Journal of Nonlinear and Convex Analysis
Volume 16, Number 6, 2015, 1131–1145

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONVEX-VALUED SELECTORS OF A NEMYTSKII OPERATOR WITH NONCONVEX VALUES AND THEIR APPLICATIONS

A. A. TOLSTONOGOV AND S. A. TIMOSHIN

Dedicated to Francesco de Blasi

Abstract. The Nemytskii operator generated by a multivalued mapping whose values are compacts from a Banach space is considered. In every point this mapping is either upper semicontinuous and has convex values or it is lower semicontinuous in a neighborhood of the point. We prove that the multivalued Nemytskii operator has a multivalued selector with convex closed values which is upper semicontinuous in the weak topology of the space of integrable functions in every point of its domain. The result we obtain is applied to prove the existence of a solution to an evolution inclusion with subdifferential operators and a multivalued perturbation, the latter having different semicontinuity types at different points of the domain.

1. Statement of the problem

Let $T = [0,1]$ be an interval of the real half-line $\mathbb{R}^+$ with the Lebesgue measure μ and the σ-algebra $\mathcal{L}$ of μ-measurable sets, $(X, \|\cdot\|)$ be a separable Banach space and $C(T,X)$ be the space of continuous mappings $x : T \to X$ with the topology of uniform convergence on T.

To prove the existence of solutions to a differential inclusion with a multivalued mapping $F : T \times X \to X$ with closed values in the right-hand side or to an evolution inclusion with the mapping F in the right-hand side considered as a perturbation one proceeds as follows. Based on a priori estimates for the solutions one takes a convex compact set $\mathcal{K} \subset C(T,X)$ and considers a multivalued mapping

$$\mathcal{F}(x) = \{u \in L^p(T,X);\ u(t) \in F(t,x(t)) \text{ a.e.}\}, \tag{1.1}$$

$1 \leq p < \infty$, $x \in \mathcal{K}$. Under suitable assumptions on $F(t,x)$ the values of the mapping $\mathcal{F}$ are nonempty closed subsets of the space $L^p(T,X)$. The mapping $\mathcal{F}$ is usually called the multivalued Nemytskii operator generated by the mapping F.

If we construct a multivalued mapping $\mathcal{M} : \mathcal{K} \to L^p(T,X)$

$$\mathcal{M}(x) \subset \mathcal{F}(x),\ \ x \in \mathcal{K}, \tag{1.2}$$

which is upper semicontinuous in the weak topology of the space $L^p(T,X)$ and has convex closed values, then the proof of the existence of solutions to both differential and evolution inclusions is completed by the well-known argument based on the Ky

2010 *Mathematics Subject Classification.* 28B20, 34G25, 47J35.

Key words and phrases. Mixed semicontinuity, evolution inclusion, sweeping process, perturbation, normal cone, subdifferential.

The research was supported by RFBR grant no. 13-01-00287 and by Federal Targeted Programme "Scientific and Scientific-Pedagogical Personnel of Innovative Russia for 2009-2013", project no. 2012-1.2.1-12-000-1001-011.

Fan fixed-point theorem for an upper semicontinuous mapping with convex closed values [5] using relations (1.1), (1.2).

If the multivalued mapping $\mathcal{F} : \mathcal{K} \to L^p(T, X)$ is lower semicontinuous, then there exists an L^p-continuous selector of the mapping $\mathcal{F}$ [6, 19], i.e. a continuous function $f : \mathcal{K} \to L^p(T, X)$ such that

$$f(x) \in \mathcal{F}(x), \quad x \in \mathcal{K}.$$

In this case as the mapping $\mathcal{M}(x)$ we take the function $f(x)$.

If the multivalued mapping $\mathcal{F}$ is not lower semicontinuous and its values are non-convex sets, then the construction of an upper semicontinuous in the weak topology of the space $L^p(T, X)$ multivalued mapping $\mathcal{M}(x)$ with convex closed values is a rather difficult task. Such a mapping was first constructed in the paper [16]. In order to state the result of this work we give next some definitions.

By a multivalued mapping we mean a mapping whose values are sets, including the empty set. Let $\mathcal{B}_X$ be the σ-algebra of Borel sets from X and $\mathcal{L} \otimes \mathcal{B}_X$ be the σ-algebra on $T \times X$ generated by the sets $A \times B$ with $A \in \mathcal{L}$ and $B \in \mathcal{B}_X$. A multivalued mapping $F : T \times X \to X$ is called (weakly) $\mathcal{L} \otimes \mathcal{B}_X$-measurable if the set

$$F^{-1}(V) = \{(t, x) \in T \times X;\ F(t, x) \cap V \neq \emptyset\}$$

is an element of the σ-algebra $\mathcal{L} \otimes \mathcal{B}_X$ for any (open) closed set $V \subset X$ [9]. Recall that since the measure μ is complete, the definitions of measurability and weak measurability for a multivalued mapping $F : T \to X$ with closed values are equivalent [9]. The same definitions we also use for single-valued mappings.

A multivalued mapping $\phi : X \to X$ has a closed graph at a point x_0 if the convergence of a sequence (x_n, y_n), $y_n \in \phi(x_n)$, $n \geq 1$ to a point (x_0, y_0) implies that $y_0 \in \phi(x_0)$. A multivalued mapping $G : \mathcal{K} \to L^1(T, X)$ has a weakly sequentially closed graph if for any sequence $z_n \in \mathcal{K}$, $n \geq 1$ converging to a point z and any sequence $u_n \in G(z_n)$, $n \geq 1$ converging to a point u in the weak topology of the space $L^1(T, X)$ one has $u \in G(z)$.

A function $f : T \times X \to \mathbb{R}^+$ is integrally bounded on bounded sets from X if for any bounded set $A \subset X$ there exists a summable function $m_A : T \to \mathbb{R}^+$ such that $f(t, x) \leq m_A(t)$ a.e. for any $x \in A$.

Let O be the zero element of the space X, $B(O, r) \subset X$ be the open ball centered at O with radius r and $\overline{B}(O, r)$ be its closure. Denote by $Cl(X)$ and $comp(X)$ the collections of all nonempty closed and compact subsets of X, respectively.

We consider a multivalued mapping $F : T \times X \to Cl(X)$ with the following properties **H(F)**:

1) the mapping $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}_X$-measurable;
2) for a.e. t and for any point $x \in X$ either the mapping $F(t, \cdot)$ has a closed graph at the point x and the set $F(t, x)$ is convex or the restriction of $F(t, \cdot)$ to some neighborhood of the point x is lower semicontinuous;
3) there exists an integrally bounded on bounded subsets from X Caratheodory function $f : T \times X \to \mathbb{R}^+$ such that

$$F(t, x) \cap \overline{B}(O, f(t, x)) \neq \emptyset$$

a.e. on T for any x.

