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GENERIC PROPERTIES OF SUCCESSIVE APPROXIMATIONS IN HILBERT SPACES

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Abstract. We investigate the convergence of successive approximations for a class of nonexpansive set-valued maps F in Hilbert spaces. We prove that the trajectories of most, in the sense of Baire category, such mappings converge to a fixed point of F.

1. Introduction

The generic behavior of sequences of successive approximations for nonexpansive single valued maps has been studied by several authors. A comprehensive account and bibliographic references can be found in Reich and Zaslavski [10]-[15] and the references therein. Apparently similar problems have not been considered for nonexpansive set valued maps.

In the present paper we investigate the generic behavior of sequences of successive approximations for a class of set valued maps of the form $\{f, g\}$ where f and g are nonexpansive maps from D into itself and D is a nonempty, closed, convex and bounded set in a Hilbert space H. If f and g are contractive i.e. they have Lipschitz constant strictly less than 1, then the trajectories relative to $\{f, g\}$ converge to a fixed point of $\{f, g\}$. A similar result is no longer true if $\{f, g\}$ are merely nonexpansive. However it will be proved that for *most* (in the sense of Baire category) maps $\{f, g\}$ the trajectories converge to a fixed point of $\{f, g\}$. It is worth noting, Theorem 4.3 , that the set of contractive maps $\{f, g\}$ is of the Baire first category in the space of nonexpansive maps.

With appropriate technical modifications our approach can be used to study the generic behavior of trajectories relative to maps of the form $\{f_1, .., f_n\}$ where each f_i is nonexpansive from D into itself. However it is not clear if it can be used to study the general case of set valued maps from D to the compact subsets of D.

The paper is divided in 4 sections including the Introduction. Section 2 contains terminology and preliminary properties. Section 3 contains some auxiliary density result. Section 4 contains the main result, namely, that for *most* nonexpansive maps $\{f, g\}$ the trajectories relative to $\{f, g\}$ are convergent.

2. Notation and preliminaries

In this section we review some preliminary properties which will be useful in what follows, some of them are known and are included for completeness.

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Let (M, d^*) be a metric space. Open and closed balls with center $a \in M$ and radius r are denoted by $B_M(a,r)$ and $B_M[a,r]$ sometimes simply $B(a,r)$ and $B[a,r]$. The closure, interior and diameter of a set $X \subset M$ are denoted by $\overline{X}$, int$X$ and diam X. We denote by $\mathcal{K}(M)$ the space of all nonempty compact subsets of M equipped with the Hausdorff metric h

$$h(X,Y) = \max\{\sup{}_{x\in X} d^*(x,Y), \sup{}_{y\in Y} d^*(y,X)\} \quad , \quad X,Y \in \mathcal{K}(M).$$

A set $X \subset M$ which is the complement of a set of the Baire first category is said to be residual. A property (P) which is enjoyed by a residual subset of M is called a generic property and, in this case, we say that *most* elements of M have the property (P). In what follows H is a real Hilbert space with inner product $<>$ and induced norm $|.|$

If $A \subset H$ is nonempty and bounded we set

$$|A| = \sup\{|a| : a \in A\}.$$

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A map $f : X \to H$, where $X \subset H$ is nonempty, is said to be nonexpansive if

$$|f(x) - f(y)| \leq |x - y|$$

for every $x, y \in X$ and is said to be contractive if there exists λ , $0 \leq \lambda < 1$ such that

$$|f(x) - f(y)| \leq \lambda|x - y|$$

for every $x, y \in X$, λ is called the Lipschitz constant of f. In analogous way a map $F : X \to \mathcal{K}(H)$, where $X \subset H$ is nonempty, is said to be nonexpansive if

$$h(F(x), F(y)) \leq |x - y|$$

for every $x, y \in X$ and is said to be contractive with Lipschitz constant $0 \leq \lambda < 1$ if

$$h(F(x), F(y)) \leq \lambda|x - y|$$

for every $x, y \in X$.

Throughout the paper D is a nonempty, bounded, closed and convex subset of H with $d = \text{diam } D > 0$.

Let

$$\mathcal{N}' = \{f : D \to D : f \text{ is nonexpansive}\}$$
$$\mathcal{C}' = \{f : D \to D : f \text{ is contractive}\}.$$

$\mathcal{N}'$ is equipped with the metric

$$|f - g| = \sup{}_{x\in D}|f(x) - g(x)|.$$

Under this metric $\mathcal{N}'$ is a complete metric space.

Let

$$\mathcal{N} = \{\{f,g\} : f, g \in \mathcal{N}'\}$$

and

$$\mathcal{C} = \{\{f,g\} : f, g \in \mathcal{C}'\}.$$

We equip $\mathcal{N}$ with the Hausdorff metric h.

Proposition 2.1. *$(\mathcal{N}, h)$ is a complete metric space.*

Proof. As $\mathcal{N}$ is closed in $\mathcal{K}(\mathcal{N})$ and $\mathcal{K}(\mathcal{N})$ is complete under the metric h, then $(\mathcal{N}, h)$ is complete. □

Proposition 2.2. *Let $\{f,g\}, \{f',g'\} \in \mathcal{N}$. Then*

$$h(\{f,g\},\{f',g'\}) = \min\left(\max(|f-f'|,|g-g'|); \max(|f-g'|,|f'-g|)\right). \tag{2.1}$$

Proof. Set

$$a = |f-f'|\ ,\ b = |f-g'|\ ,\ c = |f'-g|\ ,\ d = |g-g'|.$$

By the definition of Hausdorff distance we have

$$\alpha = h(\{f,g\},\{f',g'\}) = ((a \wedge b) \vee (c \wedge d)) \vee ((a \wedge c) \vee (b \wedge d))$$

where $p \vee q$, $p \wedge q$ mean the maximum and minimum of the real numbers p, q. By the distributive property of $\wedge, \vee$ it follows that

$$\alpha = (a \wedge (b \vee c)) \vee (d \wedge (b \vee c)) = ((b \vee c) \wedge (a \vee d)) \tag{2.2}$$

Hence (2.1) is valid. This completes the proof. □

Remark 2.3. As a consequence of Proposition 2.2 either $\alpha = a \vee d$ or $\alpha = b \vee c \leq a \vee d$. In both cases

$$h(\{f,g\},\{f',g'\}) \leq \max(|f-f'|,|g-g'|) \tag{2.3}$$

This property will be useful in the sequel of the paper.

