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FRAGMENTABILITY AND SELECTIONS OF SET-VALUED MAPPINGS

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Dedicated to the memory of Francesco de Blasi

Abstract. We consider set-valued mappings whose range spaces contain, as dense subsets, fragmentable spaces and we give sufficient conditions for these mappings to possess a single-valued continuous selection that is defined on a residual subset of their domain.

1. Introduction

Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty images acting between the topological spaces X and Y. Starting with the classical results of Michael [29–31] a question that has received much interest is the following one: under what conditions does there exist a continuous single-valued mapping $f : X \to Y$ which is a *selection of* F, that is $f(x) \in F(x)$ for any $x \in X$. Apart from a purely theoretical interest this question has an important practical consequence: when studying concrete examples of set-valued mappings whose images are solutions to different problems, the existence of a continuous selection of such a mapping allows to obtain regularity properties of the corresponding solutions. Results concerning various conditions for the existence of continuous selections of set-valued mappings have been obtained in a number of papers, see for example the above cited papers of Michael, the papers [14, 15, 32, 34] and the monograph of D. Repovš and P.V. Semenov [35] as well as the references within.

There are however important particular cases of set-valued mappings F in which the mapping may have empty images, i.e. $F(x)$ may be the empty set for some $x \in X$ (see Section 4 below for examples). In such a situation the aim is to find conditions under which the selection will be determined on a, as big as possible, subset of X. More precisely, supposing that X is a Baire space, we will be interested in the following question: under which conditions is there a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y$ such that f is a selection of F on X_1?

One approach to tackle the latter question was proposed in [7, 8]: the setting deals with densely defined set-valued mappings $F : X \rightrightarrows Y$ such that F has closed graph and is *lower demicontinuous*–the latter means that for every open set $V \subset Y$ the interior of the closure of the set $F^{-1}(V) := \{x \in X : F(x) \cap V \neq \emptyset\}$ is dense in the closure of $F^{-1}(V)$. If one supposes, in addition, that Y contains a subset Y_1 which is completely metrizable and that the mapping F embraces Y_1 (see below

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for the precise definition) then it was shown in [7, 8], that there is a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y$ which is a selection of F on X_1. A particularly important case of when F embraces Y_1 is when $Y_1 = Y$. The usefulness of selections in such a setting was demonstrated in [7,8] with the applications of the above result to geometry of Banach spaces, approximation theory, variational principles in optimization and others.

An alternative approach for obtaining such densely defined selections using topological games was considered in [24] and a further study was done in [12, 23], where the case when the mappings are lower quasicontinuous was investigated.

In two subsequent papers [13, 33] the authors showed that, in order to have selections as above in the case when $Y_1 = Y$, instead of supposing that the range spaces Y is completely metrizable, it is enough to suppose that Y is fragmentable by a special metric: a topological space Y is *fragmentable* (see [18]) if it admits a metric d (not necessarily related to the original topology) such that for every $\varepsilon > 0$ and every nonempty set $A \subset Y$ there is a nonempty relatively open subset V of A such that $d - \text{diam}(V) < \varepsilon$. Every metrizable space is fragmentable by its own metric but there are important classes of non metrizable spaces which are fragmentable. The notion of fragmentability has turned to be very useful in the study of various properties in functional analysis and topology like, single-valuedness of set-valued mappings, existence of everywhere defined Borel selections, differentiability of convex functions, validity of variational principles in optimization and others. Characterization of the fragmentable spaces as well as their applications can be found in [16, 18, 20–22, 36, 37]. See also Section 3 for more detailed discussion of fragmentability.

Supposing Y to be fragmentable by a special metric, it was shown in [33] that every lower demicontinuous densely defined mapping $F : X \rightrightarrows Y$ with closed graph has a densely defined continuous selection. A similar result was proved for lower quasicontinuous mappings in [13]. We show in this paper that the latter results remain valid in the more general case of a subset Y_1 of Y which is fragmentable by a special metric and the mapping F embraces Y_1.

The rest of the paper is organized as follows. In the next section we give some preliminaries related to set-valued mappings. In Section 3 we present and prove our main results about the existence of densely defined continuous selections of set-valued mappings. One particular case of our main result reads as follows (see Theorem 3.3): Let $F : X \rightrightarrows Y$ be a lower demicontinuous closed graph mapping with a dense domain acting between a Baire space X and a regular space Y. Suppose that Y contains a subset Y_1, which is embraced by F and which is fragmented by a complete metric whose topology contains the original topology of Y_1. Then there exists a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y$ such that f is a selection of F on X_1. In the final Section 4 we give several applications, concerning the so-called minimal (or quasicontinuous) mappings as well as applications to variational principles in optimization.

2. Some preliminaries

In the sequel all topological spaces will be assumed to be at least Hausdorff. Let $F : X \rightrightarrows Y$ be a set-valued mapping between the topological spaces X and Y. The set

$$\mathrm{Dom}\,(F) := \{x \in X : F(x) \neq \emptyset\}$$

is used to designate the *domain* of the mapping F and the set

$$\mathrm{Gr}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

will denote its *graph*. The inverse mapping of F, which acts from Y into X, will be denoted by F^{-1}, and is defined as $F^{-1}(y) := \{x \in X : y \in F(x)\}$, $y \in Y$. For a set $A \subset X$, $F(A)$ means the image of A under F, that is $F(A) := \cup\{F(x) : x \in A\}$. Given $B \subset Y$ the two possible pre-images of B under F are designated by $F^{\#}(B) := \{x \in X : F(x) \subset B\}$ and $F^{-1}(B) := \{x \in X : F(x) \cap B \neq \emptyset\}$.

