Journal of Nonlinear and Convex Analysis
Volume 16, Number 6, 2015, 1155–1183

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

EXISTENCE AND A TURNPIKE PROPERTY OF SOLUTIONS FOR A CLASS OF NONAUTONOMOUS OPTIMAL CONTROL SYSTEMS WITH DISCOUNTING

ALEXANDER J. ZASLAVSKI

Abstract. We study existence and a turnpike property of solutions of a discrete-time control system with discounting and with a compact metric space of states. To have the turnpike property means that the approximate solutions of the problems are determined mainly by the objective functions, and are essentially independent of the choice of intervals and endpoint conditions, except in regions close to the endpoints. We show that this turnpike property is stable under small perturbations of the objective functions.

1. Introduction

The study of the existence, the structure and properties of (approximate) solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research [3, 5-9, 11, 12, 15, 20-22, 24, 25, 32]. These problems arise in engineering [1, 39], in models of economic growth [2, 10, 13, 19, 23, 26, 27, 32-36], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 28], in the calculus of variations on time scales [16, 18] and in the theory of thermodynamical equilibrium for materials [14, 17].

In this paper we study the existence and structure of approximate solutions of nonautonomous discrete-time optimal control systems with discounting which are determined by sequences of lower semicontinuous objective functions.

For each nonempty set Y denote by $\mathcal{B}(Y)$ the set of all bounded functions $f : Y \to R^1$ and for each $f \in \mathcal{B}(Y)$ set

$$\|f\| = \sup\{|f(y)| :\ y \in Y\}.$$

For each nonempty compact metric space Y denote by $C(Y)$ the set of all continuous functions $f : Y \to R^1$.

Let (X, ρ) be a compact metric space with the metric ρ. The set $X \times X$ is equipped with the metric ρ_1 defined by

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2),\ (x_1, x_2),\ (y_1, y_2) \in X \times X.$$

For each integer $t \geq 0$ let Ω_t be a nonempty closed subset of the metric space $X \times X$.

Let $T \geq 0$ be an integer. A sequence $\{x_t\}_{t=T}^{\infty} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers $t \geq T$.

2010 *Mathematics Subject Classification.* 49J99.

Key words and phrases. Compact metric space, good program, infinite horizon problem, turnpike property.

Let T_1, T_2 be integers such that $0 \le T_1 < T_2$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers t satisfying $T_1 \le t < T_2$.

We assume that there exists a program $\{x_t\}_{t=0}^{\infty}$. Denote by $\mathcal{M}$ the set of all sequences of functions $\{f_t\}_{t=0}^{\infty}$ such that for each integer $t \ge 0$

$$f_t \in \mathcal{B}(\Omega_t) \tag{1.1}$$

and that

$$\sup\{\|f_t\| : t = 0, 1, \dots\} < \infty. \tag{1.2}$$

For each pair of sequences $\{f_t\}_{t=0}^{\infty}$, $\{g_t\}_{t=0}^{\infty} \in \mathcal{M}$ set

$$d(\{f_t\}_{t=0}^{\infty}, \{g_t\}_{t=0}^{\infty}) = \sup\{\|f_t - g_t\| : t = 0, 1, \dots\}. \tag{1.3}$$

It is easy to see that $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a metric on $\mathcal{M}$ and that the metric space $(\mathcal{M}, d)$ is complete.

Let $\{f_t\}_{t=0}^{\infty} \in \mathcal{M}$. We consider the following optimization problems

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \text{s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program,}$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \text{s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y,$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \text{s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, x_{T_2} = z,$$

where $y, z \in X$ and integers T_1, T_2 satisfy $0 \le T_1 < T_2$.

The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework, e. g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [13], the study of the discrete Frenkel-Kontorova model related to dislocations in one-dimensional crystals [4, 28] and the analysis of a long slender bar of a polymeric material under tension in [14, 17]. Similar optimization problems are also considered in mathematical economics [10, 13, 19, 26, 32-36]. In [29] these problems were considered in the case when $f_t = f_0$ and $\Omega_t = X \times X$ for all integers $t \ge 0$, in [30, 31] they were studied in the case when $\Omega_t = X \times X$ for all integers $t \ge 0$ and in [33-36] we studied these problems in the case when $f_t = f_0$ and $\Omega_t = \Omega_0$ for all integers $t \ge 0$. Here we study a general case when the optimal control system is determined by a nonstationary sequence of objective functions $\{f_t\}_{t=0}^{\infty}$ and by a nonstationary sequence of sets of admissible pairs $\{\Omega_t\}_{t=0}^{\infty}$. This makes the situation more realistic but more difficult and less understood.

For each $y, z \in X$ and each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$ set

$$U(\{f_t\}_{t=0}^{\infty}, T_1, T_2) = \inf\Big\{ \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\Big\}, \tag{1.4}$$

(1.5) $U(\{f_t\}_{t=0}^{\infty}, T_1, T_2, y)$

$$= \inf\Big\{\sum_{t=T_1}^{T_2-1} f_t(x_t,x_{t+1}) :\ \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1}=y\Big\},$$

$$(1.6)\quad U(\{f_t\}_{t=0}^{\infty},T_1,T_2,y,z) = \inf\Big\{\sum_{t=T_1}^{T_2-1} f_t(x_t,x_{t+1}) :\{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1}=y,\ x_{T_2}=z\Big\}.$$

Here we assume that the infimum over empty set is ∞.

Denote by $\mathcal{M}_{reg}$ the set of all sequences of functions $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}$ for which there exist a program $\{x_t^f\}_{t=0}^{\infty}$ and constants $c_f > 0$, $\gamma_f > 0$ such that the following conditions hold:

(C1) the function f_t is lower semicontinuous for all integers $t \geq 0$;

(C2) for each pair of integers $T_1 \geq 0$, $T_2 > T_1$,

$$\sum_{t=T_1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) \leq U(\{f_t\}_{t=0}^{\infty},T_1,T_2) + c_f;$$

(C3) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each integer $t \geq 0$ and each $(x,y) \in \Omega_t$ satisfying $\rho(x,x_t^f) \leq \delta$, $\rho(y,x_{t+1}^f) \leq \delta$ we have

$$|f_t(x_t^f,x_{t+1}^f) - f_t(x,y)| \leq \epsilon;$$

(C4) for each integer $t \geq 0$, each $(x_t,x_{t+1}) \in \Omega_t$ satisfying $\rho(x_t,x_t^f) \leq \gamma_f$ and each $(x'_{t+1},x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x'_{t+2},x_{t+2}^f) \leq \gamma_f$ there is $x \in X$ such that

$$(x_t,x) \in \Omega_t,\ (x,x'_{t+2}) \in \Omega_{t+1};$$

moreover, for each $\epsilon > 0$ there exists $\delta \in (0,\gamma_f)$ such that for each integer $t \geq 0$, each $(x_t,x_{t+1}) \in \Omega_t$ and each $(x'_{t+1},x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x_t,x_t^f) \leq \delta$ and $\rho(x'_{t+2},x_{t+2}^f) \leq \delta$ there is $x \in X$ such that

$$(x_t,x) \in \Omega_t,\ (x,x'_{t+2}) \in \Omega_{t+1},\ \rho(x,x_{t+1}^f) \leq \epsilon.$$

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of $\mathcal{M}_{reg}$ in $(\mathcal{M},d)$. Denote by $\mathcal{M}_{c,reg}$ the set of all sequences $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}_{reg}$ such that $f_i \in C(\Omega_i)$ for all integers $i \geq 0$ and by $\bar{\mathcal{M}}_{c,reg}$ the closure of $\mathcal{M}_{c,reg}$ in $(\mathcal{M},d)$.

We study the optimization problems stated above with the sequence of objective functions $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}_{reg}$. Our study is based on the relation between these finite horizon problems and the corresponding infinite horizon optimization problem determined by $\{f_i\}_{i=0}^{\infty}$. Note that the condition (C2) means that the program $\{x_t^f\}_{t=0}^{\infty}$ is an approximate solution of this infinite horizon problem.