Proposition 2.4. *Let $f, g : D \to D$ be lipschitzian with Lipschitz constants λ , μ respectively. Then the map $F : D \to \mathcal{K}(D)$*

$$F(x) = \{f(x), g(x)\}$$

is lipschitzian with Lipschitz constant $\gamma = \max(\lambda, \mu)$

Proof. From the previous Remark

$$\begin{aligned} h(\{f(x),g(x)\},\{f(y),g(y)\}) &\leq \max(|f(x)-f(y)|,|g(x)-g(y)|) \\ &\leq \max(\lambda|x-y|, \mu|x-y|) \\ &= \gamma|x-y|. \end{aligned}$$

The proof is complete. □

Definition 2.5. A sequence $\{x_n^{f,g}\}_{n=0}^{\infty}$ is called a sequence of successive approximations, for brevity a *trajectory* relative to $\{f, g\}$ if

$$x_{n+1}^{f,g} = \begin{cases} f(x_n^{f,g}) & \text{if} \quad |f(x_n^{f,g}) - x_n^{f,g}| < |g(x_n^{f,g}) - x_n^{f,g}| \\ g(x_n^{f,g}) & \text{if} \quad |f(x_n^{f,g}) - x_n^{f,g}| > |g(x_n^{f,g}) - x_n^{f,g}|. \end{cases} \tag{2.4}$$

When $|f(x_n^{f,g}) - x_n^{f,g}| = |g(x_n^{f,g}) - x_n^{f,g}|$ then

$$x_{n+1}^{f,g} = f(x_n^{f,g}) \quad \text{or} \quad x_{n+1}^{f,g} = g(x_n^{f,g})$$

If this does not create confusion we simply write $\{x_n\}_{n=0}^{\infty}$ in place of $\{x_n^{f,g}\}_{n=0}^{\infty}$.

A point $x \in D$ is said *regular* with respect to $a, b \in D$ if

$$|x - a| \neq |x - b|.$$

A trajectory $\{x_n\}_{n=0}^{\infty}$ is said *regular* with respect to $\{f, g\}$ if for all $n \in \mathbb{N}_0$

$$|f(x_n) - x_n| \neq |g(x_n) - x_n| \ .$$

Proposition 2.6. *Let $\{f, g\} \in \mathcal{C}$ with f, g contractive with Lipschitz constant $0 \leq \lambda < 1$ and let $\{x_n\}_{n=0}^{\infty}$ be a trajectory relative to $\{f, g\}$. Then*

$$|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}| \qquad n \in \mathbb{N} \tag{2.5}$$

Proof. Indeed

$$x_n = f(x_{n-1}) \quad \text{or} \quad x_n = g(x_{n-1}).$$

Suppose $x_n = f(x_{n-1})$ (if $x_n = g(x_{n-1})$ the argument is similar).
If $x_{n+1} = f(x_n)$ then

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq \lambda |x_n - x_{n-1}|.$$

If $x_{n+1} = g(x_n)$ then $|g(x_n) - x_n| \leq |f(x_n) - x_n|$ which implies

$$|x_{n+1} - x_n| = |g(x_n) - x_n| \leq |f(x_n) - x_n| = |f(x_n) - f(x_{n-1}| \leq \lambda |x_n - x_{n-1}|.$$

In both cases (2.5) holds. This completes the proof. □

Proposition 2.7. *Let $\{f, g\} \in \mathcal{C}$ and let ξ, η be the fixed points of f, g. Then every trajectory $\{x_n\}_{n=0}^{\infty}$ relative to $\{f, g\}$ with initial point $u \in D$ converges to ξ or η.*

The proof is an immediate consequence of Proposition 2.4 and Nadler's fixed point theorem [9].

Proposition 2.8. *Let $\{f, g\} \in \mathcal{C}$ with f, g contractive with Lipschitz constant $0 \leq \lambda < 1$ and let ξ, η be the fixed points of f, g. Let $\{x_n\}_{n=0}^{\infty}$ be a trajectory relative to $\{f, g\}$ with initial point $x_0 \in D$. If there exists $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$ such that*

$$x_n = x_{n+p} \tag{2.6}$$

Then $x_n = \xi$ or $x_n = \eta$.

Proof. Suppose that $x_{n+1} = f(x_n)$ (if $x_{n+1} = g(x_n)$ the argument is similar). We have

$$|x_{n+1} - x_n| = |f(x_n) - x_n| \leq |g(x_n) - x_n| \tag{2.7}$$

then, as $x_{n+p} = x_n$,

$$|x_{n+1} - x_n| = |f(x_n) - x_n| = |f(x_{n+p}) - x_{n+p}| \leq |g(x_{n+p}) - x_{n+p}|$$

If $x_{n+p} = f(x_{n+p-1})$ then

$$|x_{n+1} - x_n| = |f(x_{n+p}) - f(x_{n+p-1}| \leq \lambda |x_{n+p} - x_{n+p-1}|.$$

If $x_{n+p} = g(x_{n+p-1})$ then

$$|x_{n+1} - x_n| \leq |g(x_{n+p}) - g(x_{n+p-1}| \leq \lambda |x_{n+p} - x_{n+p-1}|$$

