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SINGULAR POINTS OF NON-MONOTONE POTENTIAL OPERATORS

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Dedicated to the memory of Francesco Saverio De Blasi

Abstract. In this paper, we establish some results about the singular points of certain non-monotone potential operators. Here is a sample: If X is an infinite-dimensional reflexive real Banach space and if $T : X \to X^*$ is a non-monotone, closed, continuous potential operator such that the functional $x \mapsto \int_0^1 T(sx)(x)ds$ is sequentially weakly lower semicontinuous and $\lim_{\|x\|\to+\infty}(\int_0^1 T(sx)(x)ds + \varphi(x)) = +\infty$ for all $\varphi \in X^*$, then the set of all singular points of T is not σ-compact.

Here and in what follows, $(X, \|\cdot\|)$ is a reflexive real Banach space, with topological dual X^*, and $T : X \to X^*$ is a continuous potential operator. This means that the functional

$$x \to J_T(x) := \int_0^1 T(sx)(x)ds$$

is of class C^1 in X and its Gâteaux derivative is equal to T. Let us recall a few classical definitions.

Definition 1. T is said to be monotone if

$$(T(x) - T(y))(x - y) \geq 0$$

for all $x, y \in X$.

This is equivalent to the fact that the functional J_T is convex.

Definition 2. T is said to be closed if for each closed set $C \subseteq X$, the set $T(C)$ is closed in X^*.

Definition 3. T is said to be compact if for each bounded set $B \subset X$, the set $\overline{T(B)}$ is compact in X^*.

Definition 4. T is said to be proper if for each compact set $K \subset X^*$, the set $T^{-1}(K)$ is compact in X.

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Definition 5. T is said to be a local homeomorphism at a point $x_0 \in X$ if there are a neighbourhood U of x_0 and a neighbourhood V of $T(x_0)$ such that the restriction of T to U is a homeomorphism between U and V. If T is not a local homeomorphism at x_0, we say that x_0 is a singular point of T.

We denote by S_T the set of all singular points of T. Clearly, the set S_T is closed.

Assume that the restriction of T to some open set $A \subseteq X$ is of class C^1.

We then denote by $\tilde{S}_{T_{|A}}$ the set of all $x_0 \in A$ such that the operator $T'(x_0)$ is not invertible. Since the set of all invertible operators belonging to $\mathcal{L}(X, X^*)$ is open in $\mathcal{L}(X, X^*)$, by the continuity of T', the set $\tilde{S}_{T_{|A}}$ is closed in A.

Definition 6. T is said to be a Fredholm operator of index zero in A if, for each $x \in A$, the codimension of $T'(x)(X)$ and the dimension of $(T'(x))^{-1}(0)$ are finite and equal.

Definition 7. A set in a topological space is said to be σ-compact if it is the union of an at most countable family of compact sets.

Definition 8. A functional $I : X \to \mathbf{R}$ is said to be coercive if

$$\lim_{\|x\|\to+\infty} I(x) = +\infty \ .$$

The aim of this note is to establish the following results:

Theorem 9. *If X is infinite-dimensional, T is closed and non-monotone, J_T is sequentially weakly lower semicontinuous and $J_T + \varphi$ is coercive for all $\varphi \in X^*$, then both S_T and $T(S_T)$ are not σ-compact.*

Theorem 10. *In addition to the assumptions of Theorem 9, suppose that there exists a closed, σ-compact set $B \subset X$ such that the restriction of T to $X \setminus B$ is of class C^1.*

Then, both $\tilde{S}_{T_{|(X\setminus B)}}$ and $T(\tilde{S}_{T_{|(X\setminus B)}})$ are not σ-compact.

Theorem 11. *Assume that $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space, with $\dim(X) \geq 3$, and that T is compact and of class C^1 with*

$$\liminf_{\|x\|\to+\infty} \frac{J_T(x)}{\|x\|^2} \geq 0 \tag{1}$$

and, for some $\lambda_0 \geq 0$,

$$\lim_{\|x\|\to+\infty} \|x + \lambda T(x)\| = +\infty \tag{2}$$

for all $\lambda > \lambda_0$.