Theorem 1.1 (cf. [16, Theorem 2.1]). *Let X be a finite-dimensional space and a multivalued mapping $F : T \times X \to Cl(X)$ satisfy the properties $H(F)1) - 3)$. Then for any $\varepsilon > 0$ and any compact $\mathcal{K} \subset C(T, X)$ there exists a multivalued mapping $\mathcal{M} : \mathcal{K} \to L^1(T, X)$ with convex closed values and a weakly sequentially closed graph such that*

$$v(t) \in F(t, x(t)), \tag{1.3}$$

$$\|v(t)\| \le f(t, x(t)) + \varepsilon \tag{1.4}$$

for any $x \in \mathcal{K}$ and any $v \in \mathcal{M}(x)$ a.e. on T.

Theorem 1.2 (cf. [17, Ch. 2, Theorem 6.6]). *Let X be a separable Banach space and a multivalued mapping $F : T \times X \to comp(X)$ satisfy the properties $H(F)1), 3)$ and the property* H(F) 2')*:*

2') *for a.e. $t \in T$ and for any point $x \in X$ either the multivalued mapping $F(t, \cdot)$ is upper semicontinuous at the point x and the set $F(t, x)$ is convex or the restriction of $F(t, \cdot)$ to some neighborhood of the point x is lower semicontinuous.*

Then the statements of Theorem 1.1 hold true.

In the paper [7] in the case when X is a finite-dimensional space one proved the existence of a multivalued upper semicontinuous selector $\mathcal{M} : \mathcal{K} \to L^1(T, X)$ with closed convex values of the multivalued Nemytskii operator generated by multivalued mappings $F : T \times X \to comp(X)$. The mapping F considered in [7, Examples 3.10 (4)] satisfies hypotheses H(F) 1), 2') and 3'):

3') there exists an integrable $\gamma : T \to \mathbb{R}$ such that $\|y\| \le \gamma(t)$ for every (t, x) and $y \in F(t, x)$.

The one from [7, Examples 3.10 (3)] satisfies hypotheses H(F) 1) and instead of hypotheses H(F) 2'), 3') one considered the hypotheses H(F) 2"), 3"):

2") the multivalued mapping $F(t, \cdot)$ is upper semicontinuous a.e. on T and for each (t, x) such that $F(t, \cdot)$ is nonconvex the multivalued mapping $F(t, \cdot)$ is continuous at x.

3") there exists an integrable $\gamma : T \to \mathbb{R}$ such that $\|y\| \le (1 + \|x\|)\gamma(t)$ for every (t, x) and $y \in F(t, x)$.

We now show that under hypothesis H(F) 2") for the points (t, x) in which the set $F(t, x)$ is nonconvex the multivalued mapping $F(t, \cdot)$ is continuous not only at the point x but also in some neighborhood of x. Assume that this is not true. Then, there exists a sequence of points x_n, $n \ge 1$ converging to x for which the sets $F(t, x_n)$, $n \ge 1$ are convex. Since the mapping $F(t, \cdot)$ is continuous at the point x the sequence of sets $F(t, x_n)$, $n \ge 1$ converges in the Hausdorff metric to the set $F(t, x)$. The completeness of the space of convex compacts with the Hausdorff metric should then imply that the set $F(t, x)$ is convex, which is a contradiction. Therefore, for the points (t, x) in which the set $F(t, x)$ is nonconvex the multivalued mapping $F(t, \cdot)$ is continuous in some neighborhood of x and thus is lower semicontinuous in this neighborhood. Consequently, the multivalued mappings F in [7, Examples 3.10 (3), (4)] are particular instances of the multivalued mapping F for which Theorem 1.2 holds true.

If a multivalued mapping $\mathcal{M} : \mathcal{K} \to L^1(T, X)$ with convex closed values is upper semicontinuous, then it is upper semicontinuous in the weak topology of the space $L^1(T, X)$ and hence it has a weakly sequentially closed graph. This shows that Theorems 1.1 and 1.2 have a wider range of applications than the statements in [7, Examples 3.10 (3), (4)].

Note that differential inclusions in a finite-dimensional space with multivalued mappings F having different semicontinuity type were studied in [10, 12, 14], where theorems on existence of solutions were proved by constructing approximate solutions. In [14] the multivalued mapping F satisfies hypotheses H(F) 1), 2"), 3'), in [12] and [10] hypotheses H(F) 1), 2'), 3') and H(F) 1), 2'), 3), respectively, and the results of these works are thus corollaries of Theorem 3.1 in [16] proved using Theorem 1.1 and the Ky Fan fixed-point theorem.

2. Modification of Theorems 1.1. and 1.2 for applications

In this section we revise the statements of Theorems 1.1 and 1.2 in view of applications we consider in the sequel. For convenience of the reader we give next several definitions.

Let Z and Y be metric spaces and $U : Z \to Y$ be a multivalued mapping with closed values. The multivalued mapping U is upper semicontinuous at a point z_0 if for any open set $W \subset Y$, $U(z_0) \subset W$ there exists a neighborhood $V(z_0)$ of the point z_0 such that $U(z) \subset W$ for all $z \in V(z_0)$. The multivalued mapping U is lower semicontinuous at a point z_0 if for any open set $W \subset Y$, $U(z_0) \cap W \neq \emptyset$ there exists a neighborhood $V(z_0)$ of the point z_0 such that $U(z) \cap W \neq \emptyset$ for all $z \in V(z_0)$. The mapping U is continuous at a point z_0 if it is both upper and lower semicontinuous at z_0.

If the multivalued mapping U is upper semicontinuous at a point z_0, then it has a closed graph at this point. If for some neighborhood $V(z_0)$ of a point z_0 there exists a compact set $K \subset Y$ such that $U(z) \subset K$ for all $z \in V(z_0)$, then the closedness of the graph of U at z_0 implies the upper semicontinuity of U at z_0.

A function $f : T \times X \to \mathbb{R}^+$ is p-integrally bounded on bounded sets $A \subset X$ if there exists a function $m_A \in L^p(T, \mathbb{R}^+)$, $1 \leq p < \infty$ such that $f(t, x) \leq m_A(t)$ a.e. on T for any $x \in A$.

Theorems 1.1 and 1.2 were proved for multivalued mappings with values in the space $L^1(T, X)$. However, in a large variety of problems multivalued mappings with values in the spaces $L^p(T, X)$, $1 < p < \infty$, are used. The following theorem is an analogue of Theorem 1.2 for multivalued mappings in the spaces $L^p(T, X)$, $1 \leq p < \infty$.

