition 1 and Theorem 2]. Recall that the fixed point problem for T is *well posed* if there exists a fixed point x_* of T and for any sequence (x_n) of elements of C,

$$d(x_n, Tx_n) \to 0 \quad \text{implies that} \quad x_n \to x_*.$$

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This notion was introduced by Francesco S. De Blasi and Myjak [8], and studied among others by Reich and Zaslavski ([22] and [23, Section 3.11]; also, see [21]).

Proposition 1.1. *Let (X, d) be a complete metric space, C be a nonempty closed subset of X and $T\colon C \to X$ be a continuous mapping. For $n \in \mathbb{N}$, let A_n be defined by (1.1). The following statements are equivalent:*

(i) *(A_n) satisfies the assumptions of Cantor's intersection theorem;*
(ii) *the fixed point problem for T is well posed;*
(iii) *$\inf_{x\in C} d(x, Tx) = 0$ and for any sequences (x_n) and (y_n) of elements of C,*

$$d(x_n, Tx_n) \to 0 \quad \text{and} \quad d(y_n, Ty_n) \to 0 \quad \text{imply that} \quad d(x_n, y_n) \to 0.$$

Note that in [15] T was assumed to be a selfmap of X, but this condition is not necessary in Proposition 1.1.

However, in many cases the assumption that $\operatorname{diam} A_n \to 0$ turns out to be too restrictive. In particular, it need not hold if T is a nonexpansive mapping. (Consider, e.g., the identity mapping.) Fortunately, there are several possibilities of replacing the assumption on diameters of A_n by other, often geometric type conditions, so that the intersection of A_n be nonempty. Much information on this topic can be found, e.g., in the paper of Castillo and Papini [6]. Here we recall an old result of Šmulian [24], one of basic theorems in Banach space theory (see, e.g., [11, pp. 4–5]).

Theorem 1.2 (Šmulian)**.** *A Banach space X is reflexive if and only if any decreasing sequence of nonempty closed bounded and convex subsets of X has a nonempty intersection.*

Unfortunately, for a nonexpansive mapping T, sets A_n defined by (1.1) need not be convex, so Šmulian's theorem is useless here. Our purpose is to give a partial extension of Šmulian's theorem by weakening the convexity assumption. Let us observe that if A is a closed subset of a normed linear space, then A is convex if and only if $A + A \subseteq 2A$, where

$$A + A := \{x + y : x, y \in A\} \quad \text{and} \quad \lambda A := \{\lambda x : x \in A\} \quad \text{for} \quad \lambda \in \mathbb{R}.$$

So the convexity assumption in Šmulian's theorem can be written in the following form:

$$A_n + A_n \subseteq 2A_n \quad \text{for any} \quad n \in \mathbb{N}.$$

We relax this condition in the following way:

$$\text{for any } n \in \mathbb{N}, \text{ there is } k \in \mathbb{N} \text{ such that } A_k + A_k \subseteq 2A_n. \tag{1.2}$$

Our main result says that if X is a superreflexive Banach space, then every decreasing sequence (A_n) of closed subsets of X satisfying (1.2) and such that there exists a bounded sequence (a_n) with $a_n \in A_n$, has a nonempty intersection. (For the definition of a superreflexive space, see, e.g., [1, p. 412].) Actually, our result is inspired by Goebel's [10] elementary proof of the fixed point theorem by Browder [3], Göhde [13] and Kirk [18], and particularly, by its slightly modified version given in [14, pp. 51–53] for Hilbert spaces.

We also show that in every infinite dimensional Banach space there exists a sequence (A_n) satisfying the assumptions of our theorem such that each A_n is noncompact, $\inf_{n\in\mathbb{N}} \operatorname{diam} A_n > 0$ and none of A_n contains a nontrivial segment.

Consequently, neither the compactness argument nor Cantor's theorem can be used to (A_n) to deduce the nonemptiness of the intersection, and also Šmulian's theorem is not applicable here.

In Section 3 we show that if sets A_n are defined by (1.1), where C is a nonempty convex subset of a normed linear space X, then (A_n) has property (1.2) if and only if T is *almost affine* in the sense that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\| T\left(\frac{x+y}{2}\right) - \frac{Tx+Ty}{2} \right\| < \varepsilon \quad \text{for any} \quad x, y \in F_\delta(T), \tag{1.3}$$

where $F_\delta(T) := \{x \in C : \| x - Tx \| \le \delta\}$, i.e., $F_\delta(T)$ is the set of all δ-fixed points of T. Hence we obtain a fixed point theorem for almost affine mappings which generalizes the Browder–Göhde–Kirk theorem since it can be shown that every nonexpansive mapping $T\colon C \to X$, where C is a nonempty bounded and convex subset of a uniformly convex Banach space X, is almost affine. Also, any mapping $T\colon C \to X$ of type (γ) in the sense of Bruck [4] (see also [11, p. 111]) is almost affine for an arbitrary Banach space X. Let us also note that by our Proposition 3.3, a mapping T is almost affine if and only if T is of 'convex type' in the sense of Khamsi [16, 17] who, however, considered only *nonexpansive* mappings of convex type.