Recall that F is called *upper (resp. lower) semicontinuous at* $x_0 \in X$ if for every open set $V \subset Y$ with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) the set $F^{\#}(V)$ (resp. $F^{-1}(V)$) contains an open neighborhood of x_0. F is said to be upper (resp. lower) semicontinuous in X if it is so at any point of X.

We shall denote by Int$A$ the interior and by $\overline{A}$ the closure of a set A in the corresponding topological space. A weakening of the lower semicontinuity that will be useful for us (having in mind that some of the mappings F may have empty images) is the following one, which was considered in [7, 8]: F is called *lower demicontinuous in* X if for every open set V of Y the set $\mathrm{Int}\overline{F^{-1}(V)}$ is dense in $\overline{F^{-1}(V)}$. This notion is dual (with respect to the inverse mapping) to the notion of demiopenness: a mapping $F : X \rightrightarrows Y$ is called *demiopen* ([17]) if for every open set $U \subset X$ the set $\mathrm{Int}\overline{F(U)}$ is dense in $\overline{F(U)}$. It can be proved (see, e.g., Proposition 4.2 from [8]) that F is demiopen if and only if F^{-1} is lower demicontinuous.

For our further considerations we need also a notion that relates a mapping $F : X \rightrightarrows Y$ with a subset Y_1 of Y. It is said that the mapping F *embraces* Y_1 ([7,8]) if for every open set W of Y which contains Y_1 the set $\{(x, y) \in \mathrm{Gr}(F) : y \in W\}$ is dense in the graph $\mathrm{Gr}(F)$ (with the product topology in $X \times Y$). The following fact will be useful in the sequel.

Proposition 2.1. *Let $F : X \rightrightarrows Y$ be a set-valued mapping and Y be regular. Then F embraces $Y_1 \subset Y$ if and only if for every two open sets U of X and V of Y such that $(U \times V) \cap \mathrm{Gr}(F) \neq \emptyset$ the set $\overline{F(U)} \cap V \cap Y_1$ is nonempty.*

Proof. Suppose that F embraces Y_1 and U and V be as in the statement of the proposition. Since Y is regular, there is a nonempty open set $V' \subset Y$ such that $\overline{V'} \subset V$ and $(U \times V') \cap \mathrm{Gr}(F) \neq \emptyset$. Now, if we suppose that $\overline{F(U)} \cap \overline{V'} \cap Y_1$ is empty, this will mean that Y_1 is contained in the open set $Y \setminus (\overline{F(U)} \cap \overline{V'})$ and thus, according to the definition of embracement, there will be $(\bar{x}, \bar{y}) \in (U \times V') \cap \mathrm{Gr}(F)$ such that $\bar{y} \notin \overline{F(U)} \cap \overline{V'}$. The latter is a contradiction which completes the proof.

Conversely, suppose the property of the proposition and let W be an open set of Y which contains Y_1. Let $U \subset X$ and $V \subset Y$ be open sets in X and Y, respectively, such that $(U \times V) \cap \mathrm{Gr}(F) \neq \emptyset$. Then, according to the supposed property, the set $\overline{F(U)} \cap V \cap Y_1$ is nonempty. Let $y \in \overline{F(U)} \cap V \cap Y_1$. Since $Y_1 \subset W$ there will be some nonempty open set V' of Y such that $y \in V' \subset V \cap W$. In addition, $V' \cap F(U) \neq \emptyset$ and thus, there is some $\bar{x} \in U$ and $\bar{y} \in F(\bar{x})$ with $\bar{y} \in V'$. This entails $\bar{y} \in V \cap W$ and therefore, the proof is completed. □

The following easily proved facts can be found in [8]:

Proposition 2.2. *Let $F : X \rightrightarrows Y$ be a set-valued mapping which embraces $Y_1 \subset Y$ and Y be regular. Then $F(X) \subset \overline{Y}_1$.*

Proposition 2.3. *Let $F : X \rightrightarrows Y$ be a set-valued mapping and $Y_1 \subset Y$. Each of the following implies that F embraces Y_1:*

(a) *$F(X) \subset Y_1$;*
(b) *Y_1 is dense in Y and the mapping F is demiopen.*

3. Main results

In this section we give our main results about existence of residually defined selections. Let us remind that a set A of a topological space X is *residual in X* if its complement in X is of the first Baire category in X. Typical examples of residual sets are those that contain dense G_δ-subsets of Baire spaces X.

Before formulating our first result in this section, let us introduce another piece of terminology: we say that a metric d in a given topological space Y is *conditionally complete* if every d-Cauchy sequence is convergent in the original topology in Y.