Theorem 2.1. *Let X be a separable reflexive Banach space and a multivalued mapping $F : T \times X \to comp(X)$ satisfy the properties* H(F) 1), 2'), 3) *with a Caratheodory function $f : T \times X \to \mathbb{R}^+$ which is p-integrally bounded on bounded subsets from X, $1 \leq p < \infty$. Then for any $\varepsilon > 0$ and any compact $\mathcal{K} \subset C(T, X)$ there exists an upper semicontinuous in the weak topology of the space $L^p(T, X)$ multivalued mapping $\mathcal{M} : \mathcal{K} \to L^p(T, X)$ with convex weakly compact values such that relations* (1.3), (1.4) *hold for any $x \in \mathcal{K}$ and any $v \in \mathcal{M}(x)$ a.e. on T.*

Proof. Since the set $\mathcal{K} \subset C(T,X)$ is compact there exists a p-integrable function $m : T \to \mathbb{R}^+$ such that

$$f(t,x(t)) \le m(t) \quad \text{a.e. on } T \quad \text{for any } x(\cdot) \in \mathcal{K}. \tag{2.1}$$

Consider a multivalued mapping $\Gamma : T \to X$ defined by the rule

$$\Gamma(t) = \{x \in X;\ \|x\| \le m(t) + \varepsilon \quad \text{a.e. on } T\}, \tag{2.2}$$

which is measurable with convex weakly compact values. Let

$$S^p = \{x \in L^p(T,X);\ x(t) \in \Gamma(t) \quad \text{a.e. on } T\}. \tag{2.3}$$

Then, S^p is convex weakly compact subset of the space $L^p(T,X)$, $1 \le p < \infty$. Let $\mathcal{M} : \mathcal{K} \to L^1(T,X)$ be the multivalued mapping with the properties established in Theorem 1.2. From (1.4) and (2.1) - (2.3) it follows that

$$\mathcal{M}(x) \subset S^p, \quad x \in \mathcal{K}. \tag{2.4}$$

Since on the set S^p the weak topologies of the spaces $L^1(T,X)$ and $L^p(T,X)$, $1 \le p < \infty$ coincide $\mathcal{M}(x)$ is a multivalued mapping from $\mathcal{K}$ to $L^p(T,X)$ with convex weakly compact values and a weakly sequentially closed graph. The fact that the set S^p, $1 \le p < \infty$ is metrizable in the weak topology of the space $L^p(T,X)$ implies that the multivalued mapping $\mathcal{M}(x)$ is upper semicontinuous in the weak topology of the space $L^p(T,X)$ and satisfies relations (1.3), (1.4). The theorem follows. □

Using Theorem 2.1 we give an analogue of Theorem 1.1 for multivalued mappings with values in the spaces $L^p(T,X)$, $1 \le p < \infty$.

Corollary 2.2. *Let X be a finite-dimensional space and a multivalued mapping $F : T \times X \to Cl(X)$ satisfy the properties* H(F) 1) - 3) *with a Caratheodory function $f : T \times X \to \mathbb{R}^+$ which is p-integrally bounded on bounded subsets from X. Then the statements of Theorem* 2.1 *hold true.*

Proof. Consider a multivalued mapping

$$C(t,x) = \overline{F(t,x) \cap B(O, f(t,x) + 2\,\varepsilon)}. \tag{2.5}$$

Let $\eta_n \ge 0$, $n \ge 1$ be a monotone increasing sequence converging to $2\,\varepsilon$ and

$$\overline{B}_n(t,x) = \overline{B}(O, f(t,x) + \eta_n).$$

Then

$$F(t,x) \cap B(O, f(t,x) + 2\varepsilon) = \bigcup_{n=1}^{\infty} \left(F(t,x) \cap \overline{B}_n(t,x) \right). \tag{2.6}$$

Since $f(t,x)$ is a Caratheodory function it follows from the Scorza-Dragoni theorem that there exists a sequence of compacts $T_k \subset T_{k+1}$, $k \ge 1$ such that $\mu(T \setminus \bigcup_{k=1}^{\infty} T_k) = 0$ and the restriction of $f(t,x)$ to $T_k \times X$, $k \ge 1$ is continuous. Therefore, the restriction of the multivalued mapping $\overline{B}_n(t,x)$ to $T_k \times X$, $k \ge 1$ is continuous and hence lower semicontinuous. This means that the restriction of $\overline{B}_n(t,x)$ to $T_k \times X$, $k \ge 1$ is weakly $\mathcal{L} \otimes \mathcal{B}_X$-measurable. From the equality $\mu(T \setminus \bigcup_{k=1}^{\infty} T_k) = 0$ it follows that the multivalued mapping $\overline{B}_n(t,x)$ is weakly $\mathcal{L} \otimes \mathcal{B}_X$-measurable. According to Theorem

3.5 in [9] the multivalued mapping $F(t,x)$ is weakly $\mathcal{L}\otimes\mathcal{B}_X$-measurable as well. Now from (2.6), Theorem 4.1 and Proposition 2.4 in [9] we infer that the multivalued mapping $F(t,x)\cap B(O,f(t,x)+2\varepsilon)$ is weakly $\mathcal{L}\otimes\mathcal{B}_X$-measurable. Using Proposition 2.6 and Theorem 3.5 from [9] we see that the multivalued mapping $C(t,x)$ is $\mathcal{L}\otimes\mathcal{B}_X$-measurable with nonempty compact values.

At the points in which the set $F(t,x)$ is convex we have

$$C(t,x) = F(t,x) \cap \overline{B}(O, f(t,x) + 2\varepsilon). \tag{2.7}$$

Therefore, at those points x where the multivalued mapping $F(t,x)$ has a closed graph and the set $F(t,x)$ is convex the multivalued mapping $C(t,x)$ has the same properties. According to (2.7) for all y from a neighborhood $V(x)$ of the point x the inclusion

$$C(t,y) \subset \overline{B}(O,r)$$

takes place for some $r > 0$. Since the set $\overline{B}(O,r)$ is compact the multivalued mapping $C(t,\cdot)$ is upper semicontinuous at the points x in which the multivalued mapping $C(t,\cdot)$ has a closed graph and the set $C(t,x)$ is convex.