At last our intersection theorem let us obtain a common fixed point theorem for a family of nonexpansive mappings (not necessarily commuting), which generalizes a corresponding result for a single mapping (see, e.g., [11, Proposition 10.2] or [1, Lemma 3.16]).

We close the paper with posing some questions. It seems that the following problem is particularly interesting: Let X be a Banach space in which every decreasing sequence (A_n) of nonempty closed and bounded subsets of X with property (1.2) has a nonempty intersection. Is X superreflexive?

2. Intersection theorem for superreflexive Banach spaces

We start with our main result of this section which was 'hidden' in Goebel's proof of the Browder–Göhde–Kirk theorem (see both proofs given in [10] and [14, pp. 51–53]). For $r > 0$, we denote by $B_r(0)$ the closed ball centered at 0 with radius r.

Theorem 2.1. *Let X be a superreflexive Banach space and (A_n) be a decreasing sequence of closed subsets of X such that for any $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that*

$$A_k + A_k \subseteq 2A_n.$$

Then the intersection $\bigcap_{n\in\mathbb{N}} A_n$ is nonempty closed and convex if and only if there exists a bounded sequence (a_n) such that $a_n \in A_n$ for each $n \in \mathbb{N}$.

Proof. Part 'only if' is obvious.

($\Leftarrow$): By the Enflo–James theorem (see, e.g., [1, p. 412]), there exists an equivalent uniformly convex norm $\| \cdot \|$ on X. Clearly, sets A_n are closed and (a_n) is still bounded with respect to the new norm. For $n \in \mathbb{N}$, set

$$\alpha_n := d(0, A_n) = \inf\{\| x \|\colon x \in A_n\}. \tag{2.1}$$

Since $0 \le \alpha_n \le \| a_n \|$, (α_n) is bounded. Since (A_n) is decreasing, (α_n) is increasing. Thus $\alpha_n \nearrow \alpha$ for some $\alpha \ge 0$. If $\alpha = 0$ then $\alpha_n = 0$ for each $n \in \mathbb{N}$, so $0 \in \bigcap_{n\in\mathbb{N}} A_n$ since each A_n is closed. So we assume further that $\alpha > 0$.

By (1.2), there is a sequence (k_n) of positive integers (not necessarily increasing) such that $A_{k_n} + A_{k_n} \subseteq 2A_n$ for each $n \in \mathbb{N}$. Set

$$p_1 := k_1 \quad \text{and} \quad p_{n+1} := \max\{p_n + 1, k_{n+1}\} \quad \text{for} \quad n \in \mathbb{N}.$$

Then (A_{p_n}) is a subsequence of (A_n) and

$$A_{p_n} + A_{p_n} \subseteq A_{k_n} + A_{k_n} \subseteq 2A_n$$

since $p_n \ge k_n$ and (A_n) is decreasing. For $n \in \mathbb{N}$, set

$$B_n := A_{p_n} \cap B_{\alpha+1/n}(0).$$

Since $\alpha_{p_n} \le \alpha < \alpha + 1/n$, we get that $\inf_{x\in A_{p_n}} \| x \| < \alpha + 1/n$, so there exists $x_n \in A_{p_n}$ such that $\| x_n \| < \alpha + 1/n$, which means that $x_n \in B_n$. Thus each B_n is nonempty. Moreover, sets B_n are closed and the sequence (B_n) is decreasing. We show that $\operatorname{diam} B_n \to 0$. Let $x, y \in B_n$. Then $\| x \| \le \alpha + 1/n$ and $\| y \| \le \alpha + 1/n$. Since $x, y \in A_{p_n}$, $(x+y)/2 \in A_n$, so $\| (x+y)/2 \| \ge \alpha_n$. Let η denote the inverse function to the modulus of convexity of $(X, \| \cdot \|)$. Then the above inequalities imply that (see, e.g., [25, Problem 10.1(a)])

$$\| x - y \| \le \left(\alpha + \frac{1}{n}\right) \eta \left(\frac{\alpha + 1/n - \alpha_n}{\alpha + 1/n}\right).$$

Hence we infer that

$$\operatorname{diam} B_n \le \left(\alpha + \frac{1}{n}\right) \eta \left(\frac{\alpha + 1/n - \alpha_n}{\alpha + 1/n}\right),$$

so $\lim_{n\to\infty} \operatorname{diam} B_n = 0$ since $\lim_{t\to 0^+} \eta(t) = 0$. By Cantor's intersection theorem, $\bigcap_{n\in\mathbb{N}} B_n \ne \emptyset$. Since

$$\bigcap_{n\in\mathbb{N}} B_n \subseteq \bigcap_{n\in\mathbb{N}} A_{p_n} = \bigcap_{n\in\mathbb{N}} A_n,$$

we get that $\bigcap_{n\in\mathbb{N}} A_n \ne \emptyset$. Finally, if $x, y \in \bigcap_{n\in\mathbb{N}} A_n$ then for any $n \in \mathbb{N}$, $x, y \in A_{p_n}$, so $(x+y)/2 \in A_n$ by property of (A_{p_n}). This proves the convexity of $\bigcap_{n\in\mathbb{N}} A_n$ since the intersection of A_n is closed. □