In both cases

$$|x_{n+1} - x_n| \leq \lambda |x_{n+p} - x_{n+p-1}|$$

Since, by (2.5) ,

$$|x_{n+p} - x_{n+p-1}| \leq \lambda^{p-1} |x_{n+1} - x_n|$$

it follows

$$|x_{n+1} - x_n| \leq \lambda^p |x_{n+1} - x_n|$$

Then $x_{n+1} = x_n$ which implies $f(x_n) = x_n$ and so $x_n = \xi$. □

Proposition 2.9. *Let $\{f, g\} \in \mathcal{C}$ with f, g contractive with Lipschitz constant $0 \leq \lambda < 1$ and let ξ, η , $\xi \neq \eta$ be the fixed points of f, g. Let $\{x_n\}_{n=0}^{\infty}$ be a trajectory relative to $\{f, g\}$ with initial point $x_0 \in D$. Then either*

(i) *all the x_n are pairwise distinct*
or
(ii) *there exists $m \in \mathbb{N}_0$ such that x_m is a fixed point and $x_i \neq x_j$, $i \neq j$, $0 \leq i, j \leq m$.*

Proof. Suppose that (i) is valid. Then for every $n \in \mathbb{N}_0$ x_n is not a fixed point. In fact if $x_n = \xi$, say , then $0 = |f(\xi) - \xi| < |g(\xi) - \xi|$ otherwise f and g would have equal fixed points, it follows that $x_{n+1} = f(\xi) = \xi = x_n$ a contradiction. Suppose that (i) is not valid, then there exist $i, j \in \mathbb{N}_0$, $i < j$ such that $x_i = x_j$. Proposition 2.8 implies that x_i is a fixed point, say ξ. Let m be the smallest index for which $x_m = \xi$ then $x_i \neq x_j$, $i \neq j$, $0 \leq i, j \leq m$ otherwise, by Proposition 2.8 , $x_i = \xi$ with $i < m$ a contradiction . □

3. Auxiliary results

In this Section we prove some auxiliary results which are necessary for proving the main theorem of the paper.

Proposition 3.1. *Let $\{f, g\} \in \mathcal{C}$ with f, g contractive with Lipschitz constant $0 \leq \lambda < 1$ and let ξ, η be the fixed points of f, g. Then for any $\varepsilon > 0$ there exist ϕ and ψ contractive with Lipschitz constant λ such that $\{\phi, \psi\} \in B_{\mathcal{N}}(\{f, g\}, \varepsilon)$ and ϕ, ψ have different fixed points.*

Proof. Suppose that $\xi = \eta$, otherwise there is nothing to prove. Let $\varepsilon > 0$ and $u \in D$, $u \neq \xi$ be fixed and let $0 < t < \frac{\varepsilon}{d}$. Define

$$\phi(x) = tu + (1 - t)f(x) \quad , \quad \psi(x) = g(x).$$

Clearly ϕ and ψ are contractive with Lipschitz constant λ and

$$|\phi(x) - f(x)| = |tu + (1 - t)f(x) - f(x)| = t|f(x) - u| \leq td < \varepsilon$$

then

$$h(\{\phi, \psi\}, \{f, g\}) = |\phi - f| < \epsilon.$$

Furthermore $|\phi(\xi) - \xi| = |tu + (1 - t)f(\xi) - \xi| = t|\xi - u| > 0$. As $\psi(\xi) = g(\xi) = \xi$ this implies that ϕ and ψ have different fixed points. □

Theorem 3.2. *Let $\{f, g\} \in \mathcal{C}$ f, g contractive with Lipschitz constant $0 \leq \lambda < 1$ and suppose that f, g have fixed points ξ, η , $\xi \neq \eta$. Let $\{x_n\}$ be a trajectory relative to $\{f, g\}$ with initial point $x_0 \in D$. Then for any $\epsilon > 0$ there exist $\delta > 0$ and maps ϕ, ψ contractive with Lipschitz constant $(\lambda + 1)/2$, $\{\phi, \psi\} \in B_{\mathcal{N}}(\{f, g\}, \delta)$ such that the trajectory $\{y_n\}$ relative to $\{\phi, \psi\}$ with initial point $y_0 = x_0$ is regular and $|\{y_n\} - \{x_n\}| < \varepsilon$.*

Proof. The sequence $\{x_n\}$ converges to ξ or η, to fix the ideas suppose that

$$x_n \to \xi. \tag{3.1}$$

It follows

$$0 = |f(\xi) - \xi| < |g(\xi) - \xi|$$

otherwise f and g would have equal fixed points. By a continuity argument there exists $\delta > 0$, $\delta < \varepsilon/8$ such that

(3.2)

$$|z - \xi| < \delta \ , \ |\phi(z) - f(z)| < \delta \ , \ |\psi(z) - g(z)| < \delta \quad \Rightarrow \quad |\phi(z) - z| < |\psi(z) - z|,$$

this implies that z is regular with respect to $\{\phi(z), \psi(z)\}$.

By Proposition 2.9 either

(i) $x_i \neq x_j$ if $i \neq j$, $i, j \in \mathbb{N}_0$

or

(ii) there exists $m \in \mathbb{N}_0$ such that $x_m = \xi$ and $x_i \neq x_j$, $i \neq j$, $0 \leq i, j \leq m$.

Suppose (i) holds

Let $m \in \mathbb{N}_0$ be the smallest index for which $x_m \in B(\xi, \delta)$. Clearly $|x_n - \xi| \geq \delta$ for every $n < m$. Furthermore $|x_n - \xi| < \delta$ for all $n \geq m$ in fact

$$|x_{n+1} - \xi| = |f(x_n) - f(\xi)| \leq \lambda |x_n - \xi| < \delta.$$

Note that, by (3.2), $\{x_n\}$ is regular with respect to $\{f(x_n), g(x_n)\}$ for every $n \geq m$. As the points x_i are pairwise distinct, there exists $r \in \mathbb{R}$

$$0 < r < \delta$$

such that

$$|x_m - \xi| < \delta - r \quad , \quad |x_i - x_j| > 2r \quad , \quad |x_i - \xi| \geq \delta \ , \ i \neq j \quad 0 \leq i, j < m \ . \tag{3.3}$$

Set

$$\theta = 8^{-m}(1 - \lambda) r \ . \tag{3.4}$$

We now define $y_n, \phi(y_n), \psi(y_n)$ with the following properties :

if $x_{n+1} = f(x_n)$ then $y_{n+1} = \phi(y_n)$ and if $x_{n+1} = g(x_n)$ then $y_{n+1} = \psi(y_n)$.

y_n is regular with respect to $\{\phi(y_n), \psi(y_n)\}$.

y_n is *close* to x_n.

