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FEW ALEXANDROV SURFACES ARE RIEMANN

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Abstract. We demonstrate that, in most Alexandrov surfaces of curvature bounded below, most points are not interior to any geodesic. Thus, these surfaces are not Riemannian, in contrast to the "almost Riemannian" structure found by Otsu–Shioya in any Alexandrov space [6].

Introduction and result

By a *segment* in a metric length space X we understand a shortest path between two points, by a *geodesic* a curve which is locally a segment. For two points x, y in X, we denote by xy any segment connecting them.

In spaces of second Baire category, we use the word *most* in the sense of "all, except those in a set of first Baire category".

We shall be interested in the behaviour of endpoints of segments in Alexandrov surfaces (we allow surfaces with and without boundary), that is, metric length spaces with curvature bounded from below in Alexandrov's comparison geometry, of dimension 2 [1–3]. Besides these references, see [10] for further insight into the geometry of Alexandrov surfaces.

There are many applications of Baire categories to convex surfaces [4,11]. Alexandrov surfaces simply offer a natural and more general framework. For another investigation in this framework, see [5].

Let $\mathcal{M}(n, \kappa)$ be the set of all (isometry classes of) compact Alexandrov spaces of dimension less than or equal to n and curvature bounded below by κ, equipped with the Gromov-Hausdorff metric d_G. Also, let $\mathcal{M}(n, \kappa, D)$ be the subclass of all Alexandrov spaces in $\mathcal{M}(n, \kappa)$ of diameter at most D.

We recall the following basic result.

Gromov's compactness Theorem ([3, p. 376]). *Choose $n \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and $D > 0$. The class $\mathcal{M}(n, \kappa, D)$ is compact.*

Let (X, ρ) be a metric space. For $A \subset X$, $a \in X$ and $\varepsilon > 0$, put $B(A, \varepsilon) = \{x \in X : \exists y \in A \text{ s.t. } \rho(x, y) \leq \varepsilon\}$ and $B(a, \varepsilon) = B(\{a\}, \varepsilon)$.

The next proposition is elementary.

Proposition 1. *Let M be a metric space and assume $A_i \subset M$ is compact and $B(A_i, 1/2) \subset A_{i+1}$ $(i \in \mathbb{N})$. If $M = \cup_1^\infty A_i$, then M is complete.*

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Combining Gromov's compactness Theorem and Proposition 1 we obtain the following.

Proposition 2. *The metric space $\mathcal{M}(n, \kappa)$ is complete.*

A point in the metric space $X \in \mathcal{M}(n, \kappa)$ is called *geodesically interior* if it lies on some segment, but is different from the endpoints of the segment. Is each point of X a geodesically interior point? No, certainly, no conical point is geodesically interior. Here, a point $x \in X$ is *conical* if its space of directions $T_x(X)$ has diameter less than π. Points of X which are not geodesically interior are called *endpoints.* They are of course endpoints of many geodesics. Alexandrov spaces without conical points, but possessing endpoints are also known ([1], pp. 58–59).

On the other hand, no Riemannian manifold has any endpoint. More precisely, for any point x of any Riemannian manifold, and for any tangent direction τ at x, there exists a geodesic passing through x and admitting there the tangent directions τ, $-\tau$.

We shall treat in this paper the two-dimensional case only. We take here $k \neq 0$, and find out that most Alexandrov spaces in $\mathcal{M}(2, \kappa)$ have many endpoints, and are therefore far away from being Riemann. Our main result follows.

Theorem. *In most Alexandrov spaces belonging to $\mathcal{M}(2, \kappa)$ ($\kappa \neq 0$), most points are endpoints.*

Note that the case of surfaces in $\mathcal{M}(2, 0)$ homeomorphic to the 2-sphere was already investigated (see [12]), since all such surfaces are realized by boundaries of convex bodies in $\mathbb{R}^3$ by Alexandrov's Theorem, cf. [1].

A *correspondence* between two metric spaces (X, d_X) and (Y, d_Y) is a relation $\mathcal{R}$ between their elements such that every point x in X is related to at least one point y in Y and vice versa. The *distortion* of the correspondence $\mathcal{R}$ is defined by

$$\delta_{\mathcal{R}} = \sup_{x\mathcal{R}y; x'\mathcal{R}y'} |d_X(x, x') - d_Y(y, y')|.$$

Subsequently, the *distortion distance* between X and Y is

$$d_C(X, Y) = \inf_{\mathcal{R}}(\delta_{\mathcal{R}}).$$

Proposition 3 ([3])**.** *The metrics d_C and d_G on the class of all metric spaces are equivalent; in fact, $2d_G = d_C$.*

Let μ_1 denote 1-dimensional Hausdorff measure (length).