The referee observed that the assumption of Theorem 2.1 – 'there exists a bounded sequence (a_n) such that $a_n \in A_n$ for any $n \in \mathbb{N}$' – is equivalent to the boundedness of sequence (α_n) defined by (2.1). Here we can also add yet another equivalent condition: 'there exists $r > 0$ such that each set

$$B_n := A_n \cap B_r(0) \tag{2.2}$$

is nonempty'. Clearly, sets B_n are closed and bounded, (B_n) has property (1.2) (with A_n replaced by B_n) by convexity of $B_r(0)$, and $\bigcap_{n\in\mathbb{N}} B_n \subseteq \bigcap_{n\in\mathbb{N}} A_n$. This shows that Theorem 2.1 could easily be derived from the following particular version of it:

Proposition 2.2. *Let X be a superreflexive Banach space and (A_n) be a decreasing sequence of nonempty closed and bounded subsets of X. If for any $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $A_k + A_k \subseteq 2A_n$, then the intersection $\bigcap_{n\in\mathbb{N}} A_n$ is nonempty closed and convex.*

Theorem 2.1 and Proposition 2.2 are in some sense equivalent (as explained above), but Proposition 2.2 is better comparable with Šmulian's theorem than Theorem 2.1.

On the other hand, as also observed by the referee, in reflexive spaces, any decreasing sequence (A_n) of nonempty closed and convex subsets of X such that there exists a bounded sequence (a_n) with $a_n \in A_n$ for each $n \in \mathbb{N}$, has a nonempty intersection. This follows from Šmulian's theorem applied to sequence (B_n) defined by (2.2). Actually, we observe that also the convexity assumption can be relaxed here as done in the following theorem, which in turn is in some sense equivalent to Šmulian's theorem.

Proposition 2.3. *Let X be a normed linear space. The following statements are equivalent:*

(i) *X is a reflexive Banach space;*

(ii) *any decreasing sequence (A_n) of closed subsets of X such that for any $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ with $\operatorname{conv} A_k \subseteq A_n$ and there exists a bounded sequence (a_n) with $a_n \in A_n$ for each $n \in \mathbb{N}$, has a nonempty intersection.*

Proof. (ii)⇒(i) follows from Theorem 1.2.

(i)⇒(ii): Let sets B_n be defined by (2.2). Clearly, each B_n is nonempty closed and bounded, and (B_n) is decreasing. Fix $n \in \mathbb{N}$. By hypothesis, there is $k \in \mathbb{N}$ such that $\operatorname{conv} A_k \subseteq A_n$. Then

$$\operatorname{conv} B_k \subseteq \operatorname{conv} B_r(0) \cap \operatorname{conv} A_k = B_r(0) \cap \operatorname{conv} A_k \subseteq B_r(0) \cap A_n = B_n.$$

As in the proof of Theorem 2.1, we may infer that there exists a subsequence (B_{k_n}) of (B_n) such that $\operatorname{conv} B_{k_n} \subseteq B_n$. Set $C_n := \overline{\operatorname{conv}} B_{k_n}$. Clearly, $C_n \subseteq B_n \subseteq A_n$, so $\bigcap_{n\in\mathbb{N}} C_n \subseteq \bigcap_{n\in\mathbb{N}} A_n$. Since, by Theorem 1.2, $\bigcap_{n\in\mathbb{N}} C_n$ is nonempty, so is the intersection $\bigcap_{n\in\mathbb{N}} A_n$. □

Now we hope it is convenient for the reader to compare Proposition 2.3 ((i)⇒(ii)) with Theorem 2.1.

Also, the referee observed that in Theorem 2.1 condition (1.2) cannot be omitted. We can develop this remark as follows: in every infinite dimensional Banach space there exists a decreasing sequence of nonempty closed and bounded sets having the empty intersection. This is an immediate consequence of the following result of Chelidze [7].

Proposition 2.4 (Chelidze). *A Banach space X is finite dimensional if and only if every decreasing sequence of nonempty closed and bounded subsets of X has a nonempty intersection.*

Now we construct a sequence (A_n) of subsets of the closed unit ball in l_2 satisfying the assumptions of Theorem 2.1 such that each A_n is noncompact and $\operatorname{diam} A_n = \sqrt{2}$. Thus Cantor's theorem as well as the compactness argument cannot be applied to (A_n) to deduce the nonemptiness of the intersection. Moreover, none of A_n contains a nontrivial segment, so there does not exist a subsequence (A_{k_n}) such that $\operatorname{conv} A_{k_n} \subseteq A_n$. Consequently, also Proposition 2.3 is not applicable here.