Claim 1. There exist points $y_0, y_1, ..., y_m$ and maps ϕ, ψ such that for $0 \leq n < m$ we have

$$|y_{n+1} - x_{n+1}| < 8^n \theta \ , \ |\phi(y_n) - f(y_n)| < 8^n \theta \ , \ |\psi(y_n) - g(y_n)| < 8^n \theta \tag{3.5}$$

and y_n is regular with respect to $\{\phi(y_n), \psi(y_n)\}$.

Let $y_0 = x_0$ and suppose that $x_1 = f(x_0)$ (if $x_1 = g(x_0)$ the argument is similar). Clearly

$$|f(x_0) - x_0| \leq |g(x_0) - x_0| \tag{3.6}$$

If the inequality is strict then we define

$$\phi(y_0) = f(y_0) \ \text{ and } \ \psi(y_0) = g(y_0)$$

Clearly y_0 is regular with respect to $\{\phi(y_0), \psi(y_0)\}$ and $y_1 = \phi(y_0)$. Furthermore

$$\phi(y_0) - f(y_0) = 0 \ , \ \psi(y_0) - g(y_0) = 0 \ , \ |y_1 - x_1| = |\phi(y_0) - f(x_0| = 0.$$

If (3.6) is an equality then we define

$$\phi(y_0) = (1-t)f(y_0) + ty_0 \quad \text{and} \quad \psi(y_0) = g(y_0)$$

where $0 < t < \frac{\theta}{d}$. It follows

$$|\phi(y_0) - y_0| = (1-t)|f(y_0) - y_0| = (1-t)|g(y_0) - y_0| < |\psi(y_0) - y_0|.$$

Then y_0 is regular with respect to $\{\phi(y_0), \psi(y_0)\}$ and $y_1 = \phi(y_0)$.
Furthermore

$$|\phi(y_0) - f(y_0)| = t|f(y_0) - y_0| \leq td < \theta \ \ , \ \ |\psi(y_0) - g(y_0)| = 0$$

and

$$|y_1 - x_1| = |\phi(y_0) - f(x_0)| < t|f(y_0) - y_0| \leq td < \theta.$$

Then (3.5) is satisfied for $n = 0$.

Let $0 < n < m$ and suppose that y_n , $\phi(y_{n-1})$, $\psi(y_{n-1})$ have been defined and satisfy (3.5). Let

$$x_{n+1} = f(x_n) \tag{3.7}$$

(if $x_{n+1} = g(x_n)$ the argument is similar). Clearly

$$|f(x_n) - x_n| \leq |g(x_n) - x_n|. \tag{3.8}$$

If

$$|f(y_n) - y_n| < |g(y_n) - y_n| \tag{a_1}$$

we define

$$\phi(y_n) = f(y_n) \ \text{ and } \ \psi(y_n) = g(y_n).$$

Clearly y_n is regular with respect to $\{\phi(y_n), \psi(y_n)\}$ and $y_{n+1} = \phi(y_n)$. Furthermore

$$|\phi(y_n) - f(y_n)| = |f(y_n) - f(y_n)| = 0 \quad , \quad |\psi(y_n) - g(y_n)| = |g(y_n) - g(y_n)| = 0$$

and

$$\begin{aligned} |y_{n+1} - x_{n+1}| =& |\phi(y_n) - f(x_n)| \leq |\phi(y_n) - f(y_n)| + |f(y_n) - f(x_n)| \\ \leq& 0 + \lambda|y_n - x_n| < \lambda 8^{n-1}\theta < 8^n\theta \end{aligned} \tag{3.9}$$

Hence in this case (3.5) is satisfied.
If

$$|f(y_n) - y_n| = |g(y_n) - y_n| \tag{a_2}$$

we define

$$\phi(y_n) = (1-t)f(y_n) + ty_n \quad \text{and} \quad \psi(y_n) = g(y_n)$$

where $0 < t < \frac{\theta}{d}$. We have

$$|\phi(y_n) - y_n| = (1-t)|f(y_n) - y_n| = (1-t)|g(y_n) - y_n| < |\psi(y_n) - y_n|.$$

Then y_n is regular with respect to $\{\phi(y_n), \psi(y_n)\}$ and $y_{n+1} = \phi(y_n)$. Furthermore

$$|\phi(y_n) - f(y_n)| \leq t|f(y_n) - y_n| \leq td \ < \theta$$

and

$$\begin{aligned} |y_{n+1} - x_{n+1}| &= |\phi(y_n) - f(x_n)| \le (1-t)|f(y_n) - f(x_n)| + t|f(x_n) - y_n| \\ &\le (1-t)\lambda|y_n - x_n| + t|f(x_n) - y_n| \\ &\le (1-t)\lambda 8^{n-1}\theta + td < 8^{n-1}\theta + \theta < 8^n\theta. \end{aligned} \tag{3.10}$$

Then also in this case (3.5) is satisfied.
It remains the case when

$$|f(y_n) - y_n| > |g(y_n) - y_n| \ . \tag{a_3}$$

Observe that

$$\begin{aligned} |f(y_n) - y_n| &\le |f(y_n) - f(x_n)| + |f(x_n) - x_n| + |x_n - y_n| \\ &\le (\lambda+1)|x_n - y_n| + |f(x_n) - x_n| \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} |g(y_n) - y_n| &\ge |g(x_n) - x_n| - |g(y_n) - g(x_n)| - |x_n - y_n| \\ &\ge |g(x_n) - x_n| - (\lambda+1)|x_n - y_n|. \end{aligned} \tag{3.12}$$