Theorem 3.1. *Let $F : X \rightrightarrows Y$ be a lower demicontinuous mapping between a Baire space X and a regular topological space Y such that F has a closed graph and a dense domain. Suppose that Y contains a nonempty subset Y_1 which is embraced by F and, in addition, Y_1 considered with the inherited topology is fragmented by a conditionally complete metric d, whose metric topology contains the initial topology in Y_1. Then, there exists a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y_1$ such that f is a selection of F on X_1, that is, $f(x) \in F(x)$ for every $x \in X_1$. In particular, $X_1 \subset \mathrm{Dom}\,(F)$.*

Proof. We will call a couple (U, V) of nonempty open sets of X and Y, respectively, *admissible* if $\{x \in U : F(x) \cap V \neq \emptyset\}$ is dense in U. Observe that (X, Y) is admissible because $\mathrm{Dom}\,(F)$ is dense in X. With this terminology in hand, let us put $\gamma_0 := \{(X, Y)\}$ and let $\gamma = (\gamma_n)_{n\geq 0}$ be a sequence of families of couples of open sets of X and Y which is maximal with respect to the following properties: for every $n \geq 0$

(a) every $(U_n, V_n) \in \gamma_n$ is admissible;
(b) the family $\{U_n : (U_n, V_n) \in \gamma_n$ for some nonempty open $V_n \subset Y\}$ is disjoint;
(c) for every $(U_n, V_n) \in \gamma_n$ the set $B_n := \overline{F(U_n)} \cap V_n \cap Y_1$ is nonempty and $d - \mathrm{diam}(B_n) < 1/n$;
(d) for every $(U_{n+1}, V_{n+1}) \in \gamma_{n+1}$ there exists (unique, according to (b)) couple $(U_n, V_n) \in \gamma_n$ such that $U_{n+1} \subset U_n$ and $\overline{V}_{n+1} \subset V_n$.

Conditions (a)-(c) are satisfied for γ_0 because of Proposition 2.1 and the fact that $\mathrm{Dom}\,(F)$ is dense in X.

Let $H_n := \cup\{U_n : (U_n, V_n) \in \gamma_n$ for some open $V_n \subset Y\}$, $n \geq 0$. We claim that each H_n is (open) and dense in X. Indeed, for $n = 0$ this is true, therefore, let the claim be verified up to some $k \geq 0$ and suppose that H_{k+1} is not dense in X. Thus, for some open and nonempty set U of X we will have $U \cap H_{k+1} = \emptyset$. On the other hand, there is $(U_k, V_k) \in \gamma_k$ such that $U \cap U_k \neq \emptyset$. Without loss of generality we may think that $U \subset U_k$. Observe that the couple (U, V_k) is admissible. In particular, $F(U) \cap V_k \neq \emptyset$ and hence, according to Proposition 2.1, $\overline{F(U)} \cap V_k \cap Y_1 \neq \emptyset$. Since d fragments Y_1 we can find a nonempty open (in Y) set $V_{k+1} \subset V_k$ such that the set $B := \overline{F(U)} \cap V_{k+1} \cap Y_1$ is a nonempty set whose d-diameter is less than $1/(k+1)$. Because of regularity of Y we may think that, in addition, $\overline{V}_{k+1} \subset V_k$. Further, because evidently $F(U) \cap V_{k+1} \neq \emptyset$, we use the lower demicontinuity of F to find a nonempty open set U_{k+1} of X such that $U_{k+1} \subset U$ and the couple (U_{k+1}, V_{k+1}) is admissible. In particular, again by Proposition 2.1, the set $B_{k+1} := \overline{F(U_{k+1})} \cap V_{k+1} \cap Y_1$ is nonempty and since obviously $B_{k+1} \subset B$ we conclude that the $d - \mathrm{diam}(B_{k+1}) < 1/(k+1)$.

Now let $\gamma' := (\gamma'_n)_{n\geq 0}$ be a sequence of families of couples of open sets such that $\gamma'_n = \gamma_n$ for every $n \neq k+1$ and $\gamma'_{k+1} := \gamma_{k+1} \cup \{(U_{k+1}, V_{k+1})\}$. The sequence γ' is obviously strictly larger than γ and still satisfies (a)-(d) which is a contradiction with the maximality of γ. Therefore, each H_n, $n \geq 0$, is dense in X.

Let $X_1 := \cap_{n\geq 0} H_n$. Since X is a Baire space the set X_1 is a dense G_δ-subset of X. Take and fix an arbitrary $x \in X_1$. Because of the condition (b) and (d) there is a unique sequence $\{(U_n(x), V_n(x))\}_n$ such that $x \in U_n(x)$, $(U_n(x), V_n(x)) \in \gamma_n$, $U_{n+1}(x) \subset U_n(x)$ and $\overline{V}_{n+1}(x) \subset V_n(x)$ for every $n \geq 0$. Let B_n be the sets corresponding to $(U_n(x), V_n(x))$ from (c) above and set

$$f(x) := \bigcap_{n\geq 0} B_n = \bigcap_{n\geq 0} \overline{F(U_n(x))} \cap V_n(x) \cap Y_1, \qquad x \in X_1.$$

Since for each $n \geq 1$, $d - \mathrm{diam}(B_n) < 1/n$, the intersection above is no more than a singleton (belonging necessarily to Y_1). We will show that it is always nonempty. Indeed, because of (d) above we have $\cap_{n\geq 0} B_n = \cap_{n\geq 0}(\overline{B}_n \cap Y_1)$ and $d - \mathrm{diam}(\overline{B}_{n+1} \cap Y_1) < 1/n$, for every $n \geq 0$. And since the sets $\overline{B}_n \cap Y_1$ are closed in the inherited topology in Y_1 and the metric d is conditionally complete in Y_1, it can be seen that the intersection $\cap_{n\geq 0}(\overline{B}_n \cap Y_1)$ is a singleton (in Y_1). Therefore, f is well-defined single-valued mapping from X_1 into Y_1.