Example 1. Set

$$C_1 := \{0,1\} \quad \text{and} \quad C_{n+1} := \frac{1}{2}(C_n + C_n) \quad \text{for} \quad n \in \mathbb{N},$$

so that $C_n = \{i/2^{n-1} : i = 0, 1, \ldots, 2^{n-1}\}$. Let B_1 denote the closed unit ball in the Hilbert space l_2. Define

$$A_1 := B_1 \cap \prod_{n\in\mathbb{N}} C_n.$$

Since $\prod_{n\in\mathbb{N}} C_n$ is compact, hence closed, in the product topology, we may infer that A_1 is closed in the norm topology. For $(x_n) \in l_2$, define

$$F(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots).$$

Clearly, F is an isometric isomorphism of l_2 onto $F(l_2)$, so the set $F(A_1)$ is closed and $F(B_1) \subseteq B_1$. Moreover, $F(A_1) \subseteq A_1$ since (C_n) is increasing and $0 \in C_1$. Hence, if we set

$$A_{n+1} := F(A_n) \quad \text{for} \quad n \in \mathbb{N},$$

then (A_n) is a decreasing sequence of closed subsets of B_1. By induction we show that

$$\frac{1}{2}(A_{n+1} + A_{n+1}) \subseteq A_n, \tag{2.3}$$

so (A_n) has property (1.2). Observe that if $x = (x_n) \in A_2$ and $y = (y_n) \in A_2$, then $(x_1 + y_1)/2 = 0 \in C_1$ whereas for $n \geq 2$, $x_n, y_n \in C_{n-1}$, so

$$\frac{x_n + y_n}{2} \in \frac{1}{2}(C_{n-1} + C_{n-1}) = C_n$$

by definition of C_n. Thus $(x+y)/2 \in \prod_{n\in\mathbb{N}} C_n$, and $(x+y)/2 \in B_1$ because of the convexity of B_1. This yields (2.3) for $n = 1$. Now assume that (2.3) holds for some $n \in \mathbb{N}$. Then, since F is linear, we get

$$\frac{1}{2}(A_{n+2}+A_{n+2}) = \frac{1}{2}(F(A_{n+1})+F(A_{n+1})) = F\left(\frac{1}{2}(A_{n+1} + A_{n+1})\right) \subseteq F(A_n) = A_{n+1},$$

which completes the induction.

We show that $\operatorname{diam} A_n = \sqrt{2}$. If $x, y \in A_n$, where $x = (x_k)$ and $y = (y_k)$, then $x_k, y_k \geq 0$, so $(x_k - y_k)^2 \leq x_k^2 + y_k^2$, which gives $\| x - y \| \leq \sqrt{2}$ since $x, y \in B_1$. Thus $\operatorname{diam} A_n \leq \sqrt{2}$. On the other hand, observe that $\{e_n, e_{n+1}, \ldots\} \subseteq A_n$, where $e_n := (\delta_{kn})_{k\in\mathbb{N}}$, so

$$\operatorname{diam} A_n \geq \| e_n - e_{n+1} \| = \sqrt{2}.$$

Finally, we show that none of A_n contains nontrivial segments. Let $x, y \in A_n$, $x \neq y$, and suppose, on the contrary, $\overline{xy} \subseteq A_n$. Let k be such that $x_k \neq y_k$. Then $k \geq n$ by definition of A_n. Since $\lambda x + (1-\lambda)y \in A_n$ for $\lambda \in [0,1]$, we get that $\lambda x_k + (1-\lambda)y_k \in C_{k-n+1}$ for all such λ, which yields a contradiction since C_{k-n+1} is finite.

Actually, we may adapt the above example to any infinite dimensional Banach space instead of l_2, which is done in the following

Proposition 2.5. *Let X be an infinite dimensional Banach space. Then there exists a decreasing sequence (A_n) of nonempty closed subsets of the closed unit ball in X having property (1.2) such that each A_n is noncompact and does not contain any nontrivial segment, and* $\operatorname{diam} A_n \geq 1$.