Petrunin-Alexandrov's gluing Theorem ([8])**.** *Let X, Y be Alexandrov surfaces of curvature at least κ with non-empty boundaries bd X, bd Y, and consider the arcs $a_X b_X \subset \mathrm{bd} X$, $a_Y b_Y \subset \mathrm{bd} Y$.*

1. *If bd X and bd Y are isometric, then the surface obtained by gluing X and Y along their isometric boundaries is an Alexandrov surface of curvature bounded below by κ.*

2. *If* $\mu_1(a_X b_X) = \mu_1(a_Y b_Y)$, $\mu_1 T_{a_X}(X) + \mu_1 T_{a_Y}(Y) \leq \pi$, $\mu_1 T_{b_X}(X) + \mu_1 T_{b_Y}(Y) \leq \pi$, *then the surface obtained by gluing* X *and* Y *along the isometry* $a_X b_X \cong a_Y b_Y$ *is an Alexandrov surface of curvature bounded below by* κ.

In fact, only the first assertion is (the 2-dimensional case of) Petrunin-Alexandrov's theorem. But the second assertion admits a proof which largely parallels the proof of the first.

Further, consider in a metric space (X, d_X) the distinct elements x, y and a non-negative real number ε. Define an *ε-midpoint of* $\{x, y\}$ to be a point z which fulfils both inequalities $|2d_X(x,z) - d_X(x,y)| \leq \varepsilon$ and $|2d_X(y,z) - d_X(x,y)| \leq \varepsilon$. A 0-midpoint will also be called *midpoint*, for short.

The set of all midpoints coincides with the set of all geodesically interior points in X.

Proof of the main result

We start with a proposition.

Proposition 4. *Let* X *be a compact metric length space and* $x \in X$. *Suppose for each* $\varepsilon > 0$ *there exists a metric length space* Y *with* $d_C(X, Y) < \varepsilon$ *and a correspondence* $\mathcal{R}$ *with distortion less than* ε, *such that* x *is related to some point* $y \in Y$ *which is midpoint of a segment of length* λ. *Then* x *is midpoint of a segment in* X *of length* λ.

Proof. Let ε' be a positive real number. We will prove that x is an ε'-midpoint of two points at distance $\lambda - 2\varepsilon'$. Then, by letting $\varepsilon' \to 0$, we will show that x is a midpoint of two points at distance λ. Since the metric is intrinsic, this forces x to lie in a segment between these two points.

Choose a metric space Y whose distortion distance from X is less than $\varepsilon'/5$ and which fulfills the conditions of the Proposition with $\varepsilon = \varepsilon'/5$.

Let y be a point in Y corresponding to x in X, which is midpoint of a segment $s_Y \subset Y$ of length λ, whose endpoints are e_Y and e'_Y. Let e_X and e'_X be points in X corresponding to e_Y and e'_Y, respectively. By construction, x is an ε'-midpoint of e_X, e'_X. For $\varepsilon' \to 0$, a subsequence of the resulting sequence $\{(e_X)_n, (e'_X)_n\}_{n=1}^{\infty}$ will converge to a pair of points $\{e, e'\} \subset X$, which have x as their exact midpoint, and $d_X(e, e') = \lambda$. □

By specialising to the case $X = Y$ in Proposition 4, it follows that the set of midpoints in $X \in \mathcal{M}(2, \kappa)$ of segments of a given length is closed.

Proposition 5. *Let* $X \in \mathcal{M}(2, \kappa)$. *For any* $k \in \mathbb{N}$, *the set*

$$I_k(X) = \left\{ m(s) : \exists s \in X \text{ segment, } \mu_1(s) = \frac{1}{k} \right\},$$

where $m(s)$ *is the midpoint of the segment* s, *is closed in* X.

Let S^2_κ be the Lobachevsky plane of curvature κ if $\kappa < 0$, and the 2-dimensional sphere of curvature κ, if $\kappa > 0$.

An *ε-net* in a metric space X is a subset of X meeting every compact ball of diameter $\epsilon > 0$ in X.

Proposition 6. *If $X^* \in \mathcal{M}(2,\kappa)$, $\kappa \neq 0$ and $\varepsilon > 0$, then there exists $X' \in \mathcal{M}(2,\kappa)$ containing an ε-net of conical points, such that $d_G(X^*, X') < \varepsilon$.*

Proof. At distance less than $\varepsilon/4$ from each space X^* in $\mathcal{M}(2,\kappa,D)$ we find one of dimension 2. Let this one be X. It is known that X is a topological manifold.

If $\kappa < 0$, we apply a dilation $\lambda > 1$ to X with λ close to 1, and obtain the space $\lambda X \in \mathcal{M}(2,\kappa')$, where $\kappa' > \kappa$ and $d_G(X, \lambda X) < \varepsilon/2$.

If $\kappa > 0$, we apply a contraction $\lambda < 1$ to X with λ close to 1, and obtain the space $\lambda X \in \mathcal{M}(2,\kappa')$, where still $\kappa' > \kappa$ and $d_G(X, \lambda X) < \varepsilon/2$.

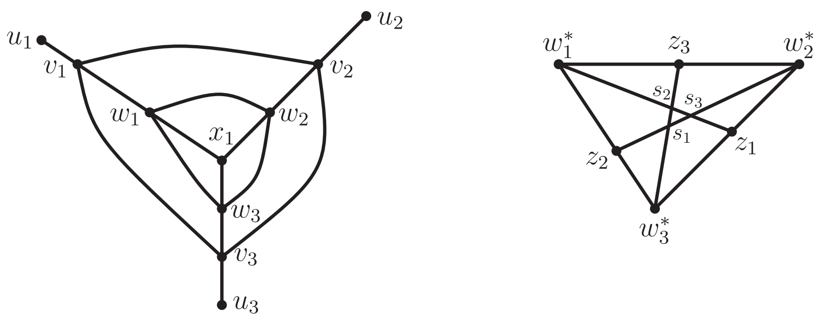
Since λX is compact, there exists a finite $(\varepsilon/8)$-net $N = \{x_1, ..., x_m\} \subset \lambda X$.

If x_1 is a conical point of λX, i.e. $\mu_1 T_x(\lambda X) < 2\pi$, then define $y_1 = x_1$.

If not, consider three directions $\tau_1, \tau_2, \tau_3 \in T_x$ at mutual distances $2\pi/3$. Close to each τ_i we find the direction of a segment x_1u_i. Let V be a neighbourhood of x_1 homeomorphic to a disc. Take $v_i \in x_1u_i$ such that

$$v_1v_2 \cup v_2v_3 \cup v_3v_1 \subset V.$$

Let w_i be the midpoint of x_1v_i. See the figure. Consider the comparison triangle



$w_1^*w_2^*w_3^*$ in S^2_κ, i.e. with $\{w_1^*, w_2^*, w_3^*\}$ isometric to $\{w_1, w_2, w_3\}$. By choosing each v_i close enough to x_1, we can arrange the diameters of both $w_1w_2w_3$ and $w_1^*w_2^*w_3^*$ to be less than $\varepsilon/8m$.

Since $\kappa < \kappa'$, we have

$$\angle w_2w_1v_1 + \angle w_3w_1v_1 + \angle w_2^*w_1^*w_3^* < 2\pi,$$

while $\angle w_iw_1v_1 < \pi \quad (i = 2, 3)$. Under these circumstances there exists a point $z_1 \in w_2^*w_3^*$ such that

$$\angle z_1w_1^*w_i^* + \angle w_iw_1v_1 < \pi \quad (i = 2, 3).$$

Take analogously $z_2 \in w_3^*w_1^*$ and $z_3 \in w_1^*w_2^*$. Let $\{s_1\} = w_2^*z_2 \cap w_3^*z_3$, and s_2, s_3 be defined analogously. The triangle $w_1^*w_2^*w_3^*$ is decomposed in three triangles, $w_1^*w_2^*s_3$, $w_2^*w_3^*s_1$, $w_3^*w_1^*s_2$, and a fourth, $s_1s_2s_3$, which can degenerate to a point.

By Petrunin-Alexandrov's gluing Theorem, the quadrilateral $v_1w_1w_2v_2$ and the triangle $s_3w_1^*w_2^*$ can be glued together along the isometry $w_1w_2 \cong w_1^*w_2^*$. Analogously, the other two pairs of one quadrilateral and one triangle can be glued together. The angle condition is satisfied at $w_1 = w_1^*$, $w_2 = w_2^*$, $w_3 = w_3^*$. This shows that the surface X_1 obtained from λX by replacing the triangle $w_1w_2w_3$ with the triangle $w_1^*w_2^*w_3^*$ is Alexandrov. The point w_1 is now conical; put $y_1 = w_1$.

Similarly, by taking x_2 instead of x_1, we obtain the surface X_2 replacing X_1 and the conical point $y_2 \in X_2$, and so on, until we obtain X_m and $y_m \in X_m$.

We relate through the relation $\mathcal{R}$ the points of $w_1w_2w_3$ in λX to all points of $w_1^*w_2^*w_3^*$ in X_m, and vice-versa. All other points of λX are related to their counterparts in X_m through the natural isometry, and vice-versa. p and p' will denote related points.

We now estimate the distance $d_C(\lambda X, X_m)$. Let $x', y' \in X_m$, and let $x\mathcal{R}x'$, $y\mathcal{R}y'$. Join $x \in \lambda X$ to $y \in \lambda X$ by a segment xy. Assume that x and y do not lie in any of the small triangles like $w_1w_2w_3$. Let t_1r_1, t_2r_2,..., t_qr_q $(q \le m)$ be the subsegments of xy which lie in triangles like $w_1w_2w_3$ around $x_1, x_2, ..., x_m$, met on the way. Since $d_{X_m}(t_i', r_i') \le \varepsilon/8m$, we have

$$\begin{aligned} d_{X_m}(x', y') &\le d_{X_m}(x', t_1') + d_{X_m}(t_1', r_1') + d_{X_m}(r_1', t_2') + \cdots + d_{X_m}(r_q', y') \\ &\le d_{\lambda X}(x, t_1) + d_{\lambda X}(t_1, r_1) + d_{\lambda X}(r_1, t_2) + \cdots + d_{\lambda X}(r_q, y) \\ &\le d_{\lambda X}(x, y) + \varepsilon/8. \end{aligned}$$

We leave to the reader the cases when x or y or both do lie in those small triangles. Analogously,

$$d_{\lambda X}(x, y) \le d_{X_m}(x', y') + \varepsilon/8.$$

Thus, $d_C(\lambda X, X_m) \le \varepsilon/8$.

This inequality together with $d_G(X^*, X) < \varepsilon/4$ and $d_G(X, \lambda X) < \varepsilon/2$, implies $d_G(X^*, X_m) < \varepsilon$.

It remains to show that X_m contains an ε-net of conical points. Indeed, the set of conical points $\{y_1, ..., y_m\}$ is such a net. For, let x' be an arbitrary point in X_m. Let $x\mathcal{R}x'$ with $x \in \lambda X$. There exists a point $x_i \in N$ at distance at most $\varepsilon/8$ from x. Since $x\mathcal{R}x'$ and $x_i\mathcal{R}y_i$, from $d_{\lambda X}(x, x_i) \le \varepsilon/8m$ we deduce $d_{X_m}(x', y_i) \le \varepsilon$. □

Proof of the Theorem. Let $I(X)$ be the set of all geodesically interior points of $X \in \mathcal{M}(2, \kappa)$. We have

$$I(X) = \bigcup_{k=1}^{\infty} I_k(X).$$

Let

$$\begin{gathered} \mathcal{A} = \{X \in \mathcal{M}(n, \kappa, D) : I(X) \text{ is of 2nd category}\}, \\ \mathcal{A}_k = \{X \in \mathcal{M}(n, \kappa, D) : I_k(X) \text{ is not nowhere dense}\}, \\ \mathcal{A}_{k,l} = \{X \in \mathcal{M}(n, \kappa, D) : \exists x \in X, \text{s.t. } B(x, 1/l) \subset I_k(X)\}. \end{gathered}$$

We have, using Proposition 5,

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \mathcal{A}_k, \quad \mathcal{A}_k = \bigcup_{l=1}^{\infty} \mathcal{A}_{k,l}.$$

It remains to show that each $\mathcal{A}_{k,l}$ is nowhere dense.

Suppose $\mathcal{A}_{k,l}$ is not nowhere dense. Proposition 5 and the compactness of X imply that each $\mathcal{A}_{k,l}$ is closed. Hence there exists a whole open set $\mathcal{O} \subset \mathcal{A}_{k,l}$. Take $X \in \mathcal{O}$.

By Proposition 6, we find a surface $\Sigma \in \mathcal{M}(2,\kappa) \cap \mathcal{O}$ with a $(1/2l)$-net on it consisting of conical points. Let x be chosen arbitrarily in Σ. Clearly, $B(x, 1/l)$ contains some point of the $(1/2l)$-net, whence $\Sigma \notin \mathcal{A}_{k,l}$, and a contradiction is obtained. □

Remark. Ironically, precisely the case $\kappa = 0$, which essentially corresponds, for orientable surfaces, to the convex ones (and was treated in [12] using extrinsic approximation), constitutes a remarkable exception! More precisely, $\mathcal{M}(2,0)$ has four components, one of which corresponds to the space of all convex surfaces, see [9].

The Theorem is not valid for $\kappa = 0$.

Indeed, in a whole neighbourhood in $\mathcal{M}(2,0)$ of a flat torus or Klein bottle, every surface is a flat torus or a flat Klein bottle (by the Perelman stability Theorem [7]). These remain, however, the only exceptions. Most surfaces in $\mathcal{M}(2,0)$ have Euler characteristic $\chi = 0$ and are flat or, on them, most points are endpoints. For $\chi = 0$, the flatness is imposed by $\kappa = 0$, and for $\chi > 0$ the universal cover being a sphere the result essentially follows from [12].

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