We show further that f is a selection of F on X_1. Suppose that for some $x_0 \in X_1$ we have $f(x_0) \notin F(x_0)$. Then $(x_0, f(x_0)) \notin \mathrm{Gr}(F)$. The graph of F is closed and Y is regular, therefore, there are open sets U of X and V of Y such that $(x_0, f(x_0)) \in U \times V$ but $(U \times \overline{V}) \cap \mathrm{Gr}(F) = \emptyset$. Since $f(x_0) \in V \cap Y_1$ and the metric topology on Y_1 contains the initial topology in Y_1 there is some $k \geq 1$ such that $B_k = \overline{F(U_k(x_0))} \cap V_k(x_0) \cap Y_1 \subset V$. On the other hand, the couple $(U_k(x_0), V_k(x_0))$ is admissible and, consequently, there is some $\bar{x} \in U \cap U_k(x_0)$ such that $F(\bar{x}) \cap V_k(x_0) \neq \emptyset$. Fix some $\bar{y} \in F(\bar{x}) \cap V_k(x_0)$. We will show that $\bar{y} \in \overline{V}$ and this will be a contradiction with $(U \times \overline{V}) \cap \mathrm{Gr}(F) = \emptyset$. Let $W \subset Y$ be a nonempty open set such that $\bar{y} \in W \subset V_k(x_0)$. By Proposition 2.1 the set $\overline{F(U_k(x_0))} \cap W \cap Y_1$ is nonempty. And since the latter set is included in $\overline{F(U_k(x_0))} \cap V_k(x_0) \cap Y_1$ which, on its turn, is a subset of V, we obtain $W \cap V \neq \emptyset$. This shows that $\bar{y} \in \overline{V}$.

It remains to show that the mapping f is continuous on X_1. To this end fix $x \in X_1$ and let V be an open set of Y which contains $f(x)$. Again by the fact that the topology determined by d on Y_1 contains the inherited topology on Y_1 there is some $k \geq 1$ such that $B_k = \overline{F(U_k(x))} \cap V_k(x) \cap Y_1 \subset V$. Let $x' \in U_k(x) \cap X_1$. Then, because of the condition (b), we will have $U_n(x') = U_n(x)$ and $V_n(x') = V_n(x)$ for any $n \geq k$. This entails that $f(x') = \cap_{n \geq 0} \overline{F(U_n(x'))} \cap V_n(x') \cap Y_1 \subset \overline{F(U_k(x))} \cap V_k(x) \cap Y_1 \subset V$ and, therefore, f is continuous at x. This completes the proof of the theorem. □

Remark 3.2. A close look at the proof of the above theorem shows that the constructed selection $f : X_1 \to Y_1$ is, in fact, continuous with respect to the topology determined by the metric d on Y_1. Observe also that our selection f takes its values in the given set Y_1 which sometimes is useful for the applications.

The following results are immediate corollaries from the previous theorem having in mind Proposition 2.3. The first one is also a consequence of Corollary 1 from [33].

Theorem 3.3. *Let $F : X \rightrightarrows Y$ be a lower demicontinuous mapping between a Baire space X and a regular topological space Y such that F has a closed graph and a dense domain. Suppose that Y is fragmented by a metric d which is conditionally complete and the metric topology contains the initial topology in Y. Then, there exists a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y$ such that f is a selection of F on X_1.*

Theorem 3.4. *Let $F : X \rightrightarrows Y$ be a demiopen lower demicontinuous mapping between a Baire space X and a regular topological space Y such that F has a closed graph and a dense domain. Suppose that Y contains a dense subset Y_1 such that the inherited topology on Y_1 is fragmented by a metric d which is conditionally complete and whose metric topology contains the initial topology of Y_1. Then, there is a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y_1$ such that f is a selection of F on X_1.*

Theorem 3.1 generalizes Theorem 4.7 from [8], where, under the same assumptions, the case when the subset Y_1 is completely metrizable was considered. Formally Theorem 3.4 also generalizes a result from [8], namely Theorem 4.9, in which, again under the same assumptions as in Theorem 3.4, the space Y_1 was considered to be completely metrizable. In fact, these two results are equivalent. Indeed, using the techniques developed by Ribarska [36] for characterization of the fragmentability, it can be proved (and this will be done elsewhere) that if a regular topological space Z is fragmented by a conditionally complete metric whose topology is stronger than the initial topology of Z, then Z is fragmentable by a complete metric whose topology again is stronger than the initial one. Hence, Theorem 3.4 can be also derived from Theorem 4.9 in [8], having in mind Proposition 2.3 and the following theorem.