Proof. By Banach's theorem (see, e.g., [19, Theorem 1.a.5]), X contains a basic sequence (e_n), i.e., (e_n) is a Schauder basis of Y, the closed linear span of $\{e_n : n \in \mathbb{N}\}$. We may assume that (e_n) is normalized, i.e., $\| e_n \| = 1$ for each $n \in \mathbb{N}$. Let (e_n^*) be the sequence of biorthogonal functionals (cf. [19, p. 7]) associated to the basis (e_n), i.e., $e_n^* \in Y^*$ and $e_n^* e_m = \delta_{nm}$. Let $e_0^* \in Y^*$ be the zero functional on Y, and let sets C_n be defined as in Example 1. Now we reformulate the definition of A_n from that example as follows: for $n \in \mathbb{N}$,

$$A_n := B_1 \cap \bigcap_{k=0}^{n-1} \operatorname{Ker} e_k^* \cap \bigcap_{k=n}^{\infty} (e_k^*)^{-1}(C_{k-n+1}),$$

i.e., $x = \sum_{k=1}^{\infty} x_k e_k \in A_n$ if and only if $\| x \| \leq 1$, $x_k = 0$ for $k = 1, \ldots, n-1$ (if $n \geq 2$), and $x_k \in C_{k-n+1}$ for $k \geq n$. (This easily implies that $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$.) Since (C_n) is increasing, we may infer that (A_n) is decreasing. The continuity of e_k^* implies that each A_n is closed in the norm topology. It is easily seen that

$$\{0\} \cup \{e_n, e_{n+1}, \ldots\} \subseteq A_n,$$

so $\operatorname{diam} A_n \geq 1$ and A_n is noncompact. (Indeed, $(e_{n+k-1})_{k \in \mathbb{N}}$ is a sequence of elements of A_n, which does not contain a convergent subsequence.) A similar argument as in Example 1 shows that none of A_n contains a nontrivial segment. Finally, if $x, y \in A_{n+1}$, $x = \sum_{k=1}^{\infty} x_k e_k$ and $y = \sum_{k=1}^{\infty} y_k e_k$, then $(x_k + y_k)/2 = 0$ for $k = 1, \ldots, n$, and $x_k, y_k \in C_{k-n}$ for $k \geq n+1$, so for all such k,

$$\frac{x_k + y_k}{2} \in \frac{1}{2}(C_{k-n} + C_{k-n}) = C_{k-n+1}.$$

This yields $(x+y)/2 \in A_n$, so we get that $(1/2)(A_{n+1} + A_{n+1}) \subseteq A_n$. Thus (A_n) has property (1.2), so Theorem 2.1 is applicable to (A_n) if X is superreflexive. □

3. Applications to metric fixed point theory

We start with the following result which is a simple consequence of Šmulian's theorem.

Proposition 3.1. *Let X be a reflexive Banach space and C be a nonempty closed bounded and convex subset of X. Let a mapping $T\colon C \to X$ be continuous and affine. Then T has a fixed point if and only if* $\inf_{x \in C} \| x - Tx \| = 0$.

Proof. Part 'only if' is obvious. So let $\inf_{x \in C} \| x - Tx \| = 0$. Then sets A_n defined by (1.1) are nonempty, closed since T is continuous, convex since T is affine, and bounded since $A_n \subseteq C$. By Theorem 1.2, $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$, so T has a fixed point. □

Our first purpose is to obtain a generalization of Proposition 3.1 by weakening the assumption that T be affine: it suffices to assume that T is almost affine, i.e., it satisfies condition (1.3). Obviously, every affine mapping is almost affine and every mapping T such that $\inf_{x\in C} \| x - Tx \| > 0$ is trivially almost affine. Also, it is clear that the class of almost affine mappings is stable under equivalent renormings. Below we give a characterization of such mappings. We need the following folklore result (cf. [15, Lemma 1]).

Lemma 3.2. *Let X be a nonempty set, and φ, ψ be nonnegative real functions on $X \times X$. The following statements are equivalent:*

(i) *for any sequences (x_n) and (y_n) of elements of X,*

$$\varphi(x_n, y_n) \to 0 \quad \textit{implies that} \quad \psi(x_n, y_n) \to 0;$$

(ii) *for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $x, y \in X$,*

$$\varphi(x, y) \leq \delta \quad \textit{implies that} \quad \psi(x, y) \leq \varepsilon.$$

Proposition 3.3. *Let C be a nonempty convex subset of a normed linear space X and $T\colon C \to X$ be a mapping. Let sets A_n be defined by (1.1). The following statements are equivalent:*

(i) *T is almost affine;*

(ii) *for any sequences (x_n) and (y_n) of elements of C,*

$$\textit{if } \| x_n - Tx_n \| \to 0 \textit{ and } \| y_n - Ty_n \| \to 0, \textit{ then } \left\| T\left(\frac{x_n + y_n}{2}\right) - \frac{Tx_n + Ty_n}{2} \right\| \to 0;$$

(iii) *for any sequences (x_n) and (y_n) of elements of C,*

$$\textit{if } \| x_n - Tx_n \| \to 0 \textit{ and } \| y_n - Ty_n \| \to 0, \textit{ then } \left\| T\left(\frac{x_n + y_n}{2}\right) - \frac{x_n + y_n}{2} \right\| \to 0;$$

(iv) *for any $\varepsilon > 0$, there is $\delta > 0$ such that if $x, y \in F_\delta(T)$, then $(x + y)/2 \in F_\varepsilon(T)$;*

(v) *the sequence (A_n) has property (1.2).*

Proof. The equivalence of (i) and (ii) follows from Lemma 3.2 with

(3.1)

$$\varphi(x, y) := \max\{\| x - Tx \|, \| y - Ty \|\}, \quad \psi(x, y) := \| T((x + y)/2) - (Tx + Ty)/2 \|$$

for $x, y \in C$.