In view of (3.8), (3.11) and (3.12) it follows

$$\begin{aligned} |f(y_n) - y_n| - |g(y_n) - y_n| &\le |f(x_n) - x_n| - |g(x_n) - x_n| + 2(\lambda+1)|x_n - y_n| \\ &< 2(\lambda+1)|x_n - y_n| < 4|x_n - y_n| \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \frac{|f(y_n) - y_n|}{|g(y_n) - y_n|} &\le \frac{|f(x_n) - x_n| + 2|y_n - x_n|}{|g(x_n) - x_n| - 2|y_n - x_n|} \le \frac{|f(x_n) - x_n| + 2|y_n - x_n|}{|f(x_n) - x_n| - 2|y_n - x_n|} \\ &\le \frac{|f(x_n) - x_n| - 2|y_n - x_n| + 4|y_n - x_n|}{|f(x_n) - x_n| - 2|y_n - x_n|} \\ &\le 1 + \frac{4|x_n - y_n|}{|f(x_n) - x_n| - 2|x_n - y_n|} \\ &\le 1 + \frac{\frac{4(1-\lambda)r}{8}}{2r - \frac{2(1-\lambda)r}{8}} < \frac{3}{2} \ . \end{aligned}$$

Set

$$\frac{|f(y_n) - y_n|}{|g(y_n) - y_n|} = 1 + \alpha \tag{3.14}$$

and define

$$\phi(y_n) = (1-\alpha)f(y_n) + \alpha y_n \quad \text{and} \quad \psi(y_n) = g(y_n).$$

It follows

$$\begin{aligned} |\phi(y_n) - y_n| = (1-\alpha)|f(y_n) - y_n| &< \frac{1}{1+\alpha}|f(y_n) - y_n| \\ &= |g(y_n) - y_n| = |\psi(y_n) - y_n| \end{aligned}$$

then y_n is regular with respect to $\{\phi(y_n), \psi(y_n)\}$ and $y_{n+1} = \phi(y_n)$. From (3.14) we obtain

$$(3.15)\quad |\phi(y_n) - f(y_n)| = \alpha|f(y_n) - y_n| = \frac{|f(y_n) - y_n| - |g(y_n) - y_n|}{|g(y_n) - y_n|}|f(y_n) - y_n|.$$

Finally from the latter, (3.12) and (3.13) we obtain

$$|\phi(y_n) - f(y_n)| < 6|x_n - y_n| < 6\, 8^{n-1}\theta < 8^n\theta$$

and

$$\begin{aligned}|y_{n+1} - x_{n+1}| &\le |\phi(y_n) - f(x_n)| \le |\phi(y_n) - f(y_n)| + \lambda|y_n - x_n| \\ &< (6+\lambda)|y_n - x_n| < 8\, 8^{n-1}\theta = 8^n\theta\end{aligned}$$

This proves Claim 1.

Observe that (3.5) implies

$$(3.16)\qquad |y_n - x_n| \le 8^{n-1}\theta < 8^{n-1}8^{-m}(1-\lambda)r < \frac{(1-\lambda)r}{8}\ ,\ 1 \le n \le m.$$

Set $E = \{y_0, y_1, .., y_{m-1}\}$ and note that

$$|y_i - y_j| > |x_i - x_j| - |y_i - x_i| - |y_j - x_j| > 2r - 2\frac{(1-\lambda)r}{8} > r$$

Claim 2. The maps $\phi, \psi : E \to D$ are contractive with Lipschitz constant $\frac{\lambda+1}{2}$. We prove the claim for the map ϕ (for the map ψ the argument is similar). In view of (3.5) it follows

$$\begin{aligned}|\phi(y_i) - \phi(y_j)| &\le |\phi(y_i) - f(y_i| + |f(y_i) - f(y_j| + |f(y_j) - \phi(y_j| \\ &\le 8^i\theta + \lambda|y_i - y_j| + 8^j\theta < 2\frac{(1-\lambda)r}{8} + \lambda|y_i - y_j| \\ &\le \left(\frac{(1-\lambda)r}{4|y_i - y_j|} + \lambda\right)|y_i - y_j| \\ &\le \left(\frac{(1-\lambda)r}{4r} + \lambda\right)|y_i - y_j| \le \frac{\lambda+1}{2}|y_i - y_j|.\end{aligned}$$

Then Claim 2 is proved.

Set

$$H = D \setminus \bigcup_{i=0}^{m-1} B(x_i, 2r)$$

and define

$$(3.17)\qquad \phi(y) = f(y)\ \text{ and }\ \psi(y) = g(y)\ ,\ y \in H.$$

Claim 3. The maps $\phi, \psi : H \cup E \to D$ are contractive with Lipschitz constant $\frac{\lambda+1}{2}$. Again, we prove the claim for the map ϕ (for the map ψ the argument is similar).

If $u, v \in H$ then

$$\phi(u) = f(u) \quad , \quad \phi(v) = f(v)$$

which implies

$$|\phi(u) - \phi(v)| \le \lambda|u - v|.$$

If $u, v \in E$ we have proved in Claim 1 that

$$|\phi(u) - \phi(v)| < \frac{\lambda + 1}{2}|u - v|$$

If $u \in H$ and $v \in E$, say $v = y_i$ then

$$\begin{aligned}|\phi(u) - \phi(y_i)| &\leq |f(u) - f(y_i)| + |f(y_i) - \phi(y_i)| \leq \lambda|u - y_i| + \frac{(1-\lambda)r}{8} \\ &\leq |u - y_i|\left(\lambda + \frac{(1-\lambda)r}{8|u - y_i|}\right) \leq |u - y_i|\left(\lambda + \frac{1-\lambda}{8}\right) \\ &\leq \frac{\lambda+1}{2}|u - y_i|.\end{aligned}$$

This completes the proof of Claim 3.