At those points x where the multivalued mapping $F(t,\cdot)$ is lower semicontinuous the multivalued mapping $x \to F(t,x) \cap B(O, f(t,x) + 2\varepsilon)$ is lower semicontinuous as well. Then, at these points the multivalued mapping $C(t,x)$ is also lower semicontinuous. Therefore, all the assumptions of Theorem 2.1 hold for the multivalued mapping $C(t,x)$. According to this theorem for an $\varepsilon > 0$ and a compact $\mathcal{K} \subset C(T,X)$ there exists an upper semicontinuous in the weak topology of the space $L^p(T,X)$ multivalued mapping $\mathcal{M} : \mathcal{K} \to L^p(T,X)$, $1 \leq p < \infty$ with convex weakly compact values such that

$$v(t) \in C(t,x(t)) \subset F(t,x(t)),$$

$$\|v(t)\| \leq f(t,x(t)) + \varepsilon,$$

and the corollary thus follows. □

3. Main results

In this section we use Theorem 2.1 to prove the existence of a solution to an evolution inclusion with a subdifferential operator and a multivalued perturbation with different semicontinuity types at different points.

Let $\overline{\mathbb{R}} = (-\infty, +\infty]$, H be a separable Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. A function $\varphi^t : H \to \overline{\mathbb{R}}$ is called proper if its effective domain $\operatorname{dom}\varphi = \{x \in H; \varphi(x) < \infty\}$ is nonempty. Denote by $\Gamma_0(H)$ the class of all proper, convex and lower semicontinuous functions $\varphi : H \to \overline{\mathbb{R}}$. For a function $\varphi \in \Gamma_0(H)$ denote by $\partial\varphi(x)$ its subdifferential at a point x and by $\operatorname{dom}\partial\varphi$ the domain of $\partial\varphi$. It is known that $\operatorname{dom}\partial\varphi \subset \operatorname{dom}\varphi$ and $\overline{\operatorname{dom}\partial\varphi} = \overline{\operatorname{dom}\varphi}$, where the bar stands for the closure in H.

By ω-H and ω-$L^2(T,H)$ we denote the spaces H and $L^2(T,H)$ equipped with the weak topologies. The same notations are used for subsets of the spaces ω-H and ω-$L^2(T,H)$.

Consider the inclusion

$$
\begin{aligned}
&-A\dot{x}(t) \in \partial\varphi^t x(t) + B(t, x(t)) + F(t, x(t)) \quad \text{a.e.,} \\
&x(0) = x_0 \in \operatorname{dom}\varphi^0.
\end{aligned} \tag{3.1}
$$

Here, $\varphi^t \in \Gamma_0(H)$, $t \in T$, $A : H \to H$ is a continuous linear operator, $B(t, \cdot) : \operatorname{dom} B(t, \cdot) \subset H \to H$, $t \in T$ is a nonlinear mapping, $F : T \times H \to H$ is a multivalued mapping with closed values.

By a solution of inclusion (3.1) we mean a function $x(\cdot) \in W^{1,2}(T, H)$, $x(0) = x_0$, such that $x(t) \in \operatorname{dom}\partial\varphi^t$ a.e. and the inclusion

$$
-A\dot{x}(t) \in \partial\varphi^t x(t) + B(t, x(t)) + f(t) \tag{3.2}
$$

holds a.e. for some $f \in L^2(T, H)$ such that

$$
f(t) \in F(t, x(t)) \quad \text{a.e.} \tag{3.3}
$$

It is known that if $x(t)$ is a solution of inclusion (3.1), then $x(t) \in \operatorname{dom}\varphi^t$, $t \in T$.

We make the following assumptions on A, φ^t, B and F:

Hypothesis H(φ). The family of functions $\varphi^t \in \Gamma_0(H)$, $t \in T$ has the property:

for each $r > 0$ there exists absolutely continuous functions $a_r, b_r : T \to \mathbb{R}$ such that $\dot{a}_r \in L^2(T, \mathbb{R})$, $\dot{b}_r \in L^1(T, \mathbb{R})$ and for any $s, t \in T$, $s \le t$ and any $x \in \operatorname{dom}\varphi^s$ with $\|x\| \le r$ there is an element $y \in \operatorname{dom}\varphi^t$ satisfying the inequalities

$$
\begin{aligned}
\|x - y\| &\le |a_r(t) - a_r(s)| \left(|\varphi^s(x)|^{\frac{1}{2}} + 1 \right), \\
\varphi^t(y) - \varphi^s(x) &\le |b_r(t) - b_r(s)| \left(|\varphi^s(x)| + 1 \right).
\end{aligned}
$$

Hypothesis H(A). The operator $A : H \to H$ is linear continuous self-adjoint and

$$
l_A\|x\|^2 \le \langle x, Ax\rangle, \quad x \in H, \ l_A > 0.
$$

Hypothesis H(B). The mapping $B(t, x)$ has the following properties

1) $B(t, \cdot)$ is a single-valued operator from H to H with the domain $\operatorname{dom} B(t, \cdot)$ such that

$$
\operatorname{dom} B(t, \cdot) \supset Q \supset \bigcup_{s \in T} \operatorname{dom}\varphi^s
$$

for each $t \in T$, where Q is a convex Borel set;

2) for each $t \in T$ the operator $B(t, \cdot)$ is demicontinuous and monotone, i.e. if $x_n \to x$ in H, then $B(t, x_n) \to B(t, x)$ in the weak topology of the space H and

$$
\langle B(t, x) - B(t, y), x - y\rangle \ge 0;
$$

3) for each $x \in Q$ the function $t \to B(t, x)$ is measurable;

4) for each $0 < \eta \le 1$ there exists a nondecreasing function $L_\eta : [0, +\infty) \to [0, +\infty)$, $L_{\eta_1}(r) \le L_{\eta_2}(r)$, $\eta_1 \le \eta_2$, $r \in [0, +\infty)$, such that

$$
\|B(t, x)\| \le \eta \| \left(\partial\varphi^t\right)^0 x\| + L_\eta(\|x\|),
$$

$t \in T$, $x \in \operatorname{dom}\varphi^t$, where $\left(\partial\varphi^t\right)^0 x$ is an element of minimal norm of the set $\partial\varphi^t x$.

Hypothesis $\mathbf{H}_1\mathbf{(F)}$. The mapping $F : T \times H \to comp(H)$ is such that

1) F satisfies the properties H(F) 1), 2'), 3) with the function
$$f(t,x) = m(t) + n(t)\|x\|, \quad m,n \in L^2(T,\mathbb{R}^+);$$
2) there exists $\varepsilon > 0$ such that for any bounded set $C \subset H$ the set
$$F(t,C) \cap \overline{B}(\Theta, m(t) + n(t)\|C\| + \varepsilon)$$
is relatively compact, where $F(t,C) = \{F(t,x);\ x \in C\}$, Θ is the zero element of the space H and $\|C\| = \sup\{\|x\|;\ x \in C\}$.

Remark 3.1. If the mapping $B(t,x)$ satisfies a traditional growth condition
$$\|B(t,x)\| \le a + b\|x\| \quad \text{a.e.},\ a,b > 0,$$
then it satisfies Hypothesis H(B) 4). In this case, we may choose the function $L_\eta(r) = (1+\eta)(a+br)$, $r \in [0,+\infty)$, $0 < \eta \le 1$.