To prove (ii)$\Leftrightarrow$(iii) it suffices to show that if $\| x_n - Tx_n \| \to 0$ and $\| y_n - Ty_n \| \to 0$, then

$$\left\| T\left(\frac{x_n + y_n}{2}\right) - \frac{Tx_n + Ty_n}{2} \right\| \to 0 \quad \Leftrightarrow \quad \left\| T\left(\frac{x_n + y_n}{2}\right) - \frac{x_n + y_n}{2} \right\| \to 0.$$

This follows from the inequalities:

$$\begin{aligned} \left\| T\left(\frac{x_n + y_n}{2}\right) - \frac{x_n + y_n}{2} \right\| \; &\leq \; \frac{1}{2}(\| x_n - Tx_n \| + \| y_n - Ty_n \|) \\ &\quad + \left\| T\left(\frac{x_n + y_n}{2}\right) - \frac{Tx_n + Ty_n}{2} \right\|; \end{aligned}$$

$$\left\|T\left(\frac{x_n+y_n}{2}\right)-\frac{Tx_n+Ty_n}{2}\right\| \leq \frac{1}{2}(\| x_n-Tx_n \| + \| y_n-Ty_n \|) + \left\|T\left(\frac{x_n+y_n}{2}\right)-\frac{x_n+y_n}{2}\right\|.$$

The equivalence of (iii) and (iv) follows from Lemma 3.2 with φ defined by (3.1) and

$$\psi(x,y) := \| (x+y)/2 - T((x+y)/2) \| \quad \text{for} \quad x,y \in C.$$

(iv)⇒(v): Fix $n \in \mathbb{N}$. By (iv), for $\varepsilon := \alpha_n$, there is $\delta > 0$ such that

$$\frac{1}{2}(F_\delta(T)+F_\delta(T)) \subseteq F_\varepsilon(T) = A_n.$$

Since $\lim_{m\to\infty}\alpha_m = 0$, there is $k \in \mathbb{N}$ such that $\alpha_k < \delta$. Clearly, $A_k = F_{\alpha_k}(T) \subseteq F_\delta(T)$, so

$$\frac{1}{2}(A_k+A_k) \subseteq \frac{1}{2}(F_\delta(T)+F_\delta(T)) \subseteq A_n.$$

(v)⇒(iv): Fix $\varepsilon > 0$. Since $\lim_{m\to\infty}\alpha_m = 0$, there is $n \in \mathbb{N}$ such that $\alpha_n < \varepsilon$. By (v), there is $k \in \mathbb{N}$ such that $A_k + A_k \subseteq 2A_n$. Set $\delta := \alpha_k$. Then

$$\frac{1}{2}(F_\delta(T)+F_\delta(T)) = \frac{1}{2}(A_k+A_k) \subseteq A_n \subseteq F_\varepsilon(T),$$

so (iv) holds. □

Thus, by Proposition 3.3, continuous almost affine mappings are exactly these continuous mappings for which a sequence (A_n) defined by (1.1) satisfies the assumptions of our intersection theorem.

Corollary 3.4. *Let C be a nonempty closed convex subset of a Banach space X and $T\colon C \to X$ be a continuous mapping. If the fixed point problem for T is well posed, then T is almost affine. In particular, this is the case if $T(C) \subseteq C$ and T is strictly contractive.*

Proof. If $\| x_n - Tx_n \| \to 0$ and $\| y_n - Ty_n \| \to 0$, then $x_n \to x_*$ and $y_n \to x_*$, where $x_* = Tx_*$, so by continuity of T, $T((x_n+y_n)/2) \to Tx_*$ and $(Tx_n+Ty_n)/2 \to Tx_*$. Hence $\| T((x_n+y_n)/2) - (Tx_n+Ty_n)/2 \| \to 0$, so by Proposition 3.3, T is almost affine. □

Let us note that in view of the Reich–Zaslavski [22] result, Corollary 3.4 also applies to mappings which are contractive in the sense of Rakotch [20]. Other classes of mappings for which the fixed point problem is well posed are described in [15].

As a consequence of Proposition 3.3 and Theorem 2.1, we obtain the following fixed point theorem for almost affine mappings. We emphasize that the whole proof of it (including the proof of Theorem 2.1) is elementary; in particular, we do not use a weak compactness argument.