We are interested in turnpike properties of approximate solutions of our optimization problems, which are independent of the length of the interval $T_2 - T_1$, for all sufficiently large intervals. To have these properties means that the approximate solutions of the problems are determined mainly by the objective functions, and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [27]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path).

In the classical turnpike theory [10, 19, 26] the space X is a compact convex subset of a finite-dimensional Euclidean space, the sets Ω_t are convex and $f_t = f_0$, $\Omega_t = \Omega_0$ for all integers $t \geq 0$, where the function f_0 is strictly convex. Under these assumptions the turnpike property can be established and the turnpike $\bar{x}$ is a unique solution of the minimization problem $f_0(x, x) \to \min$, $(x, x) \in \Omega_0$. In this situation it is shown that for each program $\{x_t\}_{t=0}^{\infty}$ either the sequence $\{\sum_{t=0}^{T-1} f_0(x_t, x_{t+1}) - T f_0(\bar{x}, \bar{x})\}_{T=1}^{\infty}$ is bounded (in this case the program $\{x_t\}_{t=0}^{\infty}$ is called (f_0)-good) or it diverges to ∞. Moreover, it is also established that any (f_0)-good program converges to the turnpike $\bar{x}$. This property is called as the asymptotic turnpike property.

In [33, 34] we studied the stationary case with $f_t = f_0$, $\Omega_t = \Omega_0$ for all integers $t \geq 0$, and showed that the turnpike property follows from the asymptotic turnpike property. More precisely, we assumed that any (f_0)-good program converges to a unique solution $\bar{x}$ of the problem $f_0(x, x) \to \min$, $(x, x) \in \Omega$ and showed that the turnpike property holds and $\bar{x}$ is the turnpike. Note that we did not use convexity assumptions.

In [37] we generalize the results of [33, 34] and establish the turnpike property for the general case when the optimal control system is determined by a nonstationary sequence of objective functions $\{f_t\}_{t=0}^{\infty}$ and by a nonstationary sequence of sets of admissible pairs $\{\Omega_t\}_{t=0}^{\infty}$. The results of [37] are also a generalization of the results of [31] obtained in the case when $\Omega_t = X \times X$ for all integers $t \geq 0$.

Let $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}_{reg}$ and let a program $\{x_i^f\}_{i=0}^{\infty}$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

It is not difficult to see that the assumption (C2) implies the following useful result obtained in [37, Proposition 2.1].

Proposition 1.1. *Let $S \geq 0$ be an integer and $\{x_i\}_{i=S}^{\infty}$ be a program. Then either the sequence $\{\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f)\}_{T=S+1}^{\infty}$ is bounded or*

$$\lim_{T\to\infty} \Big[\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f) \Big] = \infty.$$

A program $\{x_t\}_{t=S}^{\infty}$, where $S \geq 0$ is an integer, is called $(\{f_i\}_{i=0}^{\infty})$-good if the sequence

$$\Big\{ \sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f) \Big\}_{T=S+1}^{\infty}$$

is bounded [10, 13, 19, 32].

We say that the sequence $\{f_i\}_{i=0}^{\infty}$ possesses an asymptotic turnpike property (or briefly (ATP)) with $\{x_i^f\}_{i=0}^{\infty}$ being the turnpike if for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^{\infty})$-good program $\{x_i\}_{i=S}^{\infty}$,

$$\lim_{i\to\infty} \rho(x_i, x_i^f) = 0.$$

We say that the sequence $\{f_i\}_{i=0}^{\infty}$ possesses a turnpike property (or briefly (TP)) if for each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^{\infty}, T_1, T_2, x_{T_1}, x_{T_2}) + \delta,\ U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) + M\},$$

the inequality $\rho(x_i, x_i^f) \leq \epsilon$ holds for all integers $i = T_1 + L, \ldots, T_2 - L$.

The sequence $\{x_i^f\}_{i=0}^{\infty}$ is called the turnpike of $\{f_i\}_{i=0}^{\infty}$.

In [37, Theorem 2.1] we prove the following result.

Theorem 1.2. *The sequence $\{f_i\}_{i=0}^{\infty}$ possesses the turnpike property if and only if $\{f_i\}_{i=0}^{\infty}$ possesses (ATP) and the following property:*

(P) *For each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each integer $T \geq 0$ and each program $\{x_t\}_{t=T}^{T+L}$ which satisfies*

$$\sum_{i=T}^{T+L-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^{\infty}, T, T+L, x_T, x_{T+L}) + \delta,$$
$$U(\{f_i\}_{i=0}^{\infty}, T, T+L) + M\}$$

there is an integer $j \in \{T, \ldots, T+L\}$ for which $\rho(x_j, x_j^f) \leq \epsilon$.

The property (P) means that if a natural number L is large enough and a program $\{x_t\}_{t=T}^{T+L}$ is an approximate solution of the corresponding finite horizon problem, then there is $j \in \{T, \ldots, T+L\}$ such that x_j is close to x_j^f.

We also show in [37] that $\{f_i\}_{i=0}^{\infty}$ is approximated by elements of $\mathcal{M}_{reg}$ possessing (TP).

For each $r \in (0,1)$ and all integers $i \geq 0$ set

$$f_i^{(r)}(x,y) = f_i(x,y) + r\rho(x, x_i^f),\ (x,y) \in \Omega_i.$$

Clearly, $\{f_i^{(r)}\}_{i=0}^{\infty} \in \mathcal{M}_{reg}$ for all $r \in (0,1)$ and

$$\lim_{r \to 0^+} d(\{f_i^{(r)}\}_{i=0}^{\infty}, \{f_i\}_{i=0}^{\infty}) = 0.$$

In [37, Proposition 2.2] we prove the following result.

Proposition 1.3. *For each $r \in (0,1)$, $\{f_i^{(r)}\}_{i=0}^{\infty}$ possesses (TP) with $\{x_i^f\}_{i=0}^{\infty}$ being the turnpike.*

In the sequel we use the following notation. For each $y, z \in X$, each pair of integers T_1, T_2 satisfying $0 \leq T_1 < T_2$ and each finite sequence of functions $g_i \in \mathcal{B}(\Omega_i)$, $i = T_1, \ldots, T_2 - 1$ set

$$U(\{g_i\}_{i=T_1}^{T_2-1}, T_1, T_2) = \inf\Big\{\sum_{t=T_1}^{T_2-1} g_t(x_t, x_{t+1}) :\ \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\Big\}, \tag{1.7}$$

(1.8) $U(\{g_i\}_{i=T_1}^{T_2-1}, T_1, T_2, y)$

$$= \inf\Big\{ \sum_{t=T_1}^{T_2-1} g_t(x_t, x_{t+1}) : \ \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y \Big\},$$

$$(1.9) \quad U(\{g_i\}_{i=T_1}^{T_2-1}, T_1, T_2, y, z) = \inf\Big\{ \sum_{t=T_1}^{T_2-1} g_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, \ x_{T_2} = z \Big\}.$$

Here again we assume that the infimum over empty set is ∞.

In [38] we assume that $\{f_i\}_{i=0}^{\infty}$ possesses (TP) and show that the turnpike property is stable under small perturbations of the objective functions. These results generalize the results of [35] obtained in the stationary case with $f_t = f_0$, $\Omega_t = \Omega_0$ for all integers $t \geq 0$ with the stationary turnpike.

Note that in [37, 38] we consider optimal control systems, associated with $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}_{reg}$, without discounting. In the present paper we show that the turnpike property together with its stability established in [37, 38] also hold for the models with discounting.

Our results are a generalization of the results of [36] obtained for optimal control systems with discounting in the stationary case with $f_t = f_0$, $\Omega_t = \Omega_0$ for all integers $t \geq 0$ with the stationary turnpike.

2. Main results

We suppose that the sum over empty set is zero.