In what follows, we will denote for convenience
$$f(t,\|x\|) = m(t) + n(t)\|x\|. \tag{3.4}$$

Lemma 3.2. *Let the multivalued mapping $F : T \times H \to comp(H)$ satisfy the properties $H(F)\,1), 3)$ with the function $f(t,\|x\|)$ defined by* (3.4)*. Then, the multivalued mapping*
$$t \to F(t,x(t)) \cap \overline{B}\,(\Theta, f(t,\|x\|) + \varepsilon)$$
is measurable for any continuous function $x : T \to H$.

Proof. Let a function $x : T \to H$ be continuous, $V \subset H$ be a closed set and $\mathcal{T} = \{t \in T;\ F(t,x(t)) \cap V \ne \emptyset\}$. Since the multivalued mapping F is $\mathcal{L} \otimes \mathcal{B}_H$-measurable
$$F^{-1}(V) = \{(t,x);\ F(t,x) \cap V \ne \emptyset\} \in \mathcal{L} \otimes \mathcal{B}_H.$$
The fact that the graph $\operatorname{gr} x$ of the function x is a closed subset of the space $T \times H$ implies that $F^{-1}(V) \cap \operatorname{gr} x \in \mathcal{L} \otimes \mathcal{B}_H$.

Consider a mapping $v : T \to H$ defined by the rule
$$v(t) = \{y \in H;\ (t,y) \in F^{-1}(V) \cap \operatorname{gr} x\}.$$
The mapping $v(t)$ is single-valued and may admit the empty set as its values. From [9, Theorem 3.5 (iii)] it follows that the mapping $v(t)$ is measurable. Then, according to [9, Proposition 2.2] the set $\operatorname{dom} v = \{t \in T;\ v(t) \ne \emptyset\}$ is measurable. Since $\mathcal{T} = \operatorname{dom} v$, the multivalued mapping $t \to F(t,x(t))$ is measurable.

The measurability of the multivalued mapping $t \to \overline{B}\,(\Theta, f(t,\|x\|) + \varepsilon)$ is proved along the same lines as that of $t \to F(t,x(t))$. Indeed, similarly to the proof of Corollary 2.2 we may see that the multivalued mapping $\overline{B}\,(\Theta, f(t,\|x\|) + \varepsilon)$ is $\mathcal{L} \otimes \mathcal{B}_H$-measurable. Hence, according to [9, Theorem 3.5 (iii)] the graphs of the multivalued mappings $t \to F(t,x(t))$ and $t \to \overline{B}\,(\Theta, f(t,\|x\|) + \varepsilon)$ are $\mathcal{L} \otimes \mathcal{B}_H$-measurable. So is the graph of their intersection. This and [9, Theorem 3.5 (iii)] finally imply that the multivalued mapping $t \to F(t,x(t)) \cap \overline{B}\,(\Theta, f(t,\|x\|) + \varepsilon)$ is measurable and the lemma follows. □

In the sequel, we will need the following theorem

Theorem 3.3 (cf. [18, Theorem 6.2]). *Let Hypotheses $H(\varphi), H(A)$ and $H(B)$ hold true. Then for any $f \in L^2(T,H)$, $\|f\|_{L^2(T,H)} \le R$, there exists a unique solution $x(f)$ of inclusion* (3.2) *and*

$$\|x(f)(t)\| \le N(R), \;\; t \in T, \quad \|\dot{x}(f)\|_{L^2(T,H)} \le N(R), \tag{3.5}$$

$$|\varphi^t(x(f)(t))| \le N(R), \quad t \in T, \tag{3.6}$$

where $N(R) > 0$ is some constant depending only on R. For any two solutions $x(f_i)$, $f_i \in L^2(T,H)$, $i = 1,2$ we have

$$\|x(f_1)(t) - x(f_2)(t)\| \le l_A^{-1} \int_0^t \|f_1(s) - f_2(s)\| ds. \tag{3.7}$$

Let $x(\Theta)$ be the solution of inclusion (3.2) for $f(t) \equiv \Theta$, and $r_0 = \max\{\|x(\Theta)(t)\|; t \in T\}$. From (3.7) it follows that

$$\|x(f)(t)\| \le r_0 + l_A^{-1} \int_0^t \|f(s)\| ds, \;\; t \in T, \; f \in L^2(T,H). \tag{3.8}$$

Consider the differential equation

$$\begin{aligned} \dot{r}(t) &= l_A^{-1}(f(t,r(t)) + \varepsilon), \\ \dot{r}(0) &= r_0, \end{aligned} \tag{3.9}$$

with the function $f(t,r) = m(t) + n(t)r$, which has a unique solution $r(t) \ge 0$ defined on T. Let

$$S = \{g \in L^2(T,H); \; \|g(t)\| \le f(t,r(t)) + \varepsilon \quad \text{a.e.}\}. \tag{3.10}$$

The set S is a convex metrizable compact of the space ω-$L^2(T,H)$. Denote by $\mathcal{T} : L^2(T,H) \to C(T,H)$ the operator which with each $f \in L^2(T,H)$ associates the unique solution $x(f)$ of inclusion (3.2), i.e.

$$x(f) = \mathcal{T}(f).$$

Let

$$C(S) = \{x(f) \in C(T,H); \; f \in S\}$$

and

$$C(S)(t) = \{x(f)(t); \; f \in S\}$$

From inequalities (3.8) – (3.10) it follows that for any $x(\cdot) \in C(S)$ the inequality

$$\|x(t)\| \le r_0 + \int_0^t l_A^{-1}(f(s,r(s)) + \varepsilon) ds = r(t).$$

holds. Hence,

$$\|C(S)(t)\| \le r(t), \quad t \in T. \tag{3.11}$$

Let Hypotheses $H_1(F)$ hold true. Then, from (3.10) we infer that the set $F(t, C(S)(t)) \cap \overline{B}\,(\Theta, f(t, \|C(S)(t)\|) + \varepsilon)$ is relatively compact for each $t \in T$. Denote by

$$\Gamma(t) = \overline{\text{co}}\left(F(t, C(S)(t)) \cap \overline{B}\,(\Theta, f(t, \|C(S)(t)\|) + \varepsilon)\right). \tag{3.12}$$

Then, $\Gamma(t)$ is a multivalued mapping with convex compact values. Let

$$S_\Gamma = \{f \in L^2(T,H); \; f(t) \in \Gamma(t) \quad \text{a.e.}\}. \tag{3.13}$$

From our Lemma 3.2 and Theorems 5.5, 5.6 in [9] it follows that for each $x \in C(S)$ the mapping

$$t \to F(t, x(t)) \cap \overline{B}\,(\Theta, f(t, \|x(t)\|) + \varepsilon)$$

has a measurable selector which according to (3.11) is an element of the space $L^2(T, H)$. Therefore, S_Γ is a nonempty convex metrizable compact of the space ω-$L^2(T, H)$.