Set

$$\Gamma = \{(x, y) \in X \times X : \langle T'(x)(y), y\rangle < 0\}$$

and, for each $\mu \in \mathbf{R}$,

$$A_\mu = \{x \in X : T'(x)(y) = \mu y \ \ for \ \ some \ \ y \in X \setminus \{0\}\} \ .$$

When $\Gamma \neq \emptyset$, set also

$$\tilde{\mu} = \max\left\{-\frac{1}{\lambda_0}, \inf_{(x,y)\in\Gamma} \frac{\langle T'(x)(y), y\rangle}{\|y\|^2}\right\} \ .$$

Then, the following assertions are equivalent:

(i) *the operator T is not monotone ;*
(ii) *there exists $\mu < 0$ such that $A_\mu \neq \emptyset$;*
(iii) $\Gamma \neq \emptyset$ *and, for each $\mu \in]\tilde{\mu}, 0[$, the set A_μ contains an accumulation point .*

Remark 12. Of course, Theorem 10 is meaningful only when X and X^* are linearly isomorphic. Indeed, if not, the fact that $\tilde{S}_{T_{|(X \setminus B)}}$ is not σ-compact follows directly from the equality $\tilde{S}_{T_{|(X \setminus B)}} = X \setminus B$.

The previous theorems extend and improve the results of [3] in a remarkable way. The reason for this resides in the tools used to prove them. Precisely, in [3], the main tools were Theorems A and B below jointly with the minimax theorem proved in [2]. This latter theorem contains a severe restriction: one of the two variables on which the underlying function depends must run over a real interval. In the current paper, we still continue to use Theorems A and B in an essential way but, this time, jointly with a consequence of another very recent minimax theorem ([4], Theorem 3.2) which is not affected by the above recalled restriction.

So, let us recall Theorems A and B.

Theorem A ([6], Theorem 2.1)**.** *If X is infinite-dimensional, if T is closed and if S_T is σ-compact, then the restriction of T to $X \setminus S_T$ is a homeomorphism between $X \setminus S_T$ and $X^* \setminus T(S_T)$.*

Theorem B ([1], Theorem 5)**.** *If $\dim(X) \geq 3$, if T is a C^1 proper Fredholm operator of index zero and if $\tilde{S}_T$ is discrete, then T is a homeomorphism between X and X^*.*

As we said above, besides Theorems A and B, the other major tool that we will use is a consequence of the following minimax theorem (here stated in a particular version which is enough for our purposes):

Theorem C ([4], Theorem 3.2)**.** *Let Y be a convex set in a real vector space E and let $f : X \times E \to \mathbf{R}$ be sequentially weakly lower semicontinuous and coercive in X, and linear in E. Assume also that*

$$\sup_Y \inf_X f < \inf_X \sup_Y f \ .$$

Then there exists $\tilde{y} \in Y$ such that the functional $f(\cdot, \tilde{y})$ has at least two global minima.

Let us introduce the following notations. We denote by $\mathbf{R}^X$ the space of all functionals $\varphi : X \to \mathbf{R}$. For each $I \in \mathbf{R}^X$ and for each of non-empty subset A of X, we denote by $E_{I,A}$ the set of all $\varphi \in \mathbf{R}^X$ such that $I + \varphi$ is sequentially weakly lower semicontinuous and coercive, and

$$\inf_A \varphi \leq 0 \ .$$

Here is the above mentioned consequence of Theorem C:

Theorem 13. *Let $I : X \to \mathbf{R}$ be a functional and A, B two non-empty subsets of X such that*

$$\sup_A I < \inf_B I \ . \tag{3}$$

Then, for every convex set $Y \subseteq E_{I,A}$ such that

$$\inf_{x\in B} \sup_{\varphi\in Y} \varphi(x) \geq 0 \ \textit{and} \ \inf_{x\in X\setminus B} \sup_{\varphi\in Y} \varphi(x) = +\infty \ , \tag{4}$$

there exists $\tilde{\varphi} \in Y$ such that the functional $I + \tilde{\varphi}$ has at least two global minima.