Theorem 3.5. *Let Y be a regular topological space which is fragmented by a complete metric d, whose metric topology contains the initial topology in Y. Then Y contains a dense subspace Y_1 whose inherited (from Y) topology is completely metrizable.*

Proof. We will only sketch the proof. First, let us mention that if Y is fragmented by a complete metric, whose topology is stronger than the initial topology in Y, then one can prove, exactly as in the well-known Baire theorem for complete metric spaces, that the space Y with its original topology is a Baire one. Granted this, let $\beta_0 := \{Y\}$ and $\beta = (\beta_n)_{n \geq 0}$ be a sequence of families of open sets in Y which is maximal with respect to the following three properties: for every $n \geq 0$

(1) β_n is disjoint;
(2) $d - \mathrm{diam}(V_n) < 1/n$ for every $V_n \in \beta_n$;
(3) for every $V_{n+1} \in \beta_{n+1}$ there is unique $V_n \in \beta_n$ such that $\overline{V}_{n+1} \subset V_n$.

Put $O_n := \cup\{U_n : U_n \in \beta_n\}$, $n \geq 0$. One easily checks that each O_n is open and dense in Y and therefore, the set $Y_1 := \cap_{n\geq 0} O_n$ is a dense G_δ-subset of Y. It is routine matter now to check that on Y_1 the inherited topology by Y and the metric topology of d coincide. Now Y_1 is completely metrizable as a G_δ-subset of a complete metric space. □

The class of spaces Y fragmented by a complete metric whose topology contains the initial topology of Y is rather large. Apart from the obvious fact that every completely metrizable space has this kind of fragmentability, there are other non metrizable spaces which possess such a property. For instance, every *scattered topological space*, that is a topological space in which every nonempty subset A has an isolated point in inherited topology of A, is fragmented by a complete metric whose topology is finer than the initial one (this is the trivial metric on Y). Every fragmentable compact space is fragmentable by a stronger complete metric [36]. Other interesting non trivial classes of such spaces are certain subsets of Banach spaces: for example, all weak compact spaces in a Banach space are fragmentable by the norm and all weak star compact subsets of the dual of an Asplund space are fragmentable by the dual norm.

More examples of such spaces were exhibited in [9]: recall that a set H of a topological space is called *resolvable* (see e.g. [28]) if every nonempty set $A \subset Y$ contains a relatively open subset B such that either $B \subset H$ or $B \cap H = \emptyset$. Every open and every closed set is resolvable. Finite intersection, finite unions and complements of resolvable sets are resolvable as well. It was shown in [9] (see Propositions 3.7 and 3.8]) that if H_n, $n \geq 1$, are resolvable subsets of a countably compact space such that $H := \cap_n H_n$ is nonempty and fragmentable, then H is fragmentable by a stronger conditionally complete metric. In particular, all resolvable subsets of fragmentable compacta, as well as all nonempty G_δ-subsets of fragmentable compacta, are fragmentable by a conditionally complete metric whose metric topology contains the initial one (and hence, according to the remark above, fragmentable by a complete metric with stronger topology).

As it was pointed out in [8], Examples 4.10 and 4.11, all conditions supposed in the above theorems are essential and without one of them one cannot expect the existence of the obtained selections. A further study of sufficient conditions that assure densely defined selections of set-valued mappings was done in [12,23] and also in [13] where the case when the mapping F is with nonempty images was investigated (i.e., when $\mathrm{Dom}\,(F) = X$). It turned out that in such a situation, in order to obtain the same type of selections, one can drop the assumption of closedness of the graph provided a suitable semi-continuity property is used. Namely, the mapping $F : X \rightrightarrows Y$ is called *lower quasicontinuous* at $x_0 \in X$ if for every open set V of Y such that $F(x_0) \cap V \neq \emptyset$ there is an open set U such that $x_0 \in \overline{U}$ and $F(x) \cap V \neq \emptyset$ for each $x \in U$. Equivalently: if for every open set U containing x_0 and open set V of Y with $F(x_0) \cap V \neq \emptyset$ there is a nonempty open set $U' \subset U$ such that $F(x) \cap V \neq \emptyset$ for any $x \in U'$. The mapping F is lower quasicontinuous in X if it is lower quasicontinuous at any point of X. The corresponding notion of *upper quasicontinuity* of F is defined in an obvious way. If the mapping F is single-valued these notions coincide with the notion of a *quasicontinuous* single-valued mapping given by Kempisty [19] (the origins of the latter notion can be found in the works of Volterra (see [1])).

The notion of lower quasicontinuity allows us to prove the following result about existence of selections where the assumption of the closedness of the graph is replaced by closedness of the images.

Theorem 3.6. *Let $F : X \rightrightarrows Y$ be a lower quasicontinuous mapping between a Baire space X and a regular space Y such that F has nonempty closed images. Suppose that Y contains a nonempty set Y_1 which is embraced by F and such that Y_1 is fragmented by a conditionally complete metric d, whose metric topology contains the initial one in Y_1. Then there exists a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y_1$ such that f is a selection of F on X_1.*

Proof. The proof is similar to that one of Theorem 3.1. Call a couple (U, V) of open sets of X and Y *strongly admissible* if $F(x) \cap V \neq \emptyset$ for every $x \in U$ and construct exactly as above a sequence of families $\gamma = (\gamma_n)_{n\geq 0}$ which is maximal with respect to the conditions (a)-(d) where in (a) admissibility of the couples is replaced by the strong admissibility. Then the proof proceeds exactly as in the proof of Theorem 3.1 (with the only change to use lower quasicontinuity instead of lower demicontinuity when the set U_{k+1} is chosen) in order to show that the obtained mapping $f : X_1 \to Y_1$ is well-defined and continuous.