Lemma 3.4. *Let Hypotheses $H_1(F)$ hold true. Then the operator $\mathcal{T}$ is continuous from ω-S_Γ to $C(T, H)$.*

Proof. Let $f_n \in S_\Gamma$, $n \geq 1$ converges to f in the topology of the space ω-$L^2(T, H)$. From (3.10), (3.13) we see that $S_\Gamma \subset S$. Then $x(f_n), x(f) \in C(S)$, $n \geq 1$. Hence

$$\|x(f_n)(t)\| \leq r(t), \quad \|x(f)(t)\| \leq r(t), \quad t \in T. \tag{3.14}$$

From (3.12), (3.11) it follows that $\|f_n\|_{L^2(T,H)} \leq R$ for some $R > 0$. Using (3.5) and Hölder's inequality we obtain

$$\|x(f_n)(t) - x(f_n)(s)\| \leq N(R)|t - s|^{\frac{1}{2}}. \tag{3.15}$$

From this inequality it follows that the sequence $x(f_n)$, $n \geq 1$ is equicontinuous in $C(T, H)$. Then, from (3.14) and Theorem 4 in [11] we infer that the sequence $x(f_n)$, $n \geq 1$ is relatively compact in the space ω-$C(T, H)$. Since the space $C(T, H)$ is separable, any compact of the space ω-$C(T, H)$ is metrizable. Therefore, there exists a subsequence n_k, $k \geq 1$ of the sequence $n \geq 1$ such that

$$x(f_{n_k}) \to z(\cdot) \quad \text{in } \omega\text{-}C(T, H). \tag{3.16}$$

From (3.16) and Theorem 3 in [11] it follows that

$$\langle x(f_{n_k})(t) - z(t), h\rangle \to 0, \quad t \in T \quad \text{for any } h \in H.$$

Using (3.2) and Hypotheses $H(A)$, $H(B)$ 2) we obtain

$$\begin{aligned} \frac{1}{2} l_A \|x(f_{n_k})(t) - x(f)(t)\|^2 \leq& \int_0^t \langle x(f_{n_k})(\tau) - z(\tau), f(\tau) - f_{n_k}(\tau)\rangle \, d\tau \\ &+ \int_0^t \langle z(\tau) - x(f)(\tau), f(\tau) - f_{n_k}(\tau)\rangle \, d\tau \end{aligned} \tag{3.17}$$

Since $\langle x(f_{n_k})(\tau) - z(\tau), h\rangle \to 0$ for any $h \in H$, the sequence of functions $h \to \langle x(f_{n_k})(\tau) - z(\tau), h\rangle$ is equicontinuous. It is known that for each equicontinuous set the topology of pointwise convergence coincides with the topology of uniform convergence on compacts. The facts that $f(\tau), f_{n_k}(\tau) \in \Gamma(\tau)$ and that the set $\Gamma(\tau)$ is a convex compact in the space H imply that

$$\langle x(f_{n_k})(\tau) - z(\tau), f(\tau) - f_{n_k}(\tau)\rangle \to 0 \quad \text{as } k \to \infty.$$

Now, from (3.14), the inclusion $S_\Gamma \subset S$, (3.10) and Lebesgue's dominated convergence theorem we obtain

$$\int_0^t \langle x(f_{n_k})(\tau) - z(\tau), f(\tau) - f_{n_k}(\tau)\rangle \, d\tau \to 0 \quad t \in T.$$

Then, from the convergence $f_{n_k} \to f$ in the space ω-$L^2(T,H)$ and (3.17) it follows that

$$x(f_{n_k})(t) \to x(f)(t) \ \text{ in } H \quad \text{for any } t \in T.$$

Using (3.15) we further infer that

$$x(f_{n_k}) \to x(f) \ \text{ in } C(T,H).$$

If we suppose that the sequence $x(f_n)$, $n \geq 1$ does not converge to $x(f)$ in $C(T,H)$, then there would exist a subsequence $x(f_{n_k})$, $k \geq 1$ of the sequence $x(f_n)$, $n \geq 1$ such that any subsequence of the sequence $x(f_{n_k})$, $k \geq 1$ does not converge to $x(f)$. Repeating this argument to the sequence $x(f_{n_k})$, $k \geq 1$ we arrive at a contradiction. Consequently, $\mathcal{T}(f_n) \to \mathcal{T}(f)$ in $C(T,H)$ and the lemma follows. □

Theorem 3.5. *Let Hypotheses $H(\varphi), H(A), H(B)$ and $H_1(F)$ hold true. Then inclusion* (3.1) *has a solution.*

Proof. Since S_Γ is a convex compact subset of the space ω-$L^2(T,H)$, Lemma 3.4 implies that the set

$$\mathcal{K} = \{x(f) \in C(T,H);\ f \in S_\Gamma\}$$

is a compact subset of the space $C(T,H)$. According to Theorem 2.1 there exists an upper semicontinuous mapping $\mathcal{M} : \mathcal{K} \to \omega$-$L^2(T,H)$ with convex compact values such that for any $x \in \mathcal{K}$ and any $v \in \mathcal{M}(x)$ we have

$$v(t) \in F(t, x(t)) \quad \text{a.e.,} \tag{3.18}$$

$$\|v(t)\| \leq f(t, \|x(t)\| + \varepsilon) \quad \text{a.e.} \tag{3.19}$$

Consider a multivalued mapping $\mathcal{M}(\mathcal{T}) : S_\Gamma \to L^2(T,H)$ defined by the rule $\mathcal{M}(\mathcal{T})(f) = \mathcal{M}(\mathcal{T}(f))$. From Lemma 3.4 we conclude that $\mathcal{M}(\mathcal{T})$ is an upper semicontinuous mapping from ω-S_Γ to ω-$L^2(T,H)$ with convex compact values. Since $\mathcal{K} \subset C(S)$ from (3.18) it follows that

$$v(t) \in F(t, C(S)(t)) \cap \overline{B}\,(\Theta, f(t, \|C(S)(t)\|) + \varepsilon) \quad \text{a.e.}$$

Hence $v \in S_\Gamma$. Consequently $\mathcal{M}(\mathcal{T})(f) \subset S_\Gamma$ for any $f \in S_\Gamma$. Then, from the Ky Fan theorem [5] it follows that there exists a fixed point $f_* \in S_\Gamma$ of the mapping $\mathcal{M}(\mathcal{T})$, i.e.

$$f_* \in \mathcal{M}(\mathcal{T}(f_*)).$$

Setting $x_* = \mathcal{T}(f_*)$ and taking into account (3.18) we obtain

$$f_*(t) \in F(t, x_*(t)) \quad \text{a.e.}$$

Therefore, $x_* = \mathcal{T}(f_*)$ is a solution of inclusion (3.1). The theorem follows. □

Using Theorem 3.5 we now prove an existence theorem for sweeping process with perturbation under assumptions different from those traditionally used.

