Journal of Nonlinear and Convex Analysis
Volume 16, Number 6, 2015, 1083–1095

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SIMULTANEOUS FARTHEST POINTS OF SETS IN BANACH SPACES: A SURVEY

JÓZEF MYJAK AND PIER LUIGI PAPINI

This paper is dedicated to the memory of F.S. de Blasi.*

ABSTRACT. We discuss and compare the different notions of simultaneous farthest point and of simultaneous remotality for a set. We review the existing literature and we indicate new relations among them. Note that, as we shall indicate, some papers in the area contain statements that are not correct.

1. INTRODUCTION AND DEFINITIONS

Let X be a Banach space over the real field $\mathbb{R}$, B_X the closed unit ball of X and θ the origin of X. As usual $\mathbb{N}$ stands for the set of natural numbers.
Given a bounded closed set $D \subset X$ and a point $x \in X$, we say that $y_0 \in D$ is a *farthest point from x in D* if

$$\|x - y_0\| \geq \|x - y\| \ \forall\, y \in D.$$

A set D is said to be *remotal* if every $x \in X$ has in D (at least) one farthest point. We say that D is *uniquely remotal* ((ur) for short), if every x has a unique farthest point in D.

Farthest points in Banach spaces have been studied in depth along the last half century. Notwithstanding many efforts made, for example the following natural question remains open (see e.g. [16]):

Problem. Must a uniquely remotal set be a singleton?

In this paper we shall consider some extensions of the above notions, by considering, instead of a single point x, a set A. This leads to different notions of "simultaneous worst approximation". To this end, we introduce some further notations.

Let $A \subset X$ and $y \in X$; set
$d(y, A) = \inf\{\|x - y\| : x \in A\}$.
Take A bounded; we denote its *diameter* by $\delta(A) := \sup\{\|x - y\| : x, y \in A\}$.
If $D \subset X$ is a closed set, we put
$r(A, y) = \sup\{\|y - x\| : x \in A\}$;
$r(A, D) = \inf\{r(A, y) : y \in D\}$;
$F_A(y) = \{x \in A : \|x - y\| = r(A, y)\}$.
Moreover, if D is also bounded, we put:

2010 *Mathematics Subject Classification.* 41A28, 41A65, 46B99.
Key words and phrases. Farthest, remotal, simultaneous remotal, distant sets.

*Francesco was not only a colleague, but a very good friend: we met frequently, we worked together, we shared ideas; his views of the real world were always deep. Nearest and farthest points in Banach spaces was one of the many subjects he liked and that we discussed together.

$\mu(A,D) = \sup\{r(A,y) : y \in D\}$;
$Q_D(A) = \{y \in D : r(A,y) = \mu(A,D)\}$.

Note that $\mu(A,D) = \mu(D,A)$, $\mu(A,A) = \delta(A)$, while $y \in Q_A(A)$ means that y is a *diametral point* of A.
Recall that $d(y,A)$ is called the *distance from y to A*, $r(A,y)$ the *radius of A from y*, $r(A,D)$ the *radius of A from D*. F_A is the *farthest point mapping*, i.e. the map associating with $y \in X$ the set of its farthest points in A.

In Section 2 we shall indicate the different generalizations of the notions of farthest points and remotal sets. In Section 3 we make some remarks and we give several examples. An overview of the literature on the subject will be presented in Section 4, while in Section 5 we discuss some other possible extensions.
Finally, to enlighten the recent directions of research in the area, an Appendix (Section 6) will survey recent results concerning general problems on farthest points and remotality.

2. Simultaneous remotality

If instead of a single point x we consider a pair of points, the previous definition of farthest point can be extended in several ways. In particular, if $A = \{x_1, x_2\}$ and D is a bounded set, we can consider the following concepts of "simultaneous remotality" for $y_0 \in D$:

(c) $\min\{\|y_0 - x_1\|, \|y_0 - x_2\|\} = \sup_{y\in D} \min\{\|y - x_1\|, \|y - x_2\|\}$;

(C) $\max\{\|y_0 - x_1\|, \|y_0 - x_2\|\} = \sup_{y\in D} \max\{\|y - x_1\|, \|y - x_2\|\}$;

(C_p) $\|y_0 - x_1\|^p + \|y_0 - x_2\|^p = \sup_{y\in D}\{\|y - x_1\|^p + \|y - x_2\|^p\}$ $(1 \le p < \infty)$;

note that (C) is nothing else than (C_∞).

Of course, the above conditions can be extended, in an obvious way, to $A = \{x_1, ..., x_n\}$, $n \in \mathbb{N}$. Moreover, condition (c) can trivially be extended to any closed set A, and the condition (C) to any closed and bounded set A. More precisely, we can look for $y_0 \in D$ such that:

(c) $d(y_0, A) = \sup\{d(y,A) : y \in D\}$;

(C) $r(A, y_0) = \sup\{r(A,y) : y \in D\}$.

Observe that conditions (c) and (C) can be formulated in the equivalent form:

(c') $\inf_{x\in A} \|y_0 - x\| = \sup_{y\in D} \inf_{x\in A} \|y - x\|$;

(C') $\sup_{x\in A} \|y_0 - x\| = \sup_{y\in D} \sup_{x\in A} \|y - x\| = \mu(A,D)$.