It remains only to see that in this new situation the mapping f is a selection of F as well. To see this, suppose that for some $x_0 \in X_1$ one has $f(x_0) \notin F(x_0)$. Since $F(x_0)$ is closed in Y and Y is regular there is some open set V of Y such that $f(x_0) \in V$ and $\overline{V} \cap F(x_0) = \emptyset$. With the same notation as in the proof of Theorem 3.1, take k so large that $\overline{F(U_k(x_0))} \cap V_k(x_0) \cap Y_1 \subset V$. Since $(U_k(x_0), V_k(x_0))$ is strongly admissible there is some $y_0 \in F(x_0) \cap V_k(x_0)$. We show that $y_0 \in \overline{V}$ and this will be a contradiction with $\overline{V} \cap F(x_0) = \emptyset$. Let W be an open set of Y such that $y_0 \in W \subset V_k(x_0)$. Proposition 2.1 shows that the set $\overline{F(U_k(x_0))} \cap W \cap Y_1$ is nonempty and since the latter is contained evidently in V we conclude that $W \cap V \neq \emptyset$. Consequently $y_0 \in \overline{V}$. □

The above theorem is a generalization of Theorem 3.3 from [23], where the subset Y_1 was supposed to be completely metrizable. The following two results are immediate consequences of the last theorem and Proposition 2.3. The first one was proved by Giles and Moors in [13] and in particular, is improvement of a result of Giles and Bartlett [12] in which, under the same assumptions, the set Y was supposed completely metrizable..

Theorem 3.7. *Let $F : X \rightrightarrows Y$ be a lower quasicontinuous mapping between a Baire space X and a regular space Y such that F has nonempty closed images. Suppose that Y is fragmented by a conditionally complete metric d and the topology of d contains the initial one in Y. Then there exists a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y$ such that f is a selection of F on X_1.*

Theorem 3.8. *Let $F : X \rightrightarrows Y$ be a demiopen lower quasicontinuous mapping between a Baire space X and a regular space Y, such that F has nonempty closed images. Suppose that Y contains a dense set Y_1 such that Y_1 is fragmented by a conditionally complete metric d whose metric topology contains the initial topology in Y_1. Then there exists a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y_1$ such that f is a selection of F on X_1.*

4. Some applications

In this section we will give some applications of our selection theorems. However, before considering a concrete setting, let us see some of the consequences of the theorems from the preceding for the class of minimal set-valued mappings.

A set-valued mapping $F : X \rightrightarrows Y$ is called *usco* if it is upper semicontinuous in X and with nonempty compact images. An usco $F : X \rightrightarrows Y$ is *minimal* if it is usco and its graph does not contain properly the graph of any other usco mapping between X and Y. It can be seen, using Zorn's lemma, that every usco mapping $F : X \rightrightarrows Y$ contains a minimal usco one. Examples of minimal usco mappings are some solution mappings (see further in this section), subdifferentials of convex functions, or more generally, certain monotone operators between a Banach space and its continuous dual. One of the important features of this class of mappings is that often one can prove that such a mapping is single-valued on a residual part of its domain. For example, if F us usco and the space Y is fragmentable by some metric d, then the mapping F is single-valued and d-upper semicontinuous at the points of a residual subset of X (see e.g. [36]).

Minimal usco mappings are characterized by the following property (see, e.g. [3,4]): for every open sets U and V of X and Y respectively, such that $F(U) \cap V \neq \emptyset$, there is some nonempty open $U' \subset U$ such that $F(U') \subset V$. The latter property was used (i.e. in [8,23] and elsewhere) as a definition of a *minimal mapping* $F : X \rightrightarrows Y$ for a mapping F which is not necessarily usco. Observe that such a mapping is both upper and lower quasicontinuous at any point x at which $F(x)$ is nonempty. Thus, it is a generalization of the notion of quasicontinuity of single-valued mappings and this is why sometimes the minimal mappings are called also *quasicontinuous mappings.* We will use the latter term in the sequel. Observe also that if F has a dense domain $\mathrm{Dom}\,(F)$ and is quasicontinuous in X, then it is also lower demicontinuous in X. Therefore, the quasicontinuous mappings satisfy the two continuity-like properties used in the above theorems. We will see that for such mappings the conclusions of our selection theorems can be strengthened by proving that the selections, in fact, coincide with the mapping itself at the points where the selections are defined.

Indeed, the following fact, which proof we give for completeness, is well-known (see, e,g. [8, 23]).