Proof. Consider the function $f : X \times \mathbf{R}^X \to \mathbf{R}$ defined by

$$f(x, \varphi) = I(x) + \varphi(x)$$

for all $x \in X$, $\varphi \in \mathbf{R}^X$. Fix $\varphi \in Y$. In view of (3), we also can fix $\epsilon \in]0, \inf_B I - \sup_A I[$. Since $\inf_A \varphi \leq 0$, there is $\bar{x} \in A$ such that $\varphi(\bar{x}) < \epsilon$. Hence, we have

$$\inf_{x\in X}(I(x) + \varphi(x)) \leq I(\bar{x}) + \varphi(\bar{x}) < \sup_A I + \epsilon \ ,$$

from which it follows that

$$\sup_{\varphi\in Y} \inf_{x\in X}(I(x) + \varphi(x)) \leq \sup_A I + \epsilon < \inf_B I \ . \tag{5}$$

On the other hand, in view of (4), one has

$$\inf_B I \leq \inf_{x\in B}(I(x) + \sup_{\varphi\in Y} \varphi(x)) = \inf_{x\in B} \sup_{\varphi\in Y}(I(x) + \varphi(x)) = \inf_{x\in X} \sup_{\varphi\in Y}(I(x) + \varphi(x)) \ . \tag{6}$$

Finally, from (5) and (6), it follows that

$$\sup_{\varphi\in Y} \inf_{x\in X} f(x, \varphi) < \inf_{x\in X} \sup_{\varphi\in Y} f(x, \varphi) \ .$$

Therefore the function f satisfies the assumptions of Theorem C and the conclusion follows. □

More precisely, we will use the following corollary of Theorem 13:

Corollary 14. *Let $I : X \to \mathbf{R}$ be a sequentially weakly lower semicontinuous, non-convex functional such that $I + \varphi$ is coercive for all $\varphi \in X^*$.*

Then, for every convex set $Y \subseteq X^*$ dense in X^*, there exists $\tilde{\varphi} \in Y$ such that the functional $I + \tilde{\varphi}$ has at least two global minima.

Proof. Since I is not convex, there exist $x_1, x_2 \in X$ and $\lambda \in]0, 1[$ such that

$$\lambda I(x_1) + (1 - \lambda)I(x_2) < I(x_3)$$

where

$$x_3 = \lambda x_1 + (1 - \lambda)x_2 \ .$$

Fix $\psi \in X^*$ so that

$$\psi(x_1) - \psi(x_2) = I(x_1) - I(x_2)$$

and put

$$\tilde{I}(x) = I(x_3 - x) - \psi(x_3 - x)$$

for all $x \in X$. It is easy to check that

$$\tilde{I}(\lambda(x_1 - x_2)) = \tilde{I}((1 - \lambda)(x_2 - x_1)) < \tilde{I}(0) \ . \tag{7}$$

Fix a convex set $Y \subseteq X^*$ dense in X^* and put

$$\tilde{Y} = -Y - \psi \ .$$

Hence, $\tilde{Y}$ is convex and dense in X^* too. Now, set

$$A = \{\lambda(x_1 - x_2), (1-\lambda)(x_2 - x_1)\} \ .$$

Clearly, we have

$$X^* \subset E_{\tilde{I},A} \ . \tag{8}$$

Since $\tilde{Y}$ is dense in X^*, we have

$$\sup_{\varphi \in \tilde{Y}} \varphi(x) = +\infty$$

for all $x \in X \setminus \{0\}$. Hence, in view of (7) and (8), we can apply Theorem 13 with $B = \{0\}$, $I = \tilde{I}$, $Y = \tilde{Y}$. Accordingly, there exists $\tilde{\varphi} \in Y$ such that the functional $\tilde{I} - \tilde{\varphi} - \psi$ has two global minima in X, say $u_1 \neq u_2$. At this point, it is clear that $x_3 - u_1, x_3 - u_2$ are two global minima of the functional $I + \tilde{\varphi}$, and the proof is complete. □

Remark 15. We remark that Corollary 14 was also obtained very recently in [7] by means of a radically different proof.