Theorem 3.5. *Let C be a nonempty closed and convex subset of a superreflexive Banach space X. Let a mapping $T\colon C \to X$ be continuous and almost affine. The following statements are equivalent:*

(i) *there exists a bounded sequence* (x_n) *of elements of* C *such that* $\| x_n - Tx_n \| \to 0$*;*

(ii) *the set* $\operatorname{Fix} T$ *is nonempty closed and convex.*

Proof. Implication (ii)⇒(i) is obvious.

(i)⇒(ii): Let sets A_n be defined by (1.1), and (x_n) be a sequence as in (i). Then there is a subsequence (x_{k_n}) such that $x_{k_n} \in A_n$ for $n \in \mathbb{N}$, so if we set $a_n := x_{k_n}$, then (a_n) is bounded and $a_n \in A_n$. Clearly, by continuity of T, each A_n is closed. Since T is almost affine, Proposition 3.3 yields that (A_n) has property (1.2). By Theorem 2.1, $\bigcap_{n\in\mathbb{N}} A_n$ $(= \operatorname{Fix} T)$ is nonempty closed and convex. □

Remark 1. A result related to Theorem 3.5 was obtained by Garcia-Falset et al. [9, Proposition 3.2]: Here C is assumed to be a nonempty convex and weakly compact subset of an arbitrary Banach space, and T is continuous and α-*almost convex*, i.e., for any $x, y \in C$ and any $\lambda \in [0, 1]$,

$$\| \lambda x + (1-\lambda)y - T(\lambda x + (1-\lambda)y) \| \le \alpha(\max\{\| x - Tx \|, \| y - Ty \|\}),$$

where a function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and strictly increasing, and $\alpha(0) = 0$. By Proposition 3.3, it is easily seen that every α-almost convex mapping is almost affine, but the latter class of mappings seems to be wider than the former.

As a particular case of Theorem 3.5, we get the following result for nonexpansive mappings (cf. [11, Proposition 10.2] or [1, Lemma 3.16]).

Corollary 3.6. *Let* C *be a nonempty closed bounded and convex subset of a uniformly convex Banach space* X*. Let a mapping* $T\colon C \to X$ *be nonexpansive. Then* T *has a fixed point if and only if* $\inf_{x\in C} \| x - Tx \| = 0$.

Proof. Clearly, X is superreflexive. By [11, Proposition 10.1], T satisfies condition (iii) of Proposition 3.3, so T is almost affine and the result follows from Theorem 3.5. □

Let us note that if T is a selfmap of C, then Corollary 3.6 can be proved more easily by using asymptotic centers; see [12, Theorem 5.2, p. 24].

Now following Bruck [4] (see also [11, p. 111]) we denote by Γ the set of strictly increasing, continuous and convex functions $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma(0) = 0$. Further, we extend Bruck's definition of mappings of type (γ): Given $c \in (0, 1)$, we say that a mapping $T\colon C \to X$ is *of type* (γ, c) if $\gamma \in \Gamma$ and for all $x, y \in C$,

$$\gamma(\| cTx + (1-c)Ty - T(cx + (1-c)y) \| \le \| x - y \| - \| Tx - Ty \| . \tag{3.2}$$

Then T is of type (γ) in the sense of Bruck [4] if T is of type (γ, c) for any $c \in (0, 1)$. It was shown in [4] that if C is a nonempty convex and weakly compact subset of a Banach space and $T\colon C \to C$ is of type (γ), then T has a fixed point. We give the following partial extension of this result.

Theorem 3.7. *Let* C *be a nonempty closed bounded and convex subset of a superreflexive Banach space, and a mapping* $T\colon C \to C$ *be of type* $(\gamma, 1/2)$*. Then* T *has a fixed point.*

Proof. It was shown in [4] that if T is of type (γ), then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in F_\delta(T)$, then $\lambda x + (1-\lambda)y \in F_\varepsilon(T)$ for any $\lambda \in [0,1]$. The same argument as in the proof of [4, Lemma 1.2] shows that if T is of type $(\gamma, 1/2)$, then T satisfies the above condition with $\lambda = 1/2$. Thus (iv) of Proposition 3.3 holds, so T is almost affine. Since by (3.2) (with $c = 1/2$) T is nonexpansive, [11, Lemma 3.1] implies that (i) of Theorem 3.5 is satisfied, so T has a fixed point. □

We close this section with a common fixed point theorem for a family of nonexpansive mappings, which is a generalization of Corollary 3.6.

Theorem 3.8. *Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of nonexpansive mappings, $T_\lambda \colon C \to X$ for $\lambda \in \Lambda$. Then $\{T_\lambda : \lambda \in \Lambda\}$ has a common fixed point if and only if for any $\varepsilon > 0$, $\{T_\lambda : \lambda \in \Lambda\}$ has a common ε-fixed point.*

Proof. Implication $(\Rightarrow)$ is obvious.