Proposition 4.1. *Let $F : X \rightrightarrows Y$ be a quasicontinuous mapping. Suppose that there exists a dense set X_1 of X and a single-valued mapping $f : X_1 \to Y$ which is a selection of F on X_1. If f is continuous at some $x_0 \in X_1$, then $F(x_0)$ is a singleton, necessarily equal to $\{f(x_0)\}$. If, in addition, Y is regular, then F is also upper semicontinuous at x_0.*

Proof. Suppose first that there is some $y_0 \in F(x_0)$ such that $y_0 \neq f(x_0)$. Take open sets V_1 and V_2 with $y_0 \in V_1$, $f(x_0) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Because of the continuity of f at x_0 there is some open set U of x_0 such that $f(x) \in V_2$ for every $x \in U \cap X_1$. On the other hand, because of the quasicontinuity of F there is some nonempty open $U' \subset U$ such that $F(U') \subset V_1$. Since X_1 is dense in X the set $U' \cap X_1$ is nonempty and this is a contradiction because it would yield $f(x) \in V_1$ for every $x \in U' \cap X_1$. Therefore, $F(x_0) = \{f(x_0)\}$.

Let now Y be regular and take some open set V of Y such that $F(x_0) \subset V$. Let W be nonempty and open with $F(x_0) = \{f(x_0)\} \subset W$ and $\overline{W} \subset V$. There is some open set U of x_0 such that $f(x) \in W$ for every $x \in U \cap X_1$. We claim that $F(x) \subset \overline{W}$ for each $x \in U$ which will prove the upper semicontinuity of F at x_0. Suppose the contrary, that for some $x \in U$ there is $y \in F(x) \setminus \overline{W}$. Take an open set O containing y and disjoint from $\overline{W}$. By the quasicontinuity of F there is some nonempty open set $U' \subset U$ such that $F(U') \subset O$. And this is again a contradiction because $U' \cap X_1 \neq \emptyset$ and f maps the latter set into W. □

Having the latter property the following results are immediate consequences of our Theorems 3.1 and 3.6.

Theorem 4.2. *Let $F : X \rightrightarrows Y$ be a quasicontinuous mapping with closed graph and dense domain acting between a Baire space X and a regular space Y. Let Y contain a subset Y_1 which is embraced by F and the space Y_1 is fragmented by a conditionally complete metric whose metric topology contains the initial topology of Y_1. Then there is a dense G_δ-subset X_1 such that at the points of X_1 the mapping F is single-valued and upper semicontinuous. Moreover, $F(x) \in Y_1$ for each $x \in X_1$.*

Theorem 4.3. *Let $F : X \rightrightarrows Y$ be a quasicontinuous mapping with nonempty closed images acting between a Baire space X and a regular space Y. Let Y contain a subset Y_1 which is embraced by F and the space Y_1 is fragmented by a conditionally complete metric and the metric topology of d contains the initial topology of Y_1. Then there is a dense G_δ-subset X_1 of X such that at the points of X_1 the mapping F is single-valued and upper semicontinuous. Moreover, $F(x) \in Y_1$ for each $x \in X_1$.*

The last result is a generalization of Theorem 3.4 from [23], where complete metrizability of the space Y_1 was supposed. From the obvious consequences of the above two theorems that can be derived using Proposition 2.3 we will give only the next one which will be used in the sequel.

Theorem 4.4. *Let $F : X \rightrightarrows Y$ be a quasicontinuous mapping with closed graph and dense domain acting between a Baire space X and a regular space Y. Suppose that Y is fragmented by a conditionally complete metric and the metric topology in Y contains the initial one. Then there is a dense G_δ-subset X_1 of X such that at the points of X_1 the mapping F is single-valued and upper semicontinuous.*

Remark 4.5. In fact, in the above theorem, we can assure a stronger property, namely that the mapping F is upper semicontinuous at the points of X_1 with respect to the metric d in Y. This is done as follows: in the construction of the selection in Theorem 3.1, using the quasicontinuity of the mapping F, one can construct the sets U_n in such a way that $F(U_n) \subset V_n$. It is easily seen that with this choice of the sets U_n the mapping F will be d-upper semicontinuous (and single-valued) at the points of the set X_1.

Let us see now another application related to variational principles in optimization. Let X be a completely regular topological space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function in X which is bounded from below, lower semicontinuous and *proper*. The latter means that f is not identically equal to $+\infty$. Consider also the space $C(X)$ of all real-valued continuous and bounded functions in X which, equipped with the usual sup-norm $\|g\|_\infty := \sup_{x\in X} |g(x)|$, $g \in C(X)$, is a Banach space. Since the given function f is not, a priori, supposed to have a minimum in X, the question that is of interest in optimization and analysis is whether there exists a function $g \in C(X)$ such that $f + g$ attains its infimum in X. A better property would be if there exists a function $g \in C(X)$, as small with respect to the norm, as we want, with the same property. I.e., when there is a small enough perturbation of the function f by a function from $C(X)$ such that the perturbed function attains its infimum in X.

The latter question has a positive answer if the set $\{g \in C(X) : f + g$ attains its infimum in $X\}$ is dense in $C(X)$, and the validity of such a property is an example of the so-called *variational principles in optimization*. Sometimes the question is strengthened so that the perturbation has a strong minimum. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ has a *strong minimum* if it has a unique minimum and every minimizing sequence for h converges to this unique minimum. Having strong minimum for h in X is known in optimization as: *the problem to minimize h in X is Tykhonov well-posed*. Another property of interest is when the set of good perturbations is not only dense in the space $C(X)$ but also contains a dense G_δ-subset of the space $C(X)$. Examples of such principles are, e.g., the Ekeland variational principle [11], the smooth variational principles of Borwein and Preiss [2] and of Deville-Godefroy-Zizler [10], Stegall variational principle [38], the continuous principle of Čoban-Kenderov [5, 6] and others.