Let $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}_{reg}$ and let a program $\{x_i^f\}_{i=0}^{\infty}$, $c_f > 0$ and $\gamma_f \in (0,1)$ be such that (C1)-(C4) hold.

We suppose that $\{f_i\}_{i=0}^{\infty}$ possesses (ATP) and the property (P). Then by Theorem 1.2, $\{f_i\}_{i=0}^{\infty}$ possesses (TP). (Note that by Theorem 1.2 we can assume that $\{f_i\}_{i=0}^{\infty}$ possesses (TP)).

In [38, Theorem 2.1] we prove the following result which shows that the turnpike property is stable under small perturbations of the objective functions in the case without discounting.

Theorem 2.1. *Let $\epsilon \in (0,1)$ and $M > 0$. Then there exist a natural number L_0 and a real number*

$$\delta_0 \in (0, \min\{\epsilon, \gamma_f\})$$

such that for each integer $L_1 \geq L_0$ the following assertion holds with $\delta = (8L_1)^{-1}\delta_0$.

Assume that integers $T_1 \geq 0$, $T_2 > T_1 + 2L_1$, $\{g_i\}_{i=0}^{\infty} \in \mathcal{M}$ satisfies

$$d(\{f_i\}_{i=0}^{\infty}, \{g_i\}_{i=0}^{\infty}) \leq \delta$$

and that a program $\{x_t\}_{t=T_1}^{T_2}$ and a finite sequence of integers $\{S_i\}_{i=0}^{q}$ (where q is a natural number) satisfy

$$S_0 = T_1, \ T_2 \geq S_q > T_2 - L_1,$$

$$S_{i+1} - S_i \in [L_0, L_1] \text{ for all integers } i \in [0, q-1],$$

$$\sum_{t=S_i}^{S_{i+1}-1} g_t(x_t, x_{t+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_i, S_{i+1}) + M \textit{ for all } i = 0, \dots, q-1,$$

$$\sum_{t=S_{q-1}}^{T_2-1} g_t(x_t, x_{t+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_{q-1}, T_2) + M,$$

$$\sum_{t=S_i}^{S_{i+2}-1} g_t(x_t, x_{t+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_i, S_{i+2}, x_{S_i}, x_{S_{i+2}}) + \delta_0 \textit{ for all } i = 0, \dots, q-2,$$

$$\sum_{t=S_{q-2}}^{T_2-1} g_t(x_t, x_{t+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_{q-2}, T_2, x_{S_{q-2}}, x_{T_2}) + \delta_0.$$

Then

$$\rho(x_t, x_t^f) \le \epsilon \textit{ for all integers } t \in [T_1 + L_1, T_2 - L_1].$$

Let $S \ge 0$ be an integer. A point $x \in X$ is called $(\{f_i\}_{i=0}^{\infty}, S)$-good if there exists an $(\{f_i\}_{i=0}^{\infty})$-good program $\{x_t\}_{t=S}^{\infty}$ such that $x_S = x$.

A point $x \in X$ is called $(\{f_i\}_{i=0}^{\infty}, S, M)$-good, where M is a positive number, if there exists a program $\{x_t\}_{t=S}^{\infty}$ such that $x_S = x$ and for all integers $T > S$,

$$\sum_{t=S}^{T-1} f_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} f_t(x_t^f, x_{t+1}^f) \le M.$$

The following theorem is our main result.

Theorem 2.2. *Let $M > 0$ and $\epsilon \in (0, \gamma_f)$. Then there exist a number $\delta \in (0, \epsilon)$, a natural number L and $\lambda \in (0, 1)$ such that for each pair of integers $T_1 \ge 0$, $T_2 > T_1 + 2L$, each $\{g_i\}_{i=0}^{\infty} \in \mathcal{M}$ such that g_i is a lower semicontinuous function for all integers $i \ge 0$ and that*

$$d(\{f_i\}_{i=0}^{\infty}, \{g_i\}_{i=0}^{\infty}) \le \delta,$$

each finite sequence $\{\alpha_i\}_{i=T_1}^{T_2-1} \subset (0, 1]$ such that

$$\alpha_i \alpha_j^{-1} \ge \lambda \textit{ for all } i, j \in \{T_1, \dots, T_2 - 1\} \textit{ satisfying } |i - j| \le L,$$
$$\alpha_i \alpha_j^{-1} \ge \lambda \textit{ for all } i, j \in \{T_1, \dots, T_2 - 1\} \textit{ satisfying } j \ge i$$

and each program $\{x_t\}_{t=T_1}^{T_2}$ such that the point x_{T_1} is $(\{f_i\}_{i=0}^{\infty}, T_1, M)$-good and

$$\sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) = U(\{\alpha_i g_i\}_{i=T_1}^{T_2-1}, T_1, T_2, x_{T_1})$$

the following inequality holds:

$$\rho(x_t, x_t^f) \le \epsilon \textit{ for all integers } t \in [T_1 + L, T_2 - L].$$

Theorem 2.2 establishes the turnpike property of solutions of optimal finite horizon problems associated with the objective functions $\alpha_i g_i$, $i = T_1, \dots, T_2 - 1$, where the sequence of functions $\{g_i\}_{i=0}^{\infty} \in \mathcal{M}$ and the sequence of discount coefficients $\{\alpha_t\}_{t=T_1}^{T_2-1} \subset (0, 1]$ satisfy the assumptions of the theorem. Roughly speaking, the turnpike property holds if the sequence of functions $\{g_i\}_{i=0}^{\infty}$ belongs to a small neighborhood $\{f_i\}_{i=0}^{\infty}$ and the discount coefficients $\{\alpha_t\}_{t=T_1}^{T_2-1} \subset (0,1]$ are changed rather slowly.

Let $S \geq 0$ be an integer and $g_i \in \mathcal{B}(\Omega_i)$ for all integers $i \geq S$. A program $\{x_t\}_{t=S}^{\infty}$ is called $(\{g_i\}_{i=S}^{\infty})$-overtaking optimal [10, 13, 19, 32] if for each program $\{x'_t\}_{t=S}^{\infty}$ satisfying $x'_S = x_S$,

$$\limsup_{T\to\infty} \Big[\sum_{t=S}^{T-1} g_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} g_t(x'_t, x'_{t+1}) \Big] \leq 0.$$

Note that the existence of an $(\{g_i\}_{i=S}^{\infty})$-overtaking optimal program when the functions $\{g_i\}_{i=S}^{\infty}$ tends to zero rapidly is a well-known fact. Here we present a version of this result.

Theorem 2.3. *Let $\{g_i\}_{i=0}^{\infty} \subset \mathcal{M}$ be such that for each integer $t \geq 0$ the function g_t is lower semicontinuous and*

$$\sum_{i=0}^{\infty} ||g_i|| < \infty,$$

$S \geq 0$ be an integer and let $z \in X$ be such that there exists a program $\{x_t\}_{t=S}^{\infty}$ satisfying $x_S = z$.

Then there exists a $(\{g_i\}_{i=S}^{\infty})$-overtaking optimal program $\{x^*_t\}_{t=S}^{\infty}$ satisfying $x^*_S = z$.

Proof. Clearly, for any program $\{z_t\}_{t=S}^{\infty}$,

$$\sum_{t=S}^{\infty} |g_t(z_t, z_{t+1})| \leq \sum_{i=0}^{\infty} \|g_i\| < \infty.$$

Set

$$\Delta = \inf \Big\{ \sum_{t=S}^{\infty} g_t(y_t, y_{t+1}) : \ \{y_t\}_{t=S}^{\infty} \text{ is a program and } y_S = z \Big\}. \tag{2.1}$$

Clearly, Δ is well defined and finite. In order to prove Theorem 2.3 it is sufficient to show that there is a program $\{x^*_i\}_{i=S}^{\infty}$ such that

$$x^*_S = z, \ \sum_{t=S}^{\infty} g_t(x^*_t, x^*_{t+1}) = \Delta.$$

By (2.1), for each integer $k \geq 1$, there is a program $\{x^{(k)}_t\}_{t=S}^{\infty}$ such that

$$x^{(k)}_S = z, \ \sum_{t=S}^{\infty} g_t(x^{(k)}_t, x^{(k)}_{t+1}) \leq \Delta + 1/k. \tag{2.2}$$

Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that for any integer $t \geq S$ there exists

$$x^*_t = \lim_{k\to\infty} x^{(k)}_t. \tag{2.3}$$

Clearly, $\{x^*_i\}_{i=S}^{\infty}$ is a program satisfying $x^*_S = z$.