We now establish the following technical proposition:

Proposition 16. *If V is an infinite-dimensional real Banach space space and if $U \subset V$ is a σ-compact set, then there exists a convex cone $C \subset V$ dense in V, such that $U \cap C = \emptyset$.*

Proof. This proposition was proved in [3] when V is a Hilbert space ([3], Proposition 2.4). As in [3], we distinguish two cases. First, we assume that V is separable. In this case, the proof provided in [3] can be repeated word for word, and so we omit it. So, assume that V is not separable. Let $\{x_\gamma\}_{\gamma \in \Gamma}$ be a Hamel basis of V. Set

$$\Lambda = \{\gamma \in \Gamma : x_\gamma \notin \operatorname{span}(U)\}$$

and

$$L = \operatorname{span}(\{x_\gamma : \gamma \in \Lambda\}) \ .$$

Clearly, $\operatorname{span}(U)$ is separable since U is so. Hence, Λ is infinite. Introduce in Λ a total order $\leq$ with no greatest element. Next, for each $\gamma \in \Lambda$, let $\psi_\gamma : L \to \mathbf{R}$ be a linear functional such that

$$\psi_\gamma(x_\alpha) = \begin{cases} 1 & if \gamma = \alpha \\ \\ 0 & if \gamma \neq \alpha \ . \end{cases}$$

Now, set

$$D = \{x \in L : \exists \beta \in \Lambda : \psi_\beta(x) > 0 \ \text{and} \ \psi_\gamma(x) = 0 \ \forall \gamma > \beta\} \ .$$

Of course, D is a convex cone. Fix $x \in L$. So, there is a finite set $I \subset \Lambda$ such that $x = \sum_{\gamma \in I} \psi_\gamma(x) x_\gamma$. Now, fix $\beta \in \Lambda$ so that $\beta > \max I$. For each $n \in \mathbf{N}$, put

$$y_n = x + \frac{1}{n} x_\beta \ .$$

Clearly, $\psi_\beta(y_n) = \frac{1}{n}$ and $\psi_\gamma(y_n) = 0$ for all $\gamma > \beta$. Hence, $y_n \in D$. Since $\lim_{n\to\infty} y_n = x$, we infer that D is dense in L. At this point, it is immediate to check that the set $D + \mathrm{span}(U)$ is a convex cone, dense in V, which does not meet U. $\square$

Proof of Theorem 9. Let us prove that S_T is not σ-compact. Arguing by contradiction, assume the contrary. Then, by Theorem A, for each $\varphi \in X^* \setminus T(S_T)$, the equation

$$T(x) = \varphi$$

has a unique solution in X. Moreover, since T is continuous, $T(S_T)$ is σ-compact too. Therefore, in view of Proposition 16, there is a convex set $Y \subset X^*$ dense in X^*, such that $T(S_T) \cap Y = \emptyset$. On the other hand, since T is not monotone, the functional J_T is not convex and so, thanks to Corollary 14, there is $\tilde{\varphi} \in Y$ such that the functional $J_T - \tilde{\varphi}$ has at least two global minima in X which are therefore solutions of the equation

$$T(x) = \tilde{\varphi} ,$$

a contradiction. Now, let us prove that $T(S_T)$ is not σ-compact. Arguing by contradiction, assume the contrary. Consequently, since T is proper ([6], Theorem 1), $T^{-1}(T(S_T))$ would be σ-compact. But then, since S_T is closed and $S_T \subseteq T^{-1}(T(S_T))$, S_T would be σ-compact, a contradiction. The proof is complete. $\square$