Concerning condition (c): to avoid trivialities, we must of course assume that D is not contained in A. The stronger condition $A \cap D \neq \emptyset$ is introduced in [22] and in [30], but probably such restriction is not really useful. Note also that the points $y_0 \in D$ satisfying (C) form the set $Q_D(A)$.

For a bounded closed set A we can also look for $y_0 \in D$ such that

$$(\mathrm{C}_1)\quad r(y_0, A) + d(y_0, A) = \sup_{y \in D}\{r(y, A) + d(y, A)\},$$

or equivalently

$$(\mathrm{C}_1)\quad \sup_{x \in A}\|y_0 - x\| + \inf_{x \in A}\|y_0 - x\| = \sup_{y \in D}\Big\{\sup_{x \in A}\|y - x\| + \inf_{x \in A}\|y - x\|\Big\}.$$

If $A = \{x_1, x_2\}$ the last condition has the form

$$\|y_0 - x_1\| + \|y_0 - x_2\| = \sup_{y \in D}\big\{\|y - x_1\| + \|y - x_2\|\big\}.$$

Note that this is just condition (C_p) for $p = 1$. In general we cannot obtain a similar extension for (C_p) if $1 < p < \infty$.

A point $y_0 \in D$ satisfying one of the above conditions is called a *simultaneous farthest point* for A.

A set D for which simultaneous farthest points to A exist is called *simultaneously remotal* (shortly (sr)) *for* A. A set D for which simultaneous farthest points exist for any bounded set A, is called *simultaneously remotal* (shortly (sr)). If moreover farthest points are always unique, we speak of *unique simultaneous remotality* (shortly (usr)). Some authors speak, in this case, of *simultaneous F-Chebyshev sets* (see [30]).

The meaning of the above conditions is the following. Concerning (c): find points in D which are farthest from A. Concerning (C): find points in D from which it is most difficult to cover A with a ball centered at one of them.

All the above conditions, apart from (C_1) for infinite sets, have been considered in the literature. Of course, when A is a singleton, they reduce to the notion of farthest point.

Given in X the sets D, A (not necessarily bounded), the quantity

$$e(D, A) = \sup\big\{d(y, A) : y \in D\big\} = \inf\big\{\varepsilon > 0 : A \subset D + \varepsilon B_X\big\}$$

is called the *excess* or *deviation* of D from A. Recall also that

$$h(A, D) = \max\{e(A, D),\, e(D, A)\}$$

defines the *Hausdorff distance* between A and D.

The quantity $e(D, A)$, together with the question concerning existence of elements in D for which the sup is attained, had already been considered in 1963 by V.M. Tihomirov: see [46, Section 6.4] for a discussion on this, mainly for A a subspace.

If D is compact, the existence results are rather obvious. The problem becomes interesting only in the case of infinite-dimensional spaces. But in general in such framework it is difficult to obtain nice results (see for example Theorem A^+ in the Appendix).

It is known (and easy to see) that, also if X is infinite dimensional, the set B_X is remotal and, if X is strictly convex, uniquely remotal for all points different from θ. In general B_X is not simultaneously remotal (see Examples 3.4 and 3.5 in the next section).

We do not even know the answer to the following natural question.

Problem. Give an example of a not compact, simultaneous remotal set (if it exists).

Note that in general, the existence of a simultaneous farthest point $y_0 \in D$ for A does not imply that $\sup_{x\in A}\|y_0 - x\|$, or $\inf_{x\in A}\|y_0 - x\|$ is attained.

The closed bounded sets A and D are called *distant sets* if:

(rp) There exist $a_0 \in A$ and $y_0 \in D$ such that $\|a_0 - y_0\| \geq \|a - y\|$ for all $a \in A$ and $y \in D$, i.e. $\|a_o - y_o\| = \mu(A, D)$.

In this case we say that (a_0, y_0) forms a *remotal pair*. Note that the concept of remotal pair is different from the following concept of mutually farthest points. The points $a_0 \in A$ and $y_0 \in D$ are called *mutually farthest points* for A and D if y_0 is a farthest point to a_0 from D and a_0 is a farthest point to y_0 from A. Obviously if (a_0, y_0) forms a remotal pair for A and D, then a_0 and y_0 are mutually farthest points for A and D; but the converse is not necessarily true: see the next example. So the term of mutually farthest points for A and D, sometimes used to denote a remotal pair, is misleading.

Example 2.1. Consider, in the Euclidean plane, the (closed, convex, bounded) sets: $A = \{(x, 0) : -1 \leq x \leq 2\}$ and $D = \{(x, 1) : 0 \leq x \leq 2\}$.
Let $a_0 = (2, 0)$, $y_0 = (0, 1)$. Then $\|a_0 - y\| \leq \|a_0 - y_0\|$ for every $y \in D$ and $\|a - y_0\| \leq \|a_0 - y_0\|$ for every $a \in A$. Moreover $\|a_0 - y_0\| = \sqrt{5} < \|(\text{-}1, 0) - (2, 1)\| = \sqrt{10}$. Thus a_0 and y_0 are mutually farthest points for A and D in the sense of the above definition, but only $\big((\text{-}1, 0), (2, 1)\big)$ is a remotal pair for A and D.

Condition (rp) implies the existence of simultaneous farthest points, according to (C), from D to A, and vice versa. For an example of two remotal sets which do not satisfy (rp), in a Hilbert space, see [10].

Not so many and not so deep results have been given concerning this notion (see for example [14]); but it has been discussed in many papers dealing with generic results (see at the end of Section 4). For $A = D$, a remotal pair reduces to a *diametral pair* for the set: for results concerning this notion see the Appendix.

3. Some remarks and examples

It is well known that convexity of sets plays a key role in optimization (for example, in best approximation problems). We cannot say the same concerning remotal sets; but it plays some role also in this context. For example, if we consider closed convex sets A and D, then, denoting by $\partial(A)$, resp. $\partial(D)$, their boundary, we have:
$$\sup_{y\in D} d(y, A) = \sup_{y\in \partial D} d(y, \partial A).$$
This fact (not valid in general when the assumption of convexity is dropped) was proved first in [25], then again in [48].

Since
$$\sup_{y\in D}\inf_{x\in A}\|y - x\| \leq \inf_{x\in A}\sup_{y\in D}\|y - x\|,$$
for a point y_0 satisfying condition (c) we have $d(y_0, A) \leq r(D, A)$.

When a minimax theorem applies, we have equality. This shows that condition (c) seems to be tied to a simultaneous approximation problem. So we think that condition (C), or condition (C_p) for A a finite sets, are more apt to deal with

simultaneous remotality. This seems to appear also by comparing Proposition 3.6 below (dealing with (c)) with Example 3.8.
Note that $e(D,A) \leq r(D,A)$ and in general equality does not hold. For example, for the sets A and D in Example 2.1 (given in the Euclidean plane) we have: $e(D,A) = \sup_{y\in D} \inf\{\|x-y\| : x \in A\} = 1 < \sqrt{2} = \inf_{x\in A} \sup\{\|x-y\| : y \in D\} = r(D,A)$. The same estimates hold if, instead of A, we consider the line $M = \{(x,0) : x \in \mathbb{R}\}$. Unfortunately, some authors used such untrue equality. For example, in [17, Lemma 3.1] (but not only there), the following false "result" is indicated and used: if M is a proximinal subspace of X, then for every bounded set $S \subset X$ we have $e(S,M) = r(S,M)$.

For bounded sets A, A' and D, we have the following inequalities:

$$\mu(A,D) \geq \max\{r(D,A), r(A,D)\} \geq h(A,D);$$

$$\mu(A,D) \leq \delta(A) + e(D,A);\ \mu(A,D) \leq \delta(D) + e(A,D), \quad \text{so}$$

$$\mu(A,D) \leq \min\{\delta(A), \delta(D)\} + h(A,D);$$

$$|\mu(D,A') - \mu(D,A)| \leq h(A,A').$$

It is clear that if $y_0 \in D$ satisfies (c) and (C), then it satisfies also (C_1), but not necessarily condition (C_p). In fact we have the following simple remark.

Remark 3.1. Let $A = \{a_1, a_2\}$ be a two-point set. If a point $y_0 \in D$ satisfies conditions (c) and (C), then it satisfies also condition (C_p) $(1 \leq p < \infty)$. The same is not true in general if A contains more than 2 points. In fact, from our assumptions it follows that: for every $y \in D$, $\inf_{a\in A} \|y_0 - a\| \geq \inf_{a\in A} \|y-a\|$ and $\sup_{a\in A} \|y_0 - a\| \geq \sup_{a\in A} \|y-a\|$. Assume, for example, that $\|y_0 - a_1\| \geq \|y_0 - a_2\|$. Then $\|y_0 - a_1\| \geq \|y - a_1\|$ and $\|y_0 - a_1\| \geq \|y - a_2\|$ for all $y \in D$. Moreover, for every $y \in D$, either $\|y_0 - a_2\| \geq \|y - a_1\|$ or $\|y_0 - a_2\| \geq \|y - a_2\|$. So, in any case we obtain:

$$\|y_0 - a_1\|^p + \|y_0 - a_2\|^p \geq \|y - a_1\|^p + \|y - a_2\|^p \quad \text{for all } y \in D,\ 1 \leq p < \infty.$$

Now, on the plane endowed with the supremum norm, denote by D the segment joining $(-1, 0)$ with $(0, 1/2)$, and let $A = \{a_1, a_2, a_3\} = \{(-2,-1), (1-\varepsilon,-1), (1-2\varepsilon,-1)\}$. The point $y_0 = (0, 1/2)$ is the unique point in D satisfying conditions (c) and (C). But, for example

$$\|y_0 - a_1\|^2 + \|y_0 - a_2\|^2 + \|y_0 - a_3\|^2 < \|y - a_1\|^2 + \|y - a_2\|^2 + \|y - a_3\|^2$$

for $y = (-1,0)$, when ε is small. This means that the point y_0 does not satisfy condition (C_2).
Example 3.2. Let y_0 be a farthest point to $x \in X$ in D. Consider the segment $S = \{x + t(x-y_0) : 0 \leq t \leq 1\}$. Then it is simple to see that y_0 is also a simultaneous farthest point to S from D according to conditions (C) and (c). Moreover, if y_0 is the unique farthest point to x from D, then the set D is uniquely simultaneously remotal for S (but not conversely).

We can ask whether some class of sets larger than the class of closed balls is always remotal. The following example shows that if we think of closed, convex sets of constant width (see for example [40]), the answer is no.

Example 3.3. Let $X = c_0$ be the space of real sequences converging to 0 endowed with the supremum norm. Let A be the set containing all sequences from c_0 whose components belong to $[0, 1]$. The set A has constant width, but it is not remotal. For example, if $x = (1/2, 1/3, ...)$, we have $\sup\{||x - a|| : a \in A\} = 1$ and $||x - a|| < 1$ for every $a \in A$.

We know that B_X is always remotal. Next examples 3.4 and 3.5, respectively, will show, that in general it is not simultaneously remotal, neither in the sense of condition (C) nor in the sense of (c).

Example 3.4. a) Let $X = \ell_2$ be the usual Hilbert space of real sequences. Let $e_1, e_2, \ldots$ denote the natural basis of ℓ_2, i.e. $e_n = (0, \ldots, 0, 1, 0, \ldots)$.

Set $A = \{\frac{n}{n+1} e_n : n \in \mathbb{N}\}$. It is simple to see that $r(A, -e_n) \geq 1 + \frac{n}{n+1}$, $n \in \mathbb{N}$, and $r(A, x) < 2$ for $x \in B_X$. Thus $\mu(B_X, A) = 2$. This means that B_X is not simultaneously remotal for A in the sense of condition (C).

The situation is similar if, instead of A, we consider its closed convex hull. Also, we can substitute the space ℓ_2 with c_0: see next example.

b) Consider now the space c_0. Let $D = \{x = (x_1, \ldots, x_n, \ldots) : |x_n| \leq n/(n + 1), \forall\, n \in \mathbb{N}\}$.

The origin θ has no farthest point in D, so the set D is not remotal for B_X, and it is not simultaneously remotal for B_X in the sense of (C). In fact, we have $\mu(B_X, D) = 2$, but there is no point $y \in D$ such that $r(B_X, y) = 2$.

Also, B_X is not simultaneously remotal for D in the sense of condition (C). In fact, $\sup\{r(D, x) : x \in B_X\} = 2$ since $r(D, e_n) = 1 + \frac{n}{n+1}$ for $n \in \mathbb{N}$; but $r(D, x) < 2$ for every $x \in B_X$.

In particular, the sets D and B_X are not distant.

If we consider the same sets in ℓ_∞ (instead of in c_0), then state of affairs is different. In that case the points $x_0 = (-1, -1, \ldots) \in B_X$ and $y_0 = (1/2, 2/3, \ldots) \in D$ form a remotal pair.

Example 3.5. Let $X = \ell_2$ and $A = \bigcup_{n \in N} [\frac{-1}{n+1}, \frac{1}{n+1}]\, e_n$, where e_n denote the n-th element of the natural basis.

Since $\theta \in A$, we have $d(x, A) \leq 1$ for $x \in B_X$. Moreover, if $x = -e_n$ and $a \in A$, then $||x - a|| \geq 1 - \frac{1}{n+1}$. Thus $\sup\{d(x, A) : x \in B_X\} = 1$.

Observe that $d(x, A) < 1$ for every $x \in B_X$. Indeed, if $x = \theta$ this is trivial. Otherwise, let $j \in \mathbb{N}$ be such that the j-th component x_j of x is different from 0. If $x_j > 0$, take $a = e_j \min\{x_j, \frac{1}{n+1}\}$, if $x_j < 0$, take $a = e_j \max\{x_j, \frac{-1}{n+1}\}$. Obviously $a \in A$ and $||x - a|| < ||x|| \leq 1$. Thus $d(x, A) < 1$ for every $x \in B_X$. This means that B_X is not simultaneously remotal for A in the sense of condition (c).

Proposition 3.6. *Consider condition (C). Let $A = \{a_1, ..., a_n\}$ be a finite set and D a (closed, bounded) set. If D is remotal for all a_i's, then it is simultaneously remotal for A. The converse is not true.*

Proof. Let $r_i = \sup\{\|a_i - y\| : y \in D\}$ for $i = 1, ..., n$ and $r = max\{r_1, ..., r_n\}$. Let $r = r_j$, where $1 \leq j \leq n$. Since D is remotal for a_j there is $y_j \in D$ such that $\|y_j - a_j\| = r_j$. Obviously $\sup\{\|y_j - a\| : a \in A\} \geq r_i = \sup\{\|a_i - y\| : y \in D\}$ for $i = 1, ..., n$. This shows that y_j satisfies (C), so D is simultaneously remotal for A.

Concerning the converse, let $X = c_0$ and $A = \{\theta, e_1\}$. Then, the set D defined in Example 3.4 b) is simultaneously remotal for A in the sense of condition (C) (take $y_0 = -e_1/2 \in D$), but it is not remotal for θ. □

Corollary 3.7. *A closed, bounded subset of D is remotal if and only if, for every finite set in X, it is simultaneously remotal according to condition (C).*

Proof. The "if" part is trivial. The "only if" part follows from previous proposition. □

Note that previous result cannot be extended to infinite A (see Example 3.4).

Next example shows that in Proposition 3.6 (and in Corollary 3.7) we cannot substitute (C) with (c). More precisely, we first show a set remotal for the elements of a finite set A but not simultaneously remotal for A in the sense of (c). Then we indicate a set which is simultaneously remotal for a finite set in the sense of (c), but not remotal for all its elements.

Example 3.8. a) We indicate the construction of a two-point set A and of a closed, bounded set D which is remotal for the elements of A, but it is not simultaneously remotal for A in the sense of (c).

Take an infinite dimensional space X; let D_1 be a "small" set (say of diameter < 0.1); let be $a_1 \in X$ a point which has no farthest point in D_1 and $a_2 \in X$ such that $\sup\{\|a_1 - y\| : y \in D_1\} = 1 > \sup\{\|a_2 - y\| : y \in D_1\}$. Moreover let $y_1, y_2 \in X$ be such that $\|a_1 - y_1\| > 1$, $\|a_1 - y_2\| < 1$, $\|a_2 - y_1\| < 1$, $\|a_2 - y_2\| > 1$. It is easy to see that $D = D_1 \cup \{y_1, y_2\}$ is remotal for the elements of $A = \{a_1, a_2\}$. Moreover $\sup_{y \in D} d(y, A) = 1$, but 1 is not attained by any element of D.

b) Next example shows that, also concerning (c), a set can be simultaneously remotal for a set A, but not remotal for some element of A.

Take $X = \ell_\infty$. Let $D\{\frac{n}{n+1}\, e_n : n \in \mathbb{N}\}$; $A = \{\theta, y\}$, where $y = (0, 1/2, 1/2, ...)$. θ has no farthest point in D, while $\sup_{y \in D} d(y, A) = 1/2$ is attained by $e_1/2$.

Problem. Study previous facts for (C_p), or at least for (C_1) (A being a finite set).

For A a set of n elements $a_1, ..., a_n$, we could consider in $\mathbb{R}^n$ a norm $|||.|||$, and look for simultaneously farthest points according to such norm: maximize $f(y) = |||\,(\|a_1 - y\|, ..., \|a_n - y\|)\,|||$ for $y \in D$. Finding a solution amounts to looking for farthest point in $(X^n, |||.|||)$, for $(a_1, ..., a_n)$, from elements of the diagonal $D^n = (y, ..., y)$. Concerning this scheme, see [3, Section 4]; see also [33].
By considering a suitable norm on $\mathbb{R}^n$, we could study, for finite sets, conditions $(\mathrm{C}_p), 1 \le p \le \infty$, as well as weighted conditions of that type.

Studying simultaneous remotality with condition (c) (or also (C_1) for a finite set), can be given the following interpretation in terms of *location problems.* There is a dangerous area A ("obnoxious" facility); we want to "best" locate something, as a new town, in D. The interesting applications are mainly in two-dimensional spaces; see for example [41, Section 2] for a discussion on them.

A similar interpretation can be given, for $A = \{a_i, ..., a_n\}$ a finite set, concerning condition (C_1) (also a weighted procedure can be used). Or it can be asked to maximize, for $y \in D$, the following objective function, the w_i's being nonnegative weights:

$$(1-\lambda) \min_{i \in \{1,\dots,n\}} w_i \|y - a_i\| + \lambda \sum_{i=1}^{n} w_i \|y - a_i\|, \ \ 0 \le \lambda \le 1.$$

4. The literature on the subject

We list here related papers. Indeed, the existing papers (many of them recent) are not always deep and there are few applications. But the subject indicates once more the variety of situations existing in Banach spaces; also, the fact that a norm is Hilbertian seems to be not so helpful in this context: while best approximation fits well in Hilbert spaces, only finite dimensionality seems to imply good results concerning remotality.

In [28], conditions (c), (C) and (C_1) were defined, for two-point sets. Some existence results, also concerning denseness of simultaneous farthest points, were given. A simple result concerning (c) had also been given in [32]. Simple facts concerning (c) have been indicated in [15].

We note in passing that most among the general results in [27] are trivial or not new, as noticed in [33] where some of them are proved in a unified way: they regard conditions of (C_p) type, for two points.

Some papers deal with the subject in particular spaces. In [2], the condition (C_1) (for finite sets) is considered. In [1], a generalization of condition (C_1) (finite sets) by using a "modulus function" is used. In [6], condition (C) (for finite sets) is considered; some results there deal with denseness of simultaneous remotality for sets. Also, [27] contains some (doubtful) results concerning (C) in $L_1[a,b]$ and $L_\infty[a,b]$ (two-point sets), and (C_2) in Hilbert spaces (finite sets).
In [29] some estimates are given, for A consisting of two elements, concerning condition (C_p) in L_p, together with a remark concerning (C) for complex valued function in $C[a,b]$. For the last kind of result see also [28, Theorem 3.7].
In [3] some results for Köthe spaces are given: the condition used, for n elements, is a general one, that we have indicated near the end of previous section (see: suitable norm on $\mathbb{R}^n$).

In [34], uniqueness of simultaneous farthest points is studied and a characterization of strictly convex spaces, by using (C), is indicated. More precisely, the characterization is obtained by the uniqueness of simultaneous farthest points with respect to two arbitrary disjoint bounded closed balls. Since the last paper is not easily avalaible and readable, we indicate here a simple connection between strict convexity and remotality.

Proposition 4.1. *A space X is strictly convex if and only if there is no point having a nontrivial segment of remotal points in a (closed, bounded) set D.*

Proof. If X is not strictly convex, then for $D = B_X$ the origin has a set of farthest points containing a segment.

Let X contain a point x for which a segment $[a, b] \subset D$ $(a \neq b)$ consists of remotal points for it; then $\|x - a\| = \|x - b\| = \|x - (a + b)/2\|$, and this implies that X is not strictly convex. $\square$

In [30] characterizations of condition (c) are given by using (extremal) elements of B_{X^*}; then these results are applied to the space $C(Q)$, Q a compact set.

In [5], some results concerning (C) (A any bounded set) are indicated by using extreme points of B_{X^*} and the "mirror reflection property" (see next section).

In [4], remotality and unique remotality according to (C) are studied, mainly with respect to extreme points of D and comparing properties of D with properties of its closed convex hull. But the results indicated there are doubtful, since they are based on some wrong results of M. Sababheh and R. Khalil (see the discussion in the Appendix).

Many papers (in the wake of [11]) deal with generic results, often regarding also well-posedness for these problems: [38, 39] consider condition (C); papers [19, 20, 21, 22, 23, 35, 36, 37] consider remotal pairs.

5. Other possible generalizations and properties

First of all, we discuss a condition introduced recently, that we think will not foster the research in the area.

In [5], the *mirror reflection property*, (mrp) for short, defined in [**?**], was used. The property is the following.

(mrp) We say that X has (mrp) if for any closed and bounded set $D \subset X$, and any $x \in X \setminus D$, there exists a closed convex set $E \subset D$ such that $r(D, x) = r(E, x)$, and moreover the "mirror reflection" function (see [**?**] for the definition) is convex.

As noticed in [**?**, Lemma 2.3.], every finite dimensional space has (mrp). But indeed only these spaces have (mrp), as we are going to show: this fact had been suspected also by a reviewer of the above paper (see Math. Rev. 2521206). In fact, we are showing that already the first condition in defining (mrp) implies finite dimensionality of the space; so this condition can play a weak role.

Proposition 5.1. *X has the mirror reflection property if and only it is finite dimensional.*

Proof. The "if" part being known, we are proving the "only if" part.

Assume that X is infinite dimensional. We can find on S_X, the unit sphere of X, an infinite set $\{x_1, x_2, ...\}$ such that $\|x_i - x_j\| > 1$ for $i \neq j$ (see for example [12, Chapter XIV]). Now let $A = \{\frac{n-1}{n} x_n : n \in \mathbb{N}\}$. This is a discrete (thus closed) and bounded subset of X. Let $x = \theta$: if we consider a convex subset D of A, this must be a singleton; so we cannot have $r(D, \theta) = r(A, \theta) = 1$. $\square$

We conclude this section by indicating some **questions and problems**.

- "Simultaneously remotal maps" can hardly be continuous or stable (on D and/or on A). Probably a general study, at least for a finite set A, could be done.

Concerning (C), if we add a single point, probably we can only say that the change can be similar to the distance of the new point from the existing set. A better estimate can be obtained for (C_1).

And what happens if we pass from A to its closed convex hull? If we pass from D to its closed convex hull, probably changes are very small.

- Can we indicate applications that justify considering (C_1) for A infinite?

- Are the results in [6], some of the results in [27], [4] ... correct?

- Are there other connections between the notion of diametral point or pairs and simultaneous remotality?

6. Appendix - general results on remotal sets

In this section we shall discuss some general facts concerning remotality; most of these results are quite recent and general. Practically, they indicate that in infinite dimensional spaces remotality has a rather bad behaviour.

A nice, simple result was indicated in [31, Th. 4.16]: the space X is a Hilbert space, if and only if for every set D and every $y \in D$, the set $\{x \in X : y \in F_D(x)\}$ is convex.

An example of a closed bounded, not remotal set in a Hilbert space, is given in [**?**, Example 2.7].

The following result was indicated in [**?**, Theorem A].

THEOREM A. Every infinite dimensional reflexive Banach space contains a closed, bounded convex set not remotal.

Unfortunately, the proof was based on an incorrect argument (see [42, Introduction]). Note that the authors of [24], at the end of the Introduction, seem to say that Theorem A if false. What is not true is the positive answer given there to (Q2) below. In fact, later, Theorem A was proved in [42, Theorem 3], as a consequence of a similar fact for X not Schur.
Finally, in [47, Remark 2.10], then in [24, Theorem 7], and again in [18, Theorem 2], the following result was proved; it had already been obtained, for spaces with a monotone basis, in [8, Theorem 1].

THEOREM A^+. Every infinite dimensional Banach space contains a closed, bounded convex set not remotal.

Also the following fact is indicated in [24, Corollary 2]: let X be an infinite dimensional space X; then X^* contains a non remotal convex, w^* compact set.

If the space is "good", generic results can be given concerning existence of farthest points: see the long list of papers at the end of Section 4. But there are also spaces where there exist "antiremotal" sets: i.e., sets for which no point has farthest points on it; concerning this, see [13, Example 5.3] or [7, Lemma 2 and Theorem 1].

Note that in [47], the following problem was investigated. Look for "diametral pairs" of a (bounded, closed) set A: namely, look for pairs (a_1, a_2) in A such that $\|a_1 - a_2\| = \delta(A)$. The following facts were proved. Denote by BCC(X) the sets of all bounded, closed, convex subsets of A; then (see [47, Section 2]):

- In any space X, there is a dense subset of BCC(X) whose elements lack diametral pairs;

- In some, but not in all spaces X, there is a dense subset of BCC(X) whose elements have diametral pairs.

Given a closed bounded set A, denote by $co(A)$ the convex hull of A; by $\overline{co}(A)$ (resp. by $\overline{co}^w(A)$) its closed (resp.: w-closed) convex hull.

It is not difficult to see that the following facts are true:

(r) for every $x \in X,\ r(A,x) = r(\overline{co}(A),x) = r(\overline{co}^w(A),x)$
(for the first almost trivial equality, see for example [**?**, Lemma 2.1]; concerning the second one, see [**?**, Lemma 2.2]).

Consider the following questions, x being any element in X and A a bounded set:
(Q1) do exist farthest points for x in A if there exist in $co(A)$?
(Q2) do exist farthest points for x in A if there exist in $\overline{co}(A)$?
(Q3) do exist farthest points for x in A if there exist in $\overline{co}^w(A)$?
It is simple to see that the answer to (Q1) is positive (see for example [**?**, Lemma 3.1]).

Some authors thought that also (Q2) always has a positive answer: see for example [16, p.201], but also [**?**, Remark 3.2.(c')]. The question whether it is always true for w-closed, bounded sets was raised in [24, Remark 6] and in [**?**, p.62]. Indeed the answer to (Q2) is negative: an example in c_0, where there is a point which has a farthest point in $\overline{co}(A)$ but not in A, A bounded and closed, is given in [**?**, p.124]. Another similar example, with a bounded w-closed set (still in c_0), was given in [18, Example 1]. The answer is true if A is weakly compact: see [**?**, Corollary 9].

It was claimed in [**?**, Theorem 2.6] that the answer to (Q2) is true for any closed bounded set A if X is reflexive: a counterexample was given in [24, Example 5].

For a positive answer to a question similar to (Q3) (weak* version) see [24, p.393].

Questions more general than (Q1) (concerning two sets and simultaneous remotality) were studied in [4, Section 3]. More precisely, the equalities $\mu(A,D) = \mu(A,co(D))$ and $\mu(A,D) = \mu(A,\overline{co}(D))$ (D closed and bounded) were proved there (Lemmata 7 and 10). Also, it was proved (Lemmata 8 and 9) that D is (sr) (resp. D is (usr)) if (and only if) $co(D)$ is (sr) (resp.: $co(D)$ is (usr)).

In [**?**, p.63], after proving that A (ur) is equivalent to $co(A)$ (ur) (Proposition 3.3), it was natural to ask (at page 64) whether also the following equivalence is true: A is (ur) $\Longleftrightarrow \overline{co}(A)$ is (ur). Same question for (sr) in [4].

Let X be reflexive. Then for A w-closed: A is (ur) $\Longleftrightarrow$ $\overline{co}^w(A)$ is (ur) [**?**, Proposition 3.4]; for D w-closed, D is (ur) for $A \Longleftrightarrow \overline{co}^w(D)$ is (ur) for A: see [4, Theorem 14].

References

[1] A. Ababneh, Sh. Al-Sharif and J. Jawdat, *Simultaneous farthest points in vector valued function spaces*, Nonlinear Funct. Anal. Appl. **16** (2011), 63–77.

[2] E. Abu-Sirhan and O. H. H. Edely, *On simultaneously remotal in $L_1(\mu, X)$*, Int. J. Math. Anal. (Ruse) **6** (2012), 2217–2226.

[3] Sh. Al-Sharif, *On farthest and simultaneous farthest points in Köthe spaces*, Creat. Math. Inform. **21** (2012), 123–128.

[4] Sh. Al-Sharif and A. Awad, *Results on simultaneous remotal sets in reflexive spaces*, J. Nonlin. Sci. Appl., to appear.

[5] Sh. Al-Sharif and A. Awad, *New type of simultaneous remotal sets in certain Banach spaces*, Missouri J. Math. Sci., to appear.
[6] Sh. Al-Sharif and M. Rawashdeh, *On simultaneous farthest points in $L^\infty(I,X)$*, Int. J. Math. Math. Sci. 2011, Art. ID 890598, 10 pp.
[7] V. S. Balaganskiĭ, *On nearest and farthest points*, Math. Notes **63** (1998), 250–252.
[8] M. Baronti, *A note on remotal sets in Banach spaces*, Publ. Inst. Math. (Beograd) (N.S.) **53** (1993), 95–98.
[9] M. Baronti and P. L. Papini, *Remotal sets revisited*, Taiwan. J. Math. **5** (2001), 367–373.
[10] Á.P. Bosznay, *Some remarks on the farthest point problem*, Functions, series, operators, Vol. I (Budapest 1980), Colloq. Math. Soc. János Bolyai, 35, North Holland, Amsterdam, 1980, pp. 257–266.
[11] F. S. De Blasi, J. Myjak and P. L. Papini, *On mutually nearest and mutually furthest points of sets in Banach spaces*, J. Approx. Theory **70** (1992), 142–155.
[12] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Text Math. 92, Springer-Verlag, New York, 1984.
[13] S. Fitzpatrick, *Metric projections and the differentiability of distance functions*, Bull. Austral. Math. Soc. **22** (1980), 291–312.
[14] P. Govindarajulu, *On remotal points of pairs of sets*, Indian J. Pure Appl. Math. **15** (1984), 885–888.
[15] P. Govindarajulu, *On upper semicontinuity of simultaneous operators*, Indian J. Pure Appl. Math. **16** (1985), 153–156.
[16] J.-B. Hiriart-Urruty, *La conjecture des points les plus éloignés revisitée*, Ann. Sci. Math. Québec **29** (2005), 197–214.
[17] M. Iranmanesh and H. Mohebi, *On best simultaneous approximation in quotient spaces*, Anal. Theory Appl. **23** (2007), 35–49.
[18] M. Kraus, *Two remarks on remotality*, J. Approx. Theory **163** (2011), 307–310.
[19] C. Li, *On mutually nearest and mutually furthest points in reflexive Banach spaces*, J. Approx. Theory **103** (2000), 1–17.
[20] C. Li and R. X. Ni, *On well-posed mutually nearest and mutually furthest point problems in Banach spaces*, Acta Math. Sin. (Engl. Ser.) **20** (2004), 147–156.
[21] C. Li and H.-K. Xu, *On almost well posed mutually nearest and mutually farthest point problems*, Funct. Anal. Optim. **23** (2002), 323–331.
[22] C. Li and H.-K. Xu, *Porosity of mutually nearest and mutually furthest points in Banach spaces*, J. Approx. Theory **125** (2003), 10–25.
[23] C. Li and H.-K. Xu, *Ambiguous loci of mutually nearest and mutually furthest points in Banach spaces*, Nonlinear Anal. **58** (2004), 367–377.
[24] M. Martín and T. S. S. R. K. Rao, *On remotality for convex sets in Banach spaces*, J. Approx. Theory **162** (2010), 392–396.
[25] M. D. P. Monteiro Marques, *Sur la frontiére d'un convexe mobile*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Ser. VIII **77** (1984), 71–75.
[26] V. Montesinos, P. Zizler and V. Zizler, *Some remarks on farthest points*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **105** (2011), 119–131.
[27] S. Nanda, *Simultaneous farthest points in normed linear spaces*, Math. Japon. **23** (1978/79), 191–197.
[28] S. Nanda, *On simultaneous farthest points*, J. Math. Phys. Sci. **25** (1991), 13–18.
[29] S. Nanda and Ellipse, *Best simultaneous approximation and simultaneous farthest points in L_p norm*, J. Indian Math. Soc. (N.S.)**62** (1996), 51–56.
[30] E. Naraghirad, *Characterizations of simultaneous farthest point in normed linear spaces with applications*, Optim. Lett. **3** (2009), 89–100.
[31] T. D. Narang, *A study of farthest points*, Nieuw Arch. Wisk. **25** (1977), 54–79.
[32] T. D. Narang, *On simultaneous furthest points*, Math. Sem. Kobe Univ. **9** (1981), 109–112. Corrigendum: Math. Sem. Kobe Univ. **10** (1982), 238.
[33] T. D. Narang, *Some remarks on simultaneous farthest points in normed linear spaces*, (by Sudarsan Nanda-Math. Japonica, 23 (1978), 191-197), Math. Japon. **29** (1984), 123–125.

[34] R. X. Ni, *Uniqueness of simultaneous farthest points in normed linear spaces*, Numer. Math. J. Chinese Univ. **19** (1997), 357–363 (Chinese).

[35] R. X. Ni, *On well posedness of generalized mutually furthest points problems in a nonreflexive real Banach space*, J. Systems Sci. Math. Sci. **21** (2001), 335–342.

[36] R. X. Ni, *Well posedness of generalized mutually maximization problem*, Proc. 3^{rd} International Conf. on Information and Computing (ICIC 2010, Wuxi, Jiangsu, China), Vol.1, pp. 203–206.

[37] R. X. Ni, *Porosity of generalized mutually maximization problem*, Proc. 3^{rd} International Conf. on Information and Computing (ICIC 2010, Wuxi, Jiangsu, China), Vol.1, pp. 227–230.

[38] R. X. Ni and C. Li, *On well posedness problems for simultaneous farthest points in Banach spaces*, Acta Math. Sin. (Chin. Ser.) **42** (1999), 823–826 (Chinese).

[39] R. X. Ni and C. Li, *On well posedness of farthest and simultaneous farthest problems in Banach spaces*, Acta Math. Sin. (Chin. Ser.) **43** (2000), 421–426 (Chinese).

[40] P. L. Papini, *Completions and balls in Banach spaces*, Ann. Funct. Anal. **6** (2015), 24–33.

[41] F. Plastria, *Optimal location of undesirable facilities: a selective overview*, Belgian J. Oper. Res. Statist. Comput. Sc. **36** (1996), 109–127.

[42] T. S. K. K. Rao, *Remark on a paper of Sababheh and Khalil*, Numer. Funct. Anal. Optim. **30** (2009), 822–824.

[43] M. Sababheh and R. Khalil, *Remotality of closed bounded convex sets in reflexive spaces*, Numer. Funct. Anal. Optim. **29** (2008), 1166–1170.

[44] M. Sababheh and R. Khalil, *Remarks on remotal sets in vector valued function spaces*, J. Nonlinear Sci. Appl. **2** (2009), 1–10.

[45] M. Sababheh and R. Khalil, *New results on remotality in Banach spaces*, Italian J. Pure Appl. Math. **30** (2013), 59–66.

[46] I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Grundl. math. Wiss. Band 171, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[47] L. Veselý, *Convex sets without diametral pairs*, Extracta Math. **24** (2009), 271–280.

[48] M. D. Wills, *Hausdorff distance and convex sets*, J. Convex. Anal. **14** (2007), 109–117.

NOTE: Concerning [7, 8, 9, 13, 24, 26, 31, 42, 43, 44, 45, 47] are only cited in the Appendix (Section 6).

Manuscript received February 15, 2014
revised May 22, 2014

Józef Myjak
WMS AGH, al. Mickiewicza 30, 30059 Krakow, Poland
E-mail address: myjakjoz@wms.mat.agh.edu.pl

Pier Luigi Papini
Via Martucci, 19, 40136 Bologna, Italy
E-mail address: plpapini@libero.it