By the Kirszbraun-Valentine theorem [17] the maps ϕ, ψ can be extended to all of D with the same Lipschitz constant. That is there exist $\hat{\phi} : D \to D$, $\hat{\psi} : D \to D$ such that

$$\hat{\phi}(y) = \phi(y) \quad , \quad \hat{\psi}(y) = \psi(y) \ \text{ if } \ y \in H \cup E.$$

and $\hat{\phi}, \hat{\psi}$ contractive with Lipschitz constant $\frac{\lambda+1}{2}$.

Claim 4. For any $y \in D$

$$|\hat{\phi}(y) - f(y)| < \varepsilon \quad , \quad |\hat{\psi}(y) - g(y)| < \varepsilon. \tag{3.18}$$

As above we prove the Claim for the map $\hat{\phi}$ being the other case similar. Indeed, either $y \in H$ or $y \in B(x_i, 2r)$ for some $0 \leq i < m$.

If $y \in H$ then $\hat{\phi}(y) = \phi(y) = f(y)$ and (3.18) clearly holds.

If $y \in B(x_i, 2r)$ then, being $\hat{\phi}$ lipschitzian with Lipschitz constant $\frac{1+\lambda}{2}$ and $\hat{\phi}(y_i) = \phi(y_i)$ it follows

$$\begin{aligned}|\phi(y) - f(y)| &\leq |\phi(y) - \phi(y_i)| + |\phi(y_i) - f(y_i)| + |f(y_i) - f(y)| \\ &\leq \frac{1+\lambda}{2}|y - y_i| + |\phi(y_i) - f(y_i)| + \lambda|y - y_i| \leq 8r < \varepsilon.\end{aligned}$$

This proves Claim 4.

If $y_n \in B_{\mathcal{N}}(\xi, \delta)$ then, by the definition of H and (3.3), it follows that $\hat{\phi}(y_n) = \phi(y_n) = f(y_n)$ and $\hat{\psi}(y_n) = \psi(y_n) = g(y_n)$ which, in view of (3.2), implies that y_n is regular with respect to $\{\hat{\phi}(y_n), \hat{\psi}(y_n)\}$.

Claim 5. For all $n \geq m$

$$|y_n - \xi| < \delta \ . \tag{3.19}$$

We have

$$\begin{aligned}|y_m - \xi| &\leq |y_m - x_m| + |x_m - \xi| < |y_m - x_m| + \delta - r \\ &< \frac{(1-\lambda)r}{8} + \delta - r < \delta - \frac{r}{2}\end{aligned}$$

then (3.19) holds for $n = m$. Suppose that (3.19) is valid for $n = p > m$ and prove it for $n = p + 1$. As $y_n \in B(\xi, \delta) \subset H$, in view of (3.2) one has $y_{n+1} = \phi(y_n) = f(y_n)$. It follows

$$|y_{n+1} - \xi| = |\phi(y_n) - \xi| \leq |f(y_n) - f(\xi)| \leq \lambda|y_n - \xi|.$$

Then (3.19) holds for $n = p+1$ and then for any $n \geq m$.
This proves Claim 5.
From (3.19) it follows that, for any $n > m$, y_n is regular with respect to $\{\hat{\phi}(y_n), \hat{\psi}(y_n)\}$. On the other hand y_n is regular with respect to $\{\hat{\phi}(y_n), \hat{\psi}(y_n)\}$ for $0 \leq n \leq m$, then

$$\{y_n\} \text{ is regular with respect to } \{\phi, \psi\}.$$

Furthermore

$$|y_n - x_n| \leq |y_n - \xi| + |x_n - \xi| < \delta + \delta < \varepsilon \quad \text{for } n > m.$$

and, by (3.5),

$$|y_n - x_n| < \frac{(1-\lambda)r}{8} < \varepsilon \quad \text{for } n \leq m$$

it follows that

$$|\{y_n\} - \{x_n\}| < \varepsilon.$$

This completes the proof of case (i).

Suppose (ii) holds
From Proposition 2.8 the trajectory $\{x_n\}$ is equal to $\{x_0, x_1, ..., x_{m-1}, \xi, \xi, ...\}$ where $x_i \neq x_j$ if $i \neq j$, $0 \leq i, j < m$. By following step by step the previous case (i) we complete the proof of the theorem. □

4. Generic convergence of trajectory

In this section we establish the main result of the paper.

Theorem 4.1. *Let $\{f, g\} \in \mathcal{C}$ f, g contractive with Lipschitz constant $0 \leq \lambda < 1$. Suppose that f, g have fixed points ξ, η , $\xi \neq \eta$ and let $\{x_n\}$ be a regular trajectory relative to $\{f, g\}$ with initial point $u \in D$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that $\{\phi, \psi\} \in \mathcal{N}$, $\{\phi, \psi\} \in B_{\mathcal{N}}(\{f, g\}, \delta)$ imply that the trajectory $\{y_n\}$ relative to $\{\phi, \psi\}$ with initial point u, is regular and*

$$|\{x_n\} - \{y_n\}| < \epsilon .$$

Proof. The regular trajectory $\{x_n\}$ converges to ξ *or* η , to fix the ideas suppose that $x_n \to \xi$.
Clearly

$$0 = |f(\xi) - \xi| < |g(\xi) - \xi|$$

otherwise f and g would have equal fixed points. By a continuity argument there exists $\delta > 0$ such that

$$\text{(4.1)} \qquad |z - \xi| < \delta \ , \ \{\phi, \psi\} \in B_{\mathcal{N}}(\{f, g\}, \delta) \text{ imply } |\phi(z) - z| < |\psi(z) - z|.$$

From the latter it follows that z is regular with respect to $\{\phi(z), \psi(z)\}$.
By assumption the trajectory $\{x_n\}$ converges to ξ. From Proposition 2.9 there are two possible cases, either

(i) $x_i \neq x_j \quad i, j \in \mathbb{N}_0 \ \ i \neq j$
or
(ii) there exists $m \in \mathbb{N}$ such that $x_i \neq x_j \quad i, j < m$, $i \neq j$ and $x_m = \xi$.