$(\Leftarrow)$: Denote by η the inverse function to the modulus of convexity of X. It is known (see, e.g., [25, pp. 476–477]) that if $T \colon C \to X$ is nonexpansive and $\varepsilon \in (0,1]$, then for any $x, y \in F_\varepsilon(T)$, $(x+y)/2 \in F_{a(\varepsilon)}(T)$, where

$$a(\varepsilon) := \max\{2\sqrt{\varepsilon}, (\operatorname{diam} C + \varepsilon)\eta(\sqrt{\varepsilon})\}.$$

Clearly, $\lim_{\varepsilon \to 0^+} a(\varepsilon) = 0$. It is important in this proof that function $a(\cdot)$ does not depend on T. For $n \in \mathbb{N}$, set

$$A_n := \bigcap_{\lambda \in \Lambda} \left\{ x \in C : \| x - T_\lambda x \| \leq \frac{1}{n} \right\}.$$

By hypothesis, $A_n \neq \emptyset$. Clearly, each A_n is closed and bounded, and (A_n) is decreasing. We show that (A_n) satisfies (1.2). Fix $n \in \mathbb{N}$. Since $\lim_{m \to \infty} a(1/m) = 0$, there is $k \in \mathbb{N}$ such that $a(1/k) \leq 1/n$. Assume that $x, y \in A_k$, i.e., $x, y \in F_{1/k}(T_\lambda)$ for any $\lambda \in \Lambda$. Then

$$\frac{1}{2}(x+y) \in F_{a(1/k)}(T_\lambda) \subseteq F_{1/n}(T_\lambda)$$

for any $\lambda \in \Lambda$, i.e., $(x+y)/2 \in A_n$. By Theorem 2.1, $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$ which completes the proof since $\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{\lambda \in \Lambda} \operatorname{Fix} T_\lambda$. □

Remark 2. The above proof shows that Theorem 3.8 can be generalized by considering any family $\{T_\lambda : \lambda \in \Lambda\}$ of continuous mappings having the property that for any sequences (x_n) and (y_n) of elements of C, $\sup_{\lambda \in \Lambda} \| x_n - T_\lambda x_n \| \to 0$ and $\sup_{\lambda \in \Lambda} \| y_n - T_\lambda y_n \| \to 0$ imply that $\sup_{\lambda \in \Lambda} \| (x_n + y_n)/2 - T_\lambda((x_n + y_n)/2) \| \to 0$. In particular, this is the case if all T_λ are nonexpansive or all T_λ are α-almost convex with a function α which does not depend on λ. The latter case may be compared with [9, Corollary 3.5] of Garcia-Falset et al. who considered a *commuting* family of α-almost convex mappings, but with functions α depending on a parameter.

4. Four problems

Our first question deals with superreflexivity.

Question 1. Let X be a normed linear space. Assume that every decreasing sequence (A_n) of nonempty closed and bounded subsets of X with property (1.2) has a nonempty intersection. Is X a superreflexive Banach space?

Clearly, by Šmulian's theorem, every normed linear space with the above property is a reflexive Banach space.

Our second question concerns in fact relations between nonexpansive mappings and almost affine mappings.

Question 2. Let C be a nonempty closed bounded and convex subset of a superreflexive Banach space $(X, \|\cdot\|)$. Let a mapping $T\colon C \to X$ be continuous and almost affine with $\inf_{x\in C} \| x - Tx \| = 0$. Is it true that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{\text{conv}} F_\delta(T) \subseteq F_\varepsilon(T)$?

If the answer is positive, then Theorem 3.5 can be proved with the help of Šmulian's theorem. Indeed, if we consider sets A_n defined by (1.1), i.e., $A_n = F_{\alpha_n}(T)$, then the above property of the family $\{F_\varepsilon(T) : \varepsilon > 0\}$ easily implies that there is a subsequence (A_{k_n}) such that for any $n \in \mathbb{N}$, $\text{conv} A_{k_n} \subseteq A_n$. By Proposition 2.3, $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$, and hence T has a fixed point. Thus, in such a case, there would be no need to use Theorem 2.1 in a proof of Theorem 3.5.

Therefore we would prefer the negative answer to Question 2. In this case a mapping T without the above property cannot be of type (γ) in view of [5, Theorem 1.2]. (Let us note that every superreflexive Banach space is B-convex, so indeed we may refer here to that result of Bruck.) Consequently, by [4, Lemma 1.1], such a mapping T cannot be nonexpansive with respect to any uniformly convex norm on X, which is equivalent to $\|\cdot\|$. Thus Corollary 3.6 could not be applied to such a T, but Theorem 3.5 would be applicable here.

The third question was mentioned in Remark 1 in connection with the definition of Garcia-Falset et al. [9].

Question 3. Does there exist an almost affine mapping, which is not α-almost convex for any continuous and strictly increasing function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$?

Finally, we pose a question concerning mappings of type (γ, c).

Question 4. Let $c \in (0, 1)$. Does there exist a mapping of type (γ, c), which is not of type (γ)?

A special case of this question with $c = 1/2$ is also interesting since Theorem 3.7 deals with such mappings.

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