Consider the evolution inclusion

$$\begin{gathered} -A\dot{x}(t) \in \mathcal{N}_{C(t)}(x(t)) + \partial V(x(t)) + B(t,x(t)) + F(t,x(t)) \text{ a.e.,} \\ x(0) = x_0 \in C(0). \end{gathered} \tag{3.20}$$

Here, $C : T \to H$ is a multivalued mapping with closed convex values, $V : H \to \mathbb{R}$ is a convex function, $\mathcal{N}_{C(t)}(x)$ is the normal cone to the set $C(t)$ at a point $x \in C(t)$, A, $B(t,x)$ and $F(t,x)$ have the same sense as in inclusion (3.1).

By a solution of inclusion (3.20) we mean a function $x \in W^{1,2}(T,H)$, $x(t) \in C(t)$, $t \in T$, $x(0) = x_0$, such that

$$-A\dot{x}(t) \in \mathcal{N}_{C(t)}(x(t)) + \partial V(x(t)) + B(t,x(t)) + f(t) \quad \text{a.e.}$$

for some $f \in L^2(T,H)$,

$$f(t) \in F(t,x(t)) \quad \text{a.e.}$$

We make the following assumptions:

Hypothesis H(V). The function $V : H \to \mathbb{R}$ is convex and bounded from above on bounded sets from H.

Hypothesis H(C). C is a multivalued mapping from T to $Cl(H)$ with convex values and for any $r \geq 0$ there exists an absolutely continuous function $a_r : T \to \mathbb{R}$, $\dot{a}_r \in L^2(T,\mathbb{R})$ such that for any $s,t \in T$, $s \leq t$, $\|x\| \leq r$ the inequality

$$d(x,C(t)) \leq d(x,C(s)) + |a_r(s) - a_r(t)|$$

holds, where $d(x,C(t))$ is the distance from the point $x \in H$ to the set $C(t)$.

Denote by $I_{C(t)}$ the indicator function of the set $C(t)$, i.e.

$$I_{C(t)}(x) = \begin{cases} \Theta, & x \in C(t), \\ +\infty, & x \notin C(t). \end{cases}$$

Since the values of the mapping C are convex closed sets, $I_{C(t)} \in \Gamma_0(H)$, $t \in T$ and $\mathcal{N}_{C(t)}(x)$ is equal to the subdifferential $\partial I_{C(t)}(x)$ of the function $I_{C(t)}$ at the point x [1].

Consider the function

$$\varphi^t = I_{C(t)}(x) + V(x), \quad x \in H,\ t \in T. \tag{3.21}$$

From Hypothesis $H(V)$ it follows that $V : H \to \mathbb{R}$ is a convex continuous function. Then $\varphi^t \in \Gamma_0(H)$ and

$$\operatorname{dom} \varphi^t = C(t), \ \ t \in T. \tag{3.22}$$

Since $\operatorname{dom} V = H$ we have [4]

$$\partial\varphi^t(x) = \partial I_{C(t)}(x) + \partial V(x) \tag{3.23}$$

and

$$\operatorname{dom} \partial\varphi^t(x) = C(t), \ \ t \in T.$$

From (3.23) we infer that the set $\mathcal{N}_{C(t)}(x) + \partial V(x)$ is convex and closed. Therefore, it has an element of minimal norm.

Theorem 3.6. *Let Hypotheses $H(A), H(F), H(V), H(C), H(B)$ 1) – 3) hold true, and in Hypothesis $H(B)$ 4) instead of $\left(\partial\varphi^t\right)^0 x$ we have $\left(\mathcal{N}_{C(t)} + \partial V\right)^0 x$ which is an element of minimal norm of the set $\mathcal{N}_{C(t)}(x) + \partial V(x)$. Then inclusion* (3.20) *has a solution.*

Proof. In view of (3.23) we may rewrite inclusion (3.20) as inclusion (3.1). From Hypothesis $H(C)$ and (3.22) it follows that for any $r > 0$, any $s, t \in T$, $s \leq t$ and any $x \in \operatorname{dom} \varphi^s$ with $\|x\| \leq r$ there is an element $y_t \in \operatorname{dom} \varphi^t$ satisfying the inequality

$$\|x - y_t\| \leq |a_r(s) - a_r(t)|. \tag{3.24}$$

Since the function $V : H \to \mathbb{R}$ is convex and continuous, it is lower semicontinuous in the topology of the space ω-H. Hence V is bounded from below on bounded sets from H. Consequently, according to Hypothesis $H(V)$ the function V is bounded on bounded sets from H. Let

$$d_r = \int_T |\dot{a}_r(\tau)|\, d\tau, \quad m_r = r + d_r, \quad r \geq 0, \tag{3.25}$$

$$M_r = \sup\{|V(x)|;\ x \in \overline{B}(\Theta, 2m_r)\}.$$

From Proposition 5.11 in [3] it follows that there exists the constant $L_r = 2M_r/m_r$ such that

$$|V(x) - V(y)| \leq L_r \|x - y\|, \quad x, y \in \overline{B}(\Theta, m_r). \tag{3.26}$$

Let $s, t \in T$, $s \leq t$, $\|x\| \leq r$, $x \in \operatorname{dom} \varphi^s$. Then from (3.24), (3.25) we infer that $x, y_t \in \overline{B}(0, m_r)$. Hence, (3.24), (3.26) imply that

$$|V(x) - V(y_t)| \leq L_r \|a_r(s) - a_r(t)\|. \tag{3.27}$$

Let $b_r(t) = L_r a_r(t)$, $t \in T$. Then $b_r \in L^2(T, \mathbb{R})$, $r \geq 0$. From (3.22), (3.27) it follows that

$$|\varphi^t(y_t) - \varphi^s(x)| \leq L_r \|b_r(s) - b_r(t)\|. \tag{3.28}$$

Using (3.24), (3.28) we obtain

$$\|x - y_t\| \leq |a_r(t) - a_r(s)| \left(|\varphi^s(x)|^{\frac{1}{2}} + 1\right),$$

$$\varphi^t(y_t) - \varphi^s(x) \leq |b_r(t) - b_r(s)| \left(|\varphi^s(x)| + 1\right).$$

Therefore, the function φ^t defined by equality (3.21) satisfies Hypotheses $H(\varphi)$. Then, according to Theorem 3.5 inclusion (3.1) and thus inclusion (3.20) has a solution. The theorem follows. □

Remark 3.7. If we assume that for each $t \in T$ and $r \geq 0$ the set $\{x \in H;\ \|x\| \leq r,\ |\varphi^t(x)| \leq r\}$ is relatively compact in H, then Theorem 3.5 is true if Hypotheses H(F) 1), 2'), 3) hold with the function $f(t, \|x\|)$ defined by equality (3.4). In this case, the only changes in the proof are related to Lemma 3.4. In particular, from (3.6) it follows that the set $C(S)$ is relatively compact in the space $C(T, H)$. It is easy to show that the set $C(S)$ is, in fact, compact in $C(T, H)$. The further proof of the lemma is then obvious.

Remark 3.8. If H is finite-dimensional, then Theorems 3.5, 3.6 are true if Hypotheses H(F) 1) - 3) hold with the function $f(t, \|x\|)$ defined by (3.4).

A traditional assumption on the multivalued mapping $C(t)$ with both convex and nonconvex values is the validity of the inequality [2, 8, 15, and others]

$$|d(x, C(t)) - d(x, C(s))| \leq |v(t) - v(s)|, \tag{3.29}$$

for $x \in H$, $s, t \in T$, where $v : T \to \mathbb{R}$ is an absolutely continuous function. However, as the following example shows, inequality (3.29) is rather demanding from the point of view of existence theorems for sweeping processes.

Let $w : T \to H$, $\|w\| = 1$, $w \in W^{1,2}(T, H)$, $b \in W^{1,2}(T, \mathbb{R})$. Consider the multivalued mapping

$$C(t) = \{x \in H;\ \langle w(t), x\rangle - b(t) = 0\}.$$

Since

$$d(x, C(t)) = |\langle w(t), x\rangle - b(t)|/\|w(t)\|$$

we have

$$|d(x, C(t)) - d(x, C(s))| \leq \|w(t) - w(s)\| \cdot \|x\| + |b(t) - b(s)|, \tag{3.30}$$

$x \in H$. Setting

$$a_r(t) = \int_0^t \left(\|\dot{w}(s)\|r + |\dot{b}(s)|\right) ds, \quad r \geq 0$$

and using (3.30) we obtain

$$|d(x, C(t)) - d(x, C(s))| \leq |a_r(t) - a_r(s)|, \tag{3.31}$$

$s, t \in T$, $\|x\| \leq r$, $a_r \in W^{1,2}(T, \mathbb{R})$. From (3.30) it follows that inequality (3.29) cannot hold when the values of the mapping $C(t)$ are hyperplanes. Nevertheless, from (3.31) we see that the mapping $C(t)$ satisfies Hypothesis $H(C)$ under which Theorem 3.6 holds true.

Note that our results (Theorem 3.6) compliment the results of the work [8] in which the authors study a sweeping process with a perturbation $F(t, x)$ satisfying Hypotheses $H(F)$ 1)−3) with the function $f(t, \|x\|)$ as in (3.4) in a finite-dimensional space, and the sets $C(t)$ are uniformly ρ-prox-regular for some fixed $\rho \in (0, +\infty]$ and inequality (3.29) holds.

References

[1] J.-P. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, Berlin, Heidelberg, New-York, Tokyo, 1984.

[2] C. Castaing, T. X. Duc Ha and M. Valadier, *Evolution equations governed by the sweeping process*, Set-Valued Anal. **1** (1993), 109–139.

[3] G. Dal Maso, *An Introduction to Γ-Convergence*, Birkhäuser, Boston-Base-Berlin, 1993.

[4] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Elsevier publ. company, inc., New York, 1976.

[5] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Natl. Acad. Sci. USA **38** (1952) 121–126.

[6] A. Fryszkowski, *Continuous selections for a class of non-convex multivalued maps*, Stud. Math. **76** (1983) 163–174.

[7] A. Fryszkowski and L. Gorniewicz, *Mixed semicontinuous mappings and their applications to differential inclusions*, Set-Valued Anal. **8** (2000), 203–217.

[8] T. Haddad and L. Thibault, *Mixed semicontinuous perturbations of nonconvex sweeping processes*, Math. Program. **123** (2010), 225–240.

[9] C. J. Himmelberg, *Measurable relations*, Fundamenta Math. **87** (1975) 53–72.

[10] C. J. Himmelberg and F. S. van Vleck, *Existence of solutions for generalized differential equations with unbounded right-hand side*, J. Differ. Equations **61** (1986), 295–320.

[11] M. Kisielewicz, *Weak compactness in spaces* $C(S,X)$, Information theory, statistical decision functions, random processes, Trans. 11th Prague Conf. (Prague, 1990), Kluwer Acad. Publ., Dodrecht, 1992, pp. 101–106.

[12] S. Lojasiewicz, *Some theorems of Scorza Dragoni type for multifunctions with application to the problem of existence of solutions for differential multivalued equations*, Mathematical control theory, Banach Cent. Publ. **14** (1985), 625–643.

[13] J. J. Moreau, *Evolution problem associated with a moving convex set in Hilbert space*, J. Differ. Equations **26** (1977), 347–374.

[14] C. Olech, *Existence of solutions of non-convex orientor fields*, Boll. Unione Mat. Ital. **4** (1975) 189–197.

[15] L. Thibault, *Sweeping process with regular and nonregular sets*, J. Differ. Equations **193** (2003), 1–26.

[16] A. A. Tolstonogov, *Solutions of a differential inclusion with unbounded right-hand side*, Sib. Math. J. **29** (1988), 857–868.

[17] A. A. Tolstonogov, *Differential inclusions in a Banach space*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[18] A. A. Tolstonogov, *Properties of the set of "trajectory-control" pairs of a control system with subdifferential operators*, J. Math. Sci. **162** (2009), 407–442.

[19] A. A. Tolstonogov and D. A. Tolstonogov, L_p*-continuous extreme selectors of multifunctions with decomposable values: Existence theorems*, Set-Valued Anal. **4** (1996), 173–203.

Manuscript received January 7, 2014
revised May 7, 2014

A. A. Tolstonogov
Institute for System Dynamics and Control Theory, Siberian Branch, Russian Academy of Sciences, Lermontov str., 134, Irkutsk, 664033 Russia
E-mail address: aatol@icc.ru

S. A. Timoshin
Institute for System Dynamics and Control Theory, Siberian Branch, Russian Academy of Sciences, Lermontov str., 134, Irkutsk, 664033 Russia
E-mail address: sergey.timoshin@gmail.com