Suppose (i) holds.
As $x_n \to \xi$ there exists $m \in \mathbb{N}$ such that

$$|x_n - \xi| < \delta/2 \quad , \quad n \geq m \tag{4.2}$$

then $|f(x_n) - x_n| < |g(x_n) - x_n|$ which implies $x_{n+1} = f(x_n)$.
Consider the m points $x_0, x_1, ..., x_{m-1}$. By assumption the trajectory $\{x_n\}$ is regular then, by continuity argument, there exists σ , $0 < \sigma < \delta$ such that, if

$$\{\phi, \psi\} \in B_{\mathcal{N}}\left(\{f, g\}, \frac{\sigma(1-\lambda)}{2}\right) \tag{4.3}$$

then the first m terms of the trajectory $\{y_n\}$ relative to $\{\phi, \psi\}$ with initial point $y_0 = x_0 = u$ are regular and satisfy

$$|y_n - x_n| < \delta/2 \ , \ n = 0, 1, .., m-1. \tag{4.4}$$

We prove by induction that actually (4.4) is valid for every $n \in \mathbb{N}_0$. Clearly (4.4) is true for $n = m-1$. Suppose that is true for $n = p \geq m$ and prove that is true for $n = p+1$. By the inductive assumption and (4.2) we have

$$|y_p - \xi| \leq |y_p - x_p| + |x_p - \xi| < \delta/2 + \delta/2 = \delta$$

then by (4.1) $|\phi(y_p) - y_p| < |\psi(y_p) - y_p|$ which implies $y_{p+1} = \phi(y_p)$. From (4.4) it follows

$$\begin{aligned} |y_{p+1} - x_{p+1}| &= |\phi(y_p) - f(x_p)| \leq |\phi(y_p) - f(y_p)| + |f(y_p) - f(x_p)| \\ &< \frac{\delta(1-\lambda)}{2} + \lambda|y_p - x_p| < \frac{\delta(1-\lambda)}{2} + \lambda\frac{\delta}{2} = \frac{\delta}{2} \ . \end{aligned}$$

Hence (4.4) is valid for $n = p+1$ and then for any $n \in \mathbb{N}_0$. As $\delta < \varepsilon$ the proof of (i) is complete.

Suppose (ii) holds.

As in the Theorem 3.2 the trajectory $\{x_n\}$ is equal to $\{x_0, x_1, ..., x_{m-1}, \xi, \xi, ...\}$. Again, by following step by step case (i) we complete the proof of the theorem. □

Theorem 4.2. *For a typical* $\{\phi, \psi\} \in \mathcal{N}$ *any trajectory* $\{x_n\}$ *relative to* $\{\phi, \ \psi\}$ *with initial point* $u \in D$ *converges to a point* $z \in fix(\phi) \cup fix(\psi)$.

Proof. Let $\mathcal{C}_0$ be the set of all maps $\{f, g\} \in \mathcal{C}$ such that f and g have distinct fixed points and the trajectory $\{z_n\}$ relative to $\{f, g\}$ with initial point u is regular. It is easily seen that $\mathcal{C}_0$ is dense in $\mathcal{N}$ i.e.

$$\overline{\mathcal{C}_0} = \mathcal{N}. \tag{4.5}$$

In fact let $\{f_0, g_0\} \in \mathcal{N}$ and $\varepsilon > 0$ be given. By Proposition 3.1 there exists $\{f_1, g_1\} \in \mathcal{C}$ such that f_1 and g_1 have distinct fixed points and $h(\{f_0, g_0\}, \{f_1, g_1\}) < \varepsilon/2$. Furthermore, by Theorem 3.2, there exists $\{f_2, g_2\} \in \mathcal{C}$ such that $h(\{f_2, g_2\}, \{f_1, g_1\}) < \varepsilon/2$ and the trajectory $\{y_n\}$ relative to $\{f_2, g_2\}$ with initial point u is regular. Clearly $\{f_2, g_2\} \in C_0$ and $h(\{f_0, g_0\}, \{f_2, g_2\}) < \varepsilon$. This proves (4.5).
Let $\mathcal{N}^* \subset \mathcal{N}$ be defined by

$$\mathcal{N}^* = \bigcap_{i=0}^{\infty} \bigcup_{\{f,g\} \in \mathcal{C}_0} B_{\mathcal{N}}(\{f, g\}, \delta_i) \tag{4.6}$$

where δ_i corresponds to $\varepsilon_i = 1/i$ according to Theorem (4.1). $\mathcal{N}^*$ is the countable intersection of open and dense subset of the complete space $\mathcal{N}$, then is a residual set of $\mathcal{N}$. We are going to prove that each element of $\mathcal{N}^*$ satisfies the statement of the Theorem. To this end let $\{\phi, \psi\} \in \mathcal{N}^*$ be arbitrary and let $\{y_n\}$ be a trajectory relative to $\{\phi, \psi\}$ with initial point $u \in D$. By (4.6), for each $i \in \mathbb{N}$, there exists $\{f_i, g_i\} \in \mathcal{C}_0$ such that

$$\{\phi, \psi\} \in B_{\mathcal{N}}(\{f_i, g_i\}, \delta_i). \tag{4.7}$$

Let $\{x_n^i\}_{n=0}^{\infty}$ be the regular trajectory relative to $\{f_i, g_i\}$ with initial point u. From Theorem 4.1 it follows that

$$|\{y_n\}_{n=0}^{\infty} - \{x_n^i\}_{n=0}^{\infty}| < \varepsilon_i.$$

Claim. The trajectory $\{y_n\}$ is Cauchy.
Let $\varepsilon > 0$ be given and fix i such that $\varepsilon_i < \varepsilon/3$. As $\{f_i, g_i\} \in \mathcal{C}_0$ the trajectory $\{x_n^i\}_{n=0}^{\infty}$ converges, then there exists $k \in \mathbb{N}$ such that

$$|x_n^i - x_m^i| < \varepsilon_i/3 \quad n, m > k.$$

It follows

$$|y_n - y_m| \leq |y_n - x_n^i| + |x_n^i - x_m^i| + |x_m^i - y_m| < 3\varepsilon_i < \varepsilon$$

and consequently

$$|y_n - y_m| < \varepsilon \quad n, m > k$$

The claim is proved.