Proof of Theorem 10. By Theorem 9, the set S_T is not σ-compact. Now, observe that if $x \in X \setminus (\tilde{S}_{T_{|(X\setminus B)}} \cup B)$, then, by the inverse function theorem, T is a local homeomorphism at x, and so $x \notin S_T$. Hence, we have

$$S_T \subseteq \tilde{S}_{T_{|(X\setminus B)}} \cup B .$$

We then infer that $\tilde{S}_{T_{|(X\setminus B)}}$ is not σ-compact since, otherwise, $\tilde{S}_{T_{|(X\setminus B)}} \cup B$ would be so, and hence also S_T would be σ-compact being closed. Finally, the fact that $T(\tilde{S}_{T_{|(X\setminus B)}})$ is not σ-compact follows as in the final part of the proof of Theorem 9, taking into account that $\tilde{S}_{T_{|(X\setminus B)}}$ is closed in the open set $X \setminus B$ and so it turns out to be the union of an at most countable family of closed sets. $\square$

Proof of Theorem 11. Clearly, since X is a Hilbert space, we are identifying X^* to X. Let us prove that $(i) \to (iii)$. So, assume (i). Since J_T is not convex, by a classical characterization ([8], Theorem 2.1.11), the set Γ is non-empty. Fix $\mu \in]\tilde{\mu}, 0[$. For each $x \in X$, put

$$I_\mu(x) := \frac{1}{2}\|x\|^2 - \frac{1}{\mu} J_T(x) .$$

Clearly, for some $(x, y) \in \Gamma$, we have

$$\left\langle y - \frac{1}{\mu} T'(x)(y), y \right\rangle < 0$$

and so, since

$$I''_\mu(x)(y) = y - \frac{1}{\mu} T'(x)(y) ,$$

the above recalled characterization implies that the functional I_μ is not convex. Since T is compact, on the one hand, J_T is sequentially weakly continuous ([10], Corollary 41.9) and, on the other hand, in view of (2) the operator I'_μ (recall that $-\frac{1}{\mu} > \lambda_0$) is proper ([9], Example 4.43). The compactness of T also implies that, for each $x \in X$, the operator $T'(x)$ is compact ([9], Proposition 7.33) and so, for each $\lambda \in \mathbf{R}$, the operator $y \to y + \lambda T'(x)(y)$ is Fredholm of index zero ([9], Example 8.16). Therefore, the operator I'_μ is non-monotone, proper and Fredholm of index zero. Clearly, by (1), the functional $x \to I_\mu(x) + \langle z, x\rangle$ is coercive for all $z \in X$. Then, in view of Corollary 14, the operator I'_μ is not injective. At this point, we can apply Theorem B to infer that the set $\tilde{S}_{I'_\mu}$ contains an accumulation point. Finally, notice that

$$\tilde{S}_{I'_\mu} = A_\mu \ ,$$

and (iii) follows. The implication $(iii) \to (ii)$ is trivial. Finally, the implication $(ii) \to (i)$ is provided by Theorem 2.1.11 of [8] again. □

Remark 17. Some applications of the above results to weighted eigenvalue problems (which cannot be obtained by means of the results in [3]) are presented in [5].

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References

[1] R. A. Plastock, *Nonlinear Fredholm maps of index zero and their singularities*, Proc. Amer. Math. Soc. **68** (1978), 317–322.

[2] B. Ricceri, *A further improvement of a minimax theorem of Borenshtein and Shul'man*, J. Nonlinear Convex Anal. **2** (2001), 279–283.

[3] B. Ricceri, *On the singular set of certain potential operators in Hilbert spaces*, in Progr. Nonlinear Differential Equations Appl., **75**, Birkhäuser, 2007, pp. 377–391.

[4] B. Ricceri, *A strict minimax inequality criterion and some of its consequences*, Positivity, **16** (2012), 455–470.

[5] B. Ricceri, *Weights sharing the same eigenvalue*, J. Nonlinear Convex Anal., to appear.

[6] R. S. Sadyrkhanov, *On infinite dimensional features of proper and closed mappings*, Proc. Amer. Math. Soc., **98** (1986), 643–658.

[7] J. Saint Raymond, *Characterizing convex functions by variational properties*, J. Nonlinear Convex Anal. **14** (2013), 253–262.

[8] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific, 2002.

[9] E. Zeidlier, *Nonlinear Functional Analysis and its Applications*, vol. III, Springer-Verlag, New York, 1985.

[10] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol. I, Springer-Verlag, New York, 1986.

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