Let $\epsilon > 0$. Then there is a natural number $S_0 > S$ such that

$$\sum_{t=S_0}^{\infty} \|g_t\| < \epsilon. \tag{2.4}$$

By (2.2), (2.3) and (2.4) for all integers $T > S_0$,

$$\begin{aligned}
\sum_{t=S}^{T-1} g_t(x_t^*, x_{t+1}^*) &\leq \liminf_{k\to\infty} \sum_{t=S}^{T-1} g_t(x_t^{(k)}, x_{t+1}^{(k)}) \\
&\leq \liminf_{k\to\infty} \Big(\sum_{t=S}^{\infty} g_t(x_t^{(k)}, x_{t+1}^{(k)}) + \epsilon\Big) \\
&\leq \lim_{k\to\infty} (\Delta + k^{-1} + \epsilon) = \Delta + \epsilon
\end{aligned}$$

and

$$\sum_{t=S}^{\infty} g_t(x_t^*, x_{t+1}^*) \leq \Delta + \epsilon.$$

Since ϵ is any positive number we conclude that $\sum_{t=S}^{\infty} g_t(x_t^*, x_{t+1}^*) \leq \Delta$. Theorem 2.3 is proved. □

Theorem 2.4. *Let $M > 0$, $\epsilon = \gamma_f/4$ and let $\delta \in (0, \epsilon)$, a natural number L and $\lambda \in (0,1)$ be as guaranteed by Theorem 2.2.*

Let $\{g_i\}_{i=0}^{\infty} \in \mathcal{M}$ be such that for each integer $t \geq 0$, the function g_t is lower semicontinuous and that

$$d(\{f_i\}_{i=0}^{\infty}, \{g_i\}_{i=0}^{\infty}) \leq \delta \tag{2.5}$$

and let

$$\{\alpha_i\}_{i=0}^{\infty} \subset (0,1],\ \lim_{i\to\infty} \alpha_i = 0, \tag{2.6}$$

$$\alpha_i\alpha_j^{-1} \geq \lambda \text{ for all nonnegative integers } i,j \text{ satisfying } |i-j| \leq L \tag{2.7}$$

and

$$\alpha_i\alpha_j^{-1} \geq \lambda \text{ for all nonnegative integers } i,j \text{ satisfying } j \geq i. \tag{2.8}$$

Then for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^{\infty}, S, M)$-good $z \in X$ there exists a program $\{x_t^{(S,z)}\}_{t=S}^{\infty}$ such that $x_S^{(S,z)} = z$ and that the following property holds:

For each $\gamma > 0$ there is a natural number n_0 such that for each integer $S \geq 0$, each integer $T \geq S + n_0$ and each $(\{f_i\}_{i=0}^{\infty}, S, M)$-good point $z \in X$,

$$\Big|U(\{\alpha_t g_t\}_{t=S}^{T-1}, S, T, z) - \sum_{t=S}^{T-1} \alpha_t g_t(x_t^{(S,z)}, x_{t+1}^{(S,z)})\Big| \leq \gamma.$$

It is clear that Theorem 2.4 establishes the existence of $(\{\alpha_t g_t\}_{t=S}^{\infty})$-overtaking optimal program when (2.5)-(2.8) hold. Roughly speaking, an $(\{\alpha_t g_t\}_{t=S}^{\infty})$-overtaking optimal program exists if $\{g_i\}_{i=0}^{\infty}$ belongs to a small neighborhood of $\{f_i\}_{i=0}^{\infty}$ and the sequence of the discount coefficients $\{\alpha_i\}_{i=0}^{\infty}$ tends to zero slowly.

We will prove the following result which establishes the turnpike property for overtaking optimal programs.

Theorem 2.5. *Let $M > 0$ and $\epsilon \in (0, \gamma_f)$. Then there exist a number $\delta \in (0, \epsilon)$, a natural number L and $\lambda \in (0, 1)$ such that for each $\{g_i\}_{i=0}^{\infty} \in \mathcal{M}$, where g_i is a lower semicontinuous function for all integers $i \geq 0$, which satisfies*

$$d(\{f_i\}_{i=0}^{\infty}, \{g_i\}_{i=0}^{\infty}) \leq \delta,$$

each integer $T_1 \geq 0$, each sequence $\{\alpha_i\}_{i=T_1}^{\infty} \subset (0, 1]$ such that

$$\alpha_i \alpha_j^{-1} \geq \lambda \text{ for all integers } i, j \geq T_1 \text{ satisfying } |i - j| \leq L,$$
$$\alpha_i \alpha_j^{-1} \geq \lambda \text{ for all integers } i, j \text{ satisfying } j \geq i \geq T_1$$

and each $(\{\alpha_i g_i\}_{t=T_1}^{\infty})$-overtaking optimal program $\{x_t\}_{t=T_1}^{\infty}$ for which the point x_{T_1} is $(\{f_i\}_{i=0}^{\infty}, T_1, M)$-good, the following inequality holds:

$$\rho(x_t, x_t^f) \leq \epsilon \text{ for all integers } t \geq T_1 + L.$$

The paper is organized as follows. Theorems 2.2 and 2.5 are proved in Section 3. Theorem 2.4 is proved in Section 4.

3. Proof of Theorems 2.2 and 2.5

Choose

$$d_0 > \sup\{\|f_i\| : \ i = 0, 1, \dots\}. \tag{3.1}$$

We prove Theorems 2.2 and 2.5 simultaneously. Choose a number

$$M_1 > M + 4. \tag{3.2}$$

By the property (P) there exist $\delta_0 \in (0, \gamma_f/4)$ and a natural number L_0 such that the following property holds:

(P1) For each integer $T \geq 0$ and each program $\{x_t\}_{t=T}^{T+L_0}$ which satisfies

$$\sum_{i=T}^{T+L_0-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^{\infty}, T, T + L_0, x_T, x_{T+L_0}) + \delta_0,$$
$$U(\{f_i\}_{i=0}^{\infty}, T, T + L_0) + c_f + 4 + M_1\}$$

there is an integer $j \in \{T, \dots, T + L_0\}$ for which $\rho(x_j, x_j^f) \leq \gamma_f/4$.

By Theorem 2.1 there exist a natural number $L_1 \geq 4$ and a real number

$$\delta_1 \in (0, \min\{\epsilon, \gamma_f\}) \tag{3.3}$$

such that for each integer $L_2 \geq L_1$, each pair of integers $T_1 \geq 0$, $T_2 > T_1 + 2L_2$ and each $\{g_i\}_{i=0}^{\infty} \in \mathcal{M}$ satisfying

$$d(\{f_t\}_{t=0}^{\infty}, \{g_t\}_{t=0}^{\infty}) \leq (8L_2)^{-1}\delta_1 \tag{3.4}$$

the following property holds:

(P2) Assume that $\{x_t\}_{t=T_1}^{T_2}$ is a program, q is a natural number, $\{S_i\}_{i=1}^{q}$ is a finite sequence of integers such that:

$$S_1 = T_1, \ T_2 \geq S_q > T_2 - L_2,$$

for each integer i satisfying $1 \leq i \leq q - 1$,

$$S_{i+1} - S_i \in [L_1, L_2],$$

$$\sum_{t=S_i}^{S_{i+1}-1} g_t(x_t, x_{t+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_i, S_{i+1}) + M_1 + c_f + 8(d_0+1) + 4,$$

$$\sum_{t=S_{q-1}}^{T_2-1} g_t(x_t, x_{t+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_{q-1}, T_2) + M_1 + c_f + 8(d_0+1) + 4,$$

for each integer $i \in [1, q-2]$,

$$\sum_{t=S_i}^{S_{i+2}-1} g_t(x_t, x_{t+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_i, S_{i+2}, x_{S_i}, x_{S_{i+2}}) + \delta_1,$$

$$\sum_{t=S_{q-2}}^{T_2-1} g_i(x_i, x_{i+1}) \le U(\{g_j\}_{j=0}^{\infty}, S_{q-2}, T_2, x_{S_{q-2}}, x_{T_2}) + \delta_1.$$

Then

$$\rho(x_t, x_t^f) \le \epsilon \text{ for all integers } t \in [T_1 + L_2, T_2 - L_2].$$

Choose a natural number

$$p_0 \ge 6 + M_1 + 16(L_0+4)(d_0+4) + L_1. \tag{3.5}$$

Set

$$L_2 = (p_0+4)L_0 + 2L_1. \tag{3.6}$$

Choose a natural number

$$L \ge 4L_0p_0 + 4L_2, \tag{3.7}$$

a positive number

$$\delta < \min\{(16L_0)^{-1}\delta_0,\ (48L_2)^{-1}\delta_1\} \tag{3.8}$$

and a number $\lambda \in (0,1)$ such that

$$18L_0(1+d_0)(1-\lambda)\lambda^{-1} < \delta_0,$$

$$\lambda^{p_0} > 2^{-1},\ 96L_2(1+d_0)(1-\lambda) < \delta_1. \tag{3.9}$$

Assume that $\{g_i\}_{i=0}^{\infty} \in \mathcal{M}$, for each integer $i \ge 0$ the function g_i is lower semicontinuous,

$$d(\{f_i\}_{i=0}^{\infty}, \{g_i\}_{i=0}^{\infty}) \le \delta, \tag{3.10}$$

$T_1 \ge 0$ is an integer and $\tilde{x} \in X$ is an $(\{f_i\}_{i=0}^{\infty}, T_1, M)$-good point.

In the case of Theorem 2.2 we assume that an integer

$$T_2 > T_1 + 2L,\ \{\alpha_i\}_{i=T_1}^{T_2-1} \subset (0,1], \tag{3.11}$$

$$\alpha_i\alpha_j^{-1} \ge \lambda \text{ for all } i,j \in \{T_1, \dots, T_2-1\} \text{ satisfying } |i-j| \le L, \tag{3.12}$$

$$\alpha_i\alpha_j^{-1} \ge \lambda \text{ for all } i,j \in \{T_1, \dots, T_2-1\} \text{ satisfying } j \ge i \tag{3.13}$$

and that a program $\{x_t\}_{t=T_1}^{T_2}$ satisfies

$$x_{T_1} = \tilde{x}, \ \sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) = U(\{\alpha_i g_i\}_{i=T_1}^{T_2-1}, T_1, T_2, x_{T_1}). \tag{3.14}$$

In the case of Theorem 2.5 we assume that

$$\{\alpha_i\}_{i=T_1}^{\infty} \subset (0, 1], \tag{3.15}$$

$$\alpha_i \alpha_j^{-1} \geq \lambda \text{ for all integers } i, j \geq T_1 \text{ satisfying } |i - j| \leq L, \tag{3.16}$$

$$\alpha_i \alpha_j^{-1} \geq \lambda \text{ for all integers } i, j \text{ satisfying } j \geq i \geq T_1 \tag{3.17}$$

and that an $(\{\alpha_i g_i\}_{t=T_1}^{\infty})$-overtaking optimal program $\{x_t\}_{t=T_1}^{\infty}$ satisfies

$$x_{T_1} = \tilde{x}. \tag{3.18}$$

In order to complete the proof in the case of Theorem 2.2 it is sufficient to show that

$$\rho(x_t, x_t^f) \leq \epsilon \text{ for all integers } t \in [T_1 + L, T_2 - L]$$

and in the case of Theorem 2.5 it is sufficient to show that

$$\rho(x_t, x_t^f) \leq \epsilon \text{ for all integers } t \geq T_1 + L.$$

Since the point $\tilde{x}$ is $(\{f_i\}_{i=0}^{\infty}, T_1, M)$-good it follows from (3.14) and (3.18) that there is a program $\{\tilde{z}_t\}_{t=T_1}^{\infty}$ such that $\tilde{z}_{T_1} = \tilde{x}$ and that for each integer $T > T_1$,

$$\sum_{t=T_1}^{T-1} f_t(\tilde{z}_t, \tilde{z}_{t+1}) \leq \sum_{t=T_1}^{T-1} f_t(x_t^f, x_{t+1}^f) + M. \tag{3.19}$$

In the case of Theorem 2.2 set $I = [T_1, T_2]$ and in the case of Theorem 2.5 set $I = [T_1, \infty)$.

We show that the following property holds:

(P3) If an integer S satisfies

$$[S, S + L_0] \subset I, \ \min\{\rho(x_t, x_t^f) : \ t = S, \ldots, S + L_0\} > \gamma_f / 4, \tag{3.20}$$

then

$$\sum_{t=S}^{S+L_0-1} \alpha_t g_t(x_t, x_{t+1}) \geq \sum_{t=S}^{S+L_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) + 3\alpha_S.$$

Assume that an integer S satisfies (3.20). By (3.14) in the case of Theorem 2.2 and the $(\{\alpha_i g_i\}_{t=T_1}^{\infty})$-overtaking optimality of the program $\{x_t\}_{t=T_1}^{\infty}$ in the case of Theorem 2.5,

$$\sum_{t=S}^{S+L_0-1} \alpha_S^{-1} \alpha_t g_t(x_t, x_{t+1}) = U(\{\alpha_S^{-1} \alpha_i g_i\}_{i=S}^{S+L_0-1}, S, S + L_0, x_S, x_{S+L_0}). \tag{3.21}$$

By (3.10), (3.7), (3.13), (3.16), (3.8) and (3.9), for each integer $q \geq 0$ and each program $\{y_t\}_{t=q}^{q+L_0}$,

$$\Big| \sum_{t=S}^{S+L_0-1} f_t(y_t, y_{t+1}) - \sum_{t=S}^{S+L_0-1} \alpha_S^{-1} \alpha_t g_t(y_t, y_{t+1}) \Big|$$

$$\begin{aligned}
&\le L_0 \max\{\|f_t - \alpha_S^{-1}\alpha_t g_t\| :\ t = S, \dots, S + L_0 - 1\} \\
&\le L_0 \max\{\|f_t - g_t\| + \|g_t - \alpha_S^{-1}\alpha_t g_t\| :\ t = S, \dots, S + L_0 - 1\} \\
&\le L_0\delta + L_0 \max\{|1 - \alpha_S^{-1}\alpha_t|\|g_t\| :\ t = S, \dots, S + L_0 - 1\} \\
&\le L_0\delta + L_0(1 + d_0)(1 - \lambda)\lambda^{-1} < \delta_0/8.
\end{aligned} \tag{3.22}$$

By (3.22), (3.21) and (C2),

$$\begin{aligned}
\sum_{t=S}^{S+L_0-1} f_t(x_t, x_{t+1}) &< \sum_{t=S}^{S+L_0-1} \alpha_S^{-1}\alpha_t g_t(x_t, x_{t+1}) + \delta_0/8 \\
&= U(\{\alpha_S^{-1}\alpha_i g_i\}_{i=S}^{S+L_0-1}, S, S + L_0, x_S, x_{S+L_0}) + \delta_0/8 \\
&\le U(\{f_i\}_{i=0}^{\infty}, S, S + L_0, x_S, x_{S+L_0}) + \delta_0/4.
\end{aligned} \tag{3.23}$$

By (3.20), (3.23) and the property (P1),

$$\begin{aligned}
\sum_{t=S}^{S+L_0-1} f_t(x_t, x_{t+1}) &\ge U(\{f_i\}_{i=0}^{\infty}, S, S + L_0) + 4 + c_f + M_1 \\
&\ge \sum_{t=S}^{S+L_0-1} f_t(x_t^f, x_{t+1}^f) + 4.
\end{aligned} \tag{3.24}$$

By (3.23), (3.24) and (3.22),

$$\begin{aligned}
\sum_{t=S}^{S+L_0-1} \alpha_S^{-1}\alpha_t g_t(x_t, x_{t+1}) &\ge \sum_{t=S}^{S+L_0-1} f_t(x_t, x_{t+1}) - \delta_0/8 \\
&\ge \sum_{t=S}^{S+L_0-1} f_t(x_t^f, x_{t+1}^f) + 4 - \delta_0/8 \\
&\ge \sum_{t=S}^{S+L_0-1} \alpha_S^{-1}\alpha_t g_t(x_t^f, x_{t+1}^f) + 4 - \delta_0/4
\end{aligned}$$

This implies that

$$\sum_{t=S}^{S+L_0-1} \alpha_t g_t(x_t, x_{t+1}) \ge \sum_{t=S}^{S+L_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) + 3\alpha_S.$$

Thus the property (P3) holds.

By (3.19) and (C2),

$$\begin{aligned}
\sum_{t=T_1}^{T_1+L_0-1} f_t(\tilde{z}_t, \tilde{z}_{t+1}) &\le \sum_{t=T_1}^{T_1+L_0-1} f_t(x_t^f, x_{t+1}^f) + M \\
&\le U(\{f_i\}_{i=0}^{\infty}, T_1, T_1 + L_0) + M + c_f.
\end{aligned} \tag{3.25}$$

By (3.19) and (3.25) there exists a program $\{\tilde{y}_t\}_{t=T_1}^{T_1+L_0}$ such that

$$\tilde{y}_{T_1} = \tilde{z}_{T_1} = \tilde{x},\ \tilde{y}_{T_1+L_0} = \tilde{z}_{T_1+L_0},$$

$$\sum_{t=T_1}^{T_1+L_0-1} f_t(\tilde{y}_t, \tilde{y}_{t+1}) = U(\{f_i\}_{i=0}^{\infty}, T_1, T_1 + L_0, \tilde{x}, \tilde{z}_{T_1+L_0}). \tag{3.26}$$

By (3.25), (3.26) and the property (P1), there is an integer

$$\tilde{j} \in \{T_1, \dots, T_1 + L_0\} \tag{3.27}$$

for which

$$\rho(\tilde{y}_{\tilde{j}}, x^f_{\tilde{j}}) \le \gamma_f/4. \tag{3.28}$$

If $\tilde{j} < T_1 + L_0$, then $(\tilde{y}_{\tilde{j}}, \tilde{y}_{\tilde{j}+1}) \in \Omega_{\tilde{j}}$; if $\tilde{j} = T_1 + L_0$, then by (3.26) $(\tilde{y}_{\tilde{j}}, \tilde{z}_{\tilde{j}+1}) \in \Omega_{\tilde{j}}$. Thus in both cases

$$(\tilde{y}_{\tilde{j}}, y) \in \Omega_{\tilde{j}} \text{ with some } y \in X.$$

Combined with (3.28) and (C4) this implies that there is $\tilde{y} \in X$ such that

$$(\tilde{y}_{\tilde{j}}, \tilde{y}) \in \Omega_{\tilde{j}},\ (\tilde{y}, x^f_{\tilde{j}+2}) \in \Omega_{\tilde{j}+1}. \tag{3.29}$$

Set

$$\tilde{x}_i = \tilde{y}_i,\ i = T_1, \dots, \tilde{j},\ \tilde{x}_{\tilde{j}+1} = \tilde{y},\ \tilde{x}_i = x^f_i \text{ for all integers } i \ge \tilde{j} + 2. \tag{3.30}$$

Clearly, $\{\tilde{x}_i\}_{i=T_1}^{\infty}$ is a program. By (3.26), (3.30) and (3.27),

$$\tilde{x}_{T_1} = \tilde{x},\ \tilde{x}_i = x^f_i \text{ for all integers } i \ge T_1 + L_0 + 2. \tag{3.31}$$

We show that there an integer $j \in I$ such that

$$\rho(x_j, x^f_j) \le \gamma_f/4. \tag{3.32}$$

Assume the contrary. Then

$$\rho(x_i, x^f_i) > \gamma_f/4 \text{ for all integers } i \in I. \tag{3.33}$$

In the case of Theorem 2.5 set

$$p = \infty. \tag{3.34}$$

In the case of Theorem 2.2 choose a natural number p such that

$$pL_0 \le T_2 - T_1 < (p+1)L_0. \tag{3.35}$$

By (3.34), (3.35), (3.11) and (3.1),

$$p \ge 2p_0. \tag{3.36}$$

Assume that an integer j satisfies

$$0 \le j,\ j + 1 \le p. \tag{3.37}$$

By (3.33), (3.35) and the property (P3),

$$\sum_{t=T_1+jL_0}^{T_1+(j+1)L_0-1} \alpha_t g_t(x_t, x_{t+1}) \ge \sum_{t=T_1+jL_0}^{T_1+(j+1)L_0-1} \alpha_t g_t(x^f_t, x^f_{t+1}) + 3\alpha_{T_1+jL_0}. \tag{3.38}$$

Consider the case of Theorem 2.2. By (3.31), (3.14), (3.35), (3.13), (3.38), (3.10), (3.36), (3.7), (3.12), (3.9) and (3.5),

$$
\begin{aligned}
0 &\geq \sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T_2-1} \alpha_t g_t(\tilde{x}_t, \tilde{x}_{t+1}) \\
&= \sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t^f, x_{t+1}^f) \\
&\quad + \sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t^f, x_{t+1}^f) - \sum_{t=T_1}^{T_2-1} \alpha_t g_t(\tilde{x}_t, \tilde{x}_{t+1}) \\
&\geq \sum_{j=0}^{p-1} \Big(\sum_{t=T_1+jL_0}^{T_1+(j+1)L_0-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1+jL_0}^{T_1+(j+1)L_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) \Big) \\
&\quad + \sum \{ \alpha_t g_t(x_t, x_{t+1}) - \alpha_t g_t(x_t^f, x_{t+1}^f) : \\
&\qquad\qquad t \text{ is an integer such that } T_1 + pL_0 \leq t < T_2 \} \\
&\quad - 2(L_0+2)\alpha_{T_1} \lambda^{-1} \sup\{ ||g_i|| : \ i \text{ is an integer and } i \geq 0 \} \\
&\geq 3 \sum_{j=0}^{p-1} \alpha_{T_1+jL_0} - 4(L_0+2)\alpha_{T_1} \lambda^{-1} \sup\{ ||g_i|| : \ i \text{ is an integer and } i \geq 0 \} \\
&\geq 3 \sum_{j=0}^{p-1} \alpha_{T_1+jL_0} - 4(L_0+2)\alpha_{T_1} \lambda^{-1}(d_0+1) \\
&\geq 3 \sum_{j=0}^{p_0-1} \alpha_{T_1+jL_0} - 4(L_0+2)\alpha_{T_1} \lambda^{-1}(d_0+1) \\
&\geq 3\alpha_{T_1} \sum_{j=0}^{p_0-1} \lambda^j - 4(L_0+2)\alpha_{T_1} \lambda^{-1}(d_0+1) \\
&\geq 3\alpha_{T_1}(p_0/2) - 8(L_0+2)\alpha_{T_1}(d_0+1) \\
&= \alpha_{T_1}(p_0 - 8(L_0+2)(d_0+1)) > 4\alpha_{T_1},
\end{aligned}
$$

a contradiction. The contradiction we have reached proves that in the case of Theorem 2.2 there is an integer $j \in I$ such that (3.32) holds.

Consider the case of Theorem 2.5. Since $\{x_t\}_{t=T_1}^{\infty}$ is an $(\{\alpha_i g_i\}_{t=T_1}^{\infty})$-overtaking optimal program, it follows from (3.31), (3.18), (3.38), (3.10), (3.17), (3.16), (3.9) and (3.5) that

$$
\begin{aligned}
0 &\geq \limsup_{T\to\infty} \Big[\sum_{t=T_1}^{T} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T} \alpha_t g_t(\tilde{x}_t, \tilde{x}_{t+1}) \Big] \\
&\geq \limsup_{k\to\infty} \Big[\sum_{t=T_1}^{T_1+kL_0-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T_1+kL_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f)
\end{aligned}
$$

$$+\sum_{t=T_1}^{T_1+kL_0-1}\alpha_t g_t(x_t^f,x_{t+1}^f)-\sum_{t=T_1}^{T_1+kL_0-1}\alpha_t g_t(\tilde{x}_t,\tilde{x}_{t+1})\Big]$$

$$\geq \limsup_{k\to\infty}\Big[3\sum_{j=0}^{k-1}\alpha_{T_1+jL_0}-2(L_0+2)\alpha_{T_1}\lambda^{-1}$$

$$\sup\{||g_i||:\ \ i \text{ is an integer and } i\geq 0\}\Big]$$

$$\geq 3\sum_{j=0}^{p_0-1}\alpha_{T_1+jL_0}-2(L_0+2)\alpha_{T_1}\lambda^{-1}(d_0+1)$$

$$\geq 3\alpha_{T_1}\sum_{j=0}^{p_0-1}\lambda^j-2(L_0+2)\alpha_{T_1}\lambda^{-1}(d_0+1)$$

$$\geq 3\alpha_{T_1}(p_0/2)-4(L_0+2)\alpha_{T_1}(d_0+1)>\alpha_{T_1},$$

a contradiction. The contradiction we have reached proves that in the case of Theorem 2.5 there is an integer $j\in I$ such that (3.32) holds.

Assume that an integer τ satisfies

$$T_1\leq\tau,\ L_0(1+p_0)+\tau\in I,\ \rho(x_\tau,x_\tau^f)\leq\gamma_f/4. \tag{3.39}$$

We show that there is an integer S such that

$$S\in I,\ S\geq\tau+L_0,\ \rho(x_S,x_S^f)\leq\gamma_f/4. \tag{3.40}$$

Assume the contrary. Then

$$\rho(x_S,x_S^f)>\gamma_f/4 \text{ for all integers } S\in I \text{ satisfying } S\geq\tau+L_0. \tag{3.41}$$

In the case of Theorem 2.5 set

$$p=\infty. \tag{3.42}$$

In the case of Theorem 2.2 there is an integer p such that

$$pL_0\leq T_2-(\tau+L_0)<(p+1)L_0. \tag{3.43}$$

By (3.43) and (3.39),

$$p_0\leq p. \tag{3.44}$$

Assume that an integer j satisfies

$$1\leq j\leq p. \tag{3.45}$$

By (3.45), (3.43), (3.39) and the property (P3),

$$\sum_{t=\tau+jL_0}^{\tau+(j+1)L_0-1}\alpha_t g_t(x_t,x_{t+1})\geq\sum_{t=\tau+jL_0}^{\tau+(j+1)L_0-1}\alpha_t g_t(x_t^f,x_{t+1}^f)+3\alpha_{\tau+jL_0}. \tag{3.46}$$

By (3.39) and (C4) there is $\xi\in X$ such that

$$(x_\tau,\xi)\in\Omega_\tau,\ (\xi,x_{\tau+2}^f)\in\Omega_{\tau+1}. \tag{3.47}$$

By (3.47) there is a program $\{\xi_t\}_{t\in I}$ such that

$$\xi_t = x_t,\ t = T_1,\dots,\tau,\ \xi_{\tau+1} = \xi,\ \xi_t = x_t^f$$

$$\text{for all integers } t \in I \text{ satisfying } t \geq \tau + 2. \tag{3.48}$$

Consider the case of Theorem 2.2. By (3.48), (3.14), (3.13), (3.43), (3.10), (3.46), (3.12), (3.9) and (3.5),

$$\begin{aligned}
0 &\geq \sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T_2-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \\
&= \sum_{t=\tau}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau}^{T_2-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \\
&= \sum_{t=\tau}^{\tau+L_0-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau}^{\tau+L_0-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \\
&\quad + \sum_{t=\tau+L_0}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau+L_0}^{T_2-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \\
&\geq \sum_{t=\tau+L_0}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau+L_0}^{T_2-1} \alpha_t g_t(x_t^f, x_{t+1}^f) \\
&\quad - \alpha_\tau \lambda^{-1} 2(L_0+4) \sup\{\|g_t\| :\ t = 0,1,\dots\} \\
&\geq \sum_{j=1}^{p} \Bigg(\sum_{t=\tau+jL_0}^{\tau+(j+1)L_0-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau+jL_0}^{\tau+(j+1)L_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) \Bigg) \\
&\quad + \sum \{\alpha_t g_t(x_t, x_{t+1}) - \alpha_t g_t(x_t^f, x_{t+1}^f) : \\
&\qquad\qquad t \text{ is an integer such that } \tau + (p+1)L_0 \leq t < T_2\} \\
&\quad - 2\alpha_\tau \lambda^{-1}(L_0+4)(d_0+1) \\
&\geq \sum_{j=1}^{p} 3\alpha_{\tau+jL_0} - 4\alpha_\tau \lambda^{-1}(L_0+4)(d_0+1) \\
&\geq 3\alpha_\tau \sum_{j=1}^{p_0} \lambda^j - 8\alpha_\tau (L_0+4)(d_0+1) \\
&\geq 3\alpha_\tau (p_0/2) - 8\alpha_\tau (L_0+4)(d_0+1) \\
&\geq \alpha_\tau (p_0 - 8(L_0+4)(d_0+2)) \geq \alpha_\tau,
\end{aligned}$$

a contradiction. The contradiction we have reached proves that in the case of Theorem 2.2 there is an integer S which satisfies (3.40).

Consider the case of Theorem 2.5. Since $\{x_t\}_{t=T_1}^{\infty}$ is an $(\{\alpha_i g_i\}_{t=T_1}^{\infty})$-overtaking optimal program, it follows from (3.48) that

$$0 \geq \limsup_{T\to\infty} \Big[\sum_{t=T_1}^{T-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \Big]. \tag{3.49}$$

By (3.48), (3.49), (3.17), (3.10), (3.46), (3.16), (3.7), (3.5) and (3.9),

$$\begin{aligned}
0 &\geq \limsup_{T\to\infty} \Big[\sum_{t=T_1}^{T-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \Big] \\
&= \limsup_{T\to\infty} \Big[\sum_{t=\tau}^{T-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau}^{T-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \Big] \\
&= \limsup_{T\to\infty} \Big[\sum_{t=\tau}^{\tau+L_0-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau}^{\tau+L_0-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \\
&\qquad + \sum_{t=\tau+L_0}^{T-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau+L_0}^{T-1} \alpha_t g_t(\xi_t, \xi_{t+1}) \Big] \\
&\geq \limsup_{k\to\infty} \Big[- 2L_0 \alpha_\tau \lambda^{-1} \sup\{||g_i|| :\ i \text{ is an integer and } i \geq 0\} \\
&\qquad + \sum_{t=\tau+L_0}^{\tau+(k+1)L_0-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau+L_0}^{\tau+(k+1)L_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) \\
&\qquad - 4\alpha_\tau \lambda^{-1} \sup\{||g_i|| :\ i \text{ is an integer and } i \geq 0\} \Big] \\
&\geq \limsup_{k\to\infty} \Big[- 2(d_0+1)(L_0+2)\alpha_\tau \lambda^{-1} \\
&\qquad + \sum_{j=1}^{k} \Big(\sum_{t=\tau+jL_0}^{\tau+(j+1)L_0-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=\tau+jL_0}^{\tau+(j+1)L_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) \Big) \Big] \\
&\geq -2(d_0+1)(L_0+2)\alpha_\tau \lambda^{-1} + \limsup_{k\to\infty} \sum_{j=1}^{k} 3\alpha_{\tau+jL_0} \\
&\geq -2(d_0+1)(L_0+2)\alpha_\tau \lambda^{-1} + \sum_{j=1}^{p_0} 3\alpha_{\tau+jL_0} \\
&\geq -2(d_0+1)(L_0+2)\alpha_\tau \lambda^{-1} + 3\alpha_\tau \sum_{j=1}^{p_0} \lambda^j \\
&\geq -4(d_0+1)(L_0+2)\alpha_\tau + 3\alpha_\tau (p_0/2) \\
&\geq \alpha_\tau (p_0 - 4(d_0+1)(L_0+2)) > \alpha_\tau,
\end{aligned}$$

a contradiction. The contradiction we have reached proves that in the case of Theorem 2.5 there is an integer S which satisfies (3.40). Thus we have shown that the following property holds:

(P4) for each integer τ satisfying (3.39) there is an integer S satisfying (3.40).

We show that the following property holds:

(P5) for each integer τ satisfying

$$\tau \in I,\ \tau - L_0(1+p_0) \geq T_1,\ \rho(x_\tau, x_\tau^f) \leq \gamma_f/4 \tag{3.50}$$

there is an integer S such that

$$T_1 \leq S \leq \tau - L_0,\ \rho(x_S, x_S^f) \leq \gamma_f/4. \tag{3.51}$$

Let an integer τ satisfy (3.50). We show that there is an integer S satisfying (3.51). Assume the contrary. Then

$$\rho(x_t, x_t^f) > \gamma_f/4 \text{ for all integers } t = T_1, \dots, \tau - L_0. \tag{3.52}$$

By (3.50), there is a natural number p such that

$$p_0 \leq p,\ pL_0 \leq \tau - L_0 - T_1 < (p+1)L_0. \tag{3.53}$$

Assume that an integer

$$j \in [1, p]. \tag{3.54}$$

By (3.54), (3.52), (3.53) and the property (P3),

$$\sum_{t=\tau-(j+1)L_0}^{\tau-jL_0-1} \alpha_t g_t(x_t, x_{t+1}) \geq \sum_{t=\tau-(j+1)L_0}^{\tau-jL_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) + 3\alpha_{\tau-(j+1)L_0}. \tag{3.55}$$

We continue to consider the program $\{\tilde{x}_t\}_{t=T_1}^\infty$ satisfying (3.31).

By (3.50) and (C4) there is $\eta \in X$ such that

$$(\eta, x_\tau) \in \Omega_{\tau-1},\ (x_{\tau-2}^f, \eta) \in \Omega_{\tau-2}. \tag{3.56}$$

By (3.31), (3.50), (3.56) and (3.5) there is a program $\{\eta_t\}_{t=T_1}^\tau$ such that

$$\eta_t = \tilde{x}_t,\ t = T_1, \dots, \tau-2,\ \eta_{\tau-1} = \eta,\ \eta_\tau = x_\tau. \tag{3.57}$$

By (3.57), (3.31), (3.14), (3.18), (3.50) and (3.5),

$$\eta_{T_1} = \tilde{x}_{T_1} = \tilde{x} = x_{T_1},\ \eta_t = x_t^f \text{ for all integers } t = T_1 + L_0 + 2, \dots, \tau - 2. \tag{3.58}$$

By (3.58) and (3.14) (in the case of Theorem 2.2) and ($\{\alpha_i g_i\}_{t=T_1}^\infty$)-overtaking optimality of the program $\{x_t\}_{t=T_1}^\infty$ (in the case of Theorem 2.5), (3.57), (3.50), (3.52), (3.13), (3.17), (3.55), (3.10), (3.12), (3.16), (3.9) and (3.5),

$$\begin{aligned}
0 &\geq \sum_{t=T_1}^{\tau-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{\tau-1} \alpha_t g_t(\eta_t, \eta_{t+1}) \\
&= \sum_{j=1}^{p} \sum_{t=\tau-(j+1)L_0}^{\tau-jL_0-1} (\alpha_t g_t(x_t, x_{t+1}) - \alpha_t g_t(\eta_t, \eta_{t+1})) \\
&\quad + \sum \{\alpha_t g_t(x_t, x_{t+1}) - \alpha_t g_t(\eta_t, \eta_{t+1}) : \\
&\qquad\qquad t \text{ is an integer such that } T_1 \leq t < \tau - (p+1)L_0\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum\{\alpha_t g_t(x_t,x_{t+1})-\alpha_t g_t(\eta_t,\eta_{t+1}):\\
&\qquad t \text{ is an integer such that } \tau-L_0\le t<\tau\}\\
&=\sum_{j=1}^{p}\Big(\sum_{t=\tau-(j+1)L_0}^{\tau-jL_0-1}\alpha_t g_t(x_t,x_{t+1})-\sum_{t=\tau-(j+1)L_0}^{\tau-jL_0-1}\alpha_t g_t(x_t^f,x_{t+1}^f)\Big)\\
&\quad+\sum_{t=\tau-(p+1)L_0}^{\tau-L_0-1}\alpha_t g_t(x_t^f,x_{t+1}^f)-\sum_{t=\tau-(p+1)L_0}^{\tau-L_0-1}\alpha_t g_t(\eta_t,\eta_{t+1})\\
&\quad-4L_0\sup\{\|g_t\|:\ t=0,1,\dots\}\alpha_{T_1}\lambda^{-1}\\
&\ge\sum_{j=1}^{p}3\alpha_{\tau-(j+1)L_0}-2\sup\{\|g_i\|:\ i=0,1,\dots\}\alpha_{T_1}\lambda^{-1}(L_0+4)\\
&\quad-4L_0\alpha_{T_1}\lambda^{-1}\sup\{||g_i||:\ i \text{ is a nonnegative integer}\}\\
&\ge 3\alpha_{\tau-(p+1)L_0}\sum_{j=0}^{p-1}\lambda^j-\alpha_{T_1}\lambda^{-1}(d_0+1)8(L_0+2)\\
&\ge 3\alpha_{T_1}\lambda\sum_{j=0}^{p-1}\lambda^j-2\alpha_{T_1}(d_0+1)8(L_0+2)\\
&\ge 3\alpha_{T_1}(p_0/2)-16\alpha_{T_1}(d_0+1)(L_0+2)\\
&\ge\alpha_{T_1}((p_0-16(d_0+1)(L_0+2))>\alpha_{T_1},
\end{aligned}
$$

a contradiction. The contradiction we have reached proves that there is an integer S satisfying (3.51) and that the property (P5) holds.

In the case of Theorem 2.2 it follows from (3.11), (3.37), (3.32) and the properties (P4) and (P5) that there exist a natural number q and a finite strictly increasing sequence of integers $\{S_i\}_{i=1}^q\subset[T_1,T_2]$ such that for each $t\in\{T_1,\dots,T_2\}$,

$$\rho(x_t,x_t^f)\le\gamma_f/4 \text{ if and only if } t\in\{S_1,\dots,S_q\},\tag{3.59}$$

$$q\ge 2,\ S_1<T_1+L_0(1+p_0),\ S_q>T_2-L_0(1+p_0).\tag{3.60}$$

In the case of Theorem 2.5 it follows from (3.32) and the properties (P4) and (P5) that there exists a strictly increasing sequence of integers $\{S_i\}_{i=1}^\infty\subset[T_1,\infty)$ such that for each integer $t\in[T_1,\infty)$,

$$\rho(x_t,x_t^f)\le\gamma_f/4 \text{ if and only if } t\in\{S_i:\ i \text{ is a natural number}\},\tag{3.61}$$

$$S_1<T_1+L_0(1+p_0).$$

In the case of Theorem 2.5 set

$$q=\infty.\tag{3.62}$$

Let an integer k satisfy

$$1\le k \text{ and } k+1\le