When the function f is continuous, conditions under which the set $E(f) := \{g \in C(X) : f + g$ attains its minimum in $X\}$ or the set $S(f) := \{g \in C(X) : f + g$ attains its strong minimum in $X\}$ are residual in $C(X)$ were investigated in [5,6,25]. We also used the selection theorems from [7, 8] to see how to ensure some of these conditions. Recently, in the case when f is not necessarily continuous, we obtained sufficient conditions for the residually of the sets $E(f)$ and $S(f)$ which were in terms of topological games in [9, 26] and by using fragmentability of the underlying space in [27]. We will see how some of our theorems, obtained in this article, can be used to derive the corresponding variational principles when the underlying space is fragmentable.

For a given f as above set $M_f(g) := \{x \in X : x$ is a minimum of $f + g$ on $X\}$, $g \in C(X)$. M_f is a set-valued mapping between $C(X)$ and X which puts into correspondence to each $g \in C(X)$ the set of minimizers (possibly empty) of the perturbation $f + g$. It was shown in [9], Proposition 2.4, that the mapping M_f has a closed graph and is quasicontinuous. Moreover, it was proved in [26] that M_f has a dense domain. The latter is a consequence of the following useful lemma that was obtained again in [26] and whose proof we sketch here for completeness:

Lemma 4.6 ([26, Lemma 2.1])**.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which is proper and bounded from below. Let $x_0 \in X$ and $\varepsilon > 0$ be such that $f(x_0) < \inf_X f + \varepsilon$. Then there is a continuous bounded function $g : X \to \mathbb{R}^+$ such that $\|g\|_\infty < \varepsilon$ and $f + g$ attains its infimum at x_0.*

Proof. Set $\delta := f(x_0) - \inf_X f$ (we may suppose that $\delta > 0$) and consider the sets

$$L_n := \Big\{x \in X : f(x) \leq \inf_X f + \delta - \frac{\delta}{2^n}\Big\}, \qquad n \geq 1.$$

The sets L_n are nonempty and closed, increasing (by inclusion) with n and such that $x_0 \notin L_n$ for each $n \geq 1$. For every $n \geq 1$, let $h_n : X \to [0, 1]$ be a continuous function such that $h_n(x_0) = 0$ and $h_n|_{L_n} \equiv 1$ and consider the (well-defined and continuous) function $h(x) = \sum_{n=1}^{\infty}(1/2^n)h_n(x)$, $x \in X$.

The function $g := \delta h$ is continuous and bounded and $\|g\|_\infty \leq \delta < \varepsilon$. Therefore, to complete the proof, we need to check that $(f + g)(x) \geq (f + g)(x_0)$ for every $x \in X$.

To verify the latter inequality for each fixed $x \in X$ one considers two cases:

Case 1. $f(x) \geq \inf_X f + \delta = f(x_0)$ (this is easy since g is positive and $g(x_0) = 0$); and

Case 2. $f(x) < \inf_X f + \delta = f(x_0)$.

The second case needs more careful attention–to verify the inequality in this situation one uses the fact that in this case $x \in \cup_{n=1}^{\infty} L_n$ and finds the smallest integer $k \geq 1$ for which $x \in L_k$. To finish, one has to take into account the definition of g (and h), the fact that $x \in L_n$ for each $n \geq k$ and that $f(x) \geq \inf_X f + \delta - (\delta/2^{k-1})$. □

With the above properties of M_f in hand, we can apply Theorem 4.4 to obtain the following result from [9], where it was proved via an approach involving topological games.

Theorem 4.7 ([9, Corollary 3.7])**.** *Let X be a completely regular topological space which is fragmented by a conditionally complete metric d whose metric topology contains the initial topology in Y. Then, for every lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, which is proper and bounded from below, the set $S(f) = \{g \in C(X) : f + g$ attains a strong minimum in $X\}$ contains a dense G_δ-subset of $C(X)$.*

Proof. According to Theorem 4.4 and the properties of M_f listed above the mapping M_f is single-valued and upper semicontinuous at the points of a residual subset H of $C(X)$. On the other hand, it can be checked that the upper semicontinuity of M_f at any $g \in H$ entails that the minimum of $f + g$ is strong. Indeed, let $x_0 \in X$ be the unique minimum of $f + g$, $g \in H$, and $(x_k)_{k\geq 1}$ be a minimizing sequence for $f + g$. We may think that $\varepsilon_k := (f + g)(x_k) - \inf_X(f + g)$ is strictly positive for every $k \geq 1$. Take some open set V containing $\{x_0\} = M_f(g)$. By Lemma 4.6 for every $k \geq 1$ there is $g_k \in C(X)$ such that $\|g_k\|_\infty < \varepsilon_k$ and $x_k \in M_f(f + g + g_k)$. Since $\varepsilon_k \to 0$ we obviously have $g + g_k \to g$ and thus by the upper semicontinuity of M_f at g it follows that $x_k \in V$ for large k. This completes the proof. □

Under the assumption that the metric d is complete the result above was derived in [27, Theorem 2.3] as a consequence of a general variational principle proved in the same paper.

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