We now prove that $\{y_n\}$ converges to $z \in fix\phi \ \cup \ fix\psi$. By the previous claim the Cauchy sequence $\{y_n\}$ converges to a $z \in D$. For each n, either $y_{n+1} = \phi(y_n)$ or $y_{n+1} = \psi(y_n)$. Let

$$A = \{n \in \mathbb{N} : y_{n+1} = \phi(y_n)\} \qquad B = \{n \in \mathbb{N} : y_{n+1} = \psi(y_n)\}.$$

Clearly $A \cup B = \mathbb{N}$ and at least one of the sets A and B is infinite, suppose A. Let $\{y_{k_n}\}$ be a subsequence of $\{y_n\}$ whose elements are in A. Thus

$$y_{k_n+1} = \phi(y_{k_n}) \tag{4.8}$$

As $\{y_{k_n+1}\}$ and $\{y_{k_n}\}$ are subsequences of the sequence $\{y_n\}$ which converges to z, they also converge to z. From (4.8) it follows that $z = \phi(z)$ and then $z \in fix\ \phi$. Since $\{\phi, \psi\}$ is an arbitrary element of $\mathcal{N}^*$ the proof is complete. □

Example. Let $D \subset B[0, 1]$ and let $f, g : D \to \mathbb{R}^2$ be given by

$$f(x, y) = (-y, x) \qquad g(x, y) = (y, -x).$$

The maps f and g are non expansive, thus $\{f, g\} \in \mathcal{N}$ and each of them has a unique fixed point $(0, 0)$. Let $z_n(x, y)$ be any trajectory relative to $\{f, g\}$ with initial point $(x_0, y_0) \neq (0, 0)$. From the definition of trajectory

$$z_n(x, y) = \sqrt{x_0^2 + y_0^2}$$

and then $z_n(x, y)$ does not converge to $(0, 0)$. Actually $z_n(x, y)$ does not converge at all because

$$z_{n+1}(x, y) - z_n(x, y) = 2\sqrt{x_0^2 + y_0^2}.$$

By adapting an argument of [3] one obtains the following result.

Theorem 4.3. *The set $\mathcal{C}$ is of the Baire first category in $\mathcal{N}$.*

Proof. Let

$$C_n = \{\{f, g\} \in \mathcal{N} \ : \ h(\{f(x), g(x)\}, \{f(y), g(y)\}) \leq \lambda_n |x - y|\}$$

where $\lambda_n = \frac{n}{n+1}$.
Clearly C_n is closed and

$$C = \bigcup_{n=1}^{\infty} C_n$$

We claim the C_n has empty interior. Suppose, by contradiction, that $\operatorname{int} C_n \neq \emptyset$, that is there exist $\{f, g\} \in \operatorname{int} C_n$ and $\varepsilon > 0$ such that $B_{\mathcal{N}}(\{f, g\}, \varepsilon) \subset C_n$. Let ξ, η be the fixed points of f, g and set

$$\delta = \frac{\varepsilon(1 - \lambda_n)}{4}$$

Define

$$\begin{cases} \phi(x) = x \quad \text{if} \quad x \in B(\xi, \delta) \quad , \quad \phi(x) = f(x) \ \text{ if } \ x \in D \setminus B(\xi, \varepsilon) \\ \psi(x) = g(x) \end{cases}$$

It is easily seen that ϕ is nonexpansive in the set $B(\xi, \delta) \cup (D \setminus B(\xi, \varepsilon))$. In fact, if $x, y \in B(\xi, \delta)$ then the Lipschitz constant of ϕ is equal to 1, while if $x \in D \setminus B(\xi, \varepsilon)$ the Lipschitz constant is equal to λ_n. If $x \in B(\xi, \delta)$ and $y \in D \setminus B(\xi, \varepsilon)$ then

$$|\phi(x) - \phi(y)| = |x - f(y)| \leq |x - \xi| + \lambda_n |y - \xi| \leq (1 + \lambda_n |x - \xi| + \lambda_n |x - y|.$$

Then

$$\frac{|\phi(x) - \phi(y)|}{|x - y|} \leq (1 + \lambda_n) \frac{|x - \xi|}{|x - y|} + \lambda_n.$$

As $|x - \xi| < \delta$ and $|x - y| > \varepsilon/2 - \delta$ it follows

$$\frac{|\phi(x) - \phi(y)|}{|x - y|} \leq 1$$

By Kirszbraun-Valentine theorem there exists $\hat{\phi}$ which extend ϕ to the whole D with the same Lipschitz constant, equal to 1. Furthermore $|\hat{\phi}(x) - f(x)| = 0$ if $x \in D \setminus B(\xi, \varepsilon)$ while if $x \in B(\xi, \varepsilon)$ then $|\hat{\phi}(x) - f(x)| = |\hat{\phi}(x) - \hat{\phi}(\xi)| + |\hat{\phi}(\xi) - f(x) \leq |x - \xi| + \lambda_n |x - \xi| < 2\varepsilon/2 = \varepsilon$ By setting $\hat{\psi}(x) = g(x)$ it follows that $\{\hat{\phi}, \hat{\psi}\} \in B_{\mathcal{N}}(\{f, g\}, \varepsilon)$ and $\{\hat{\phi}, \hat{\psi}\} \notin C_n$, a contradiction. Then C is a countable union of nowhere dense subsets of $\mathcal{N}$. This implies that C is of the Baire first category. This completes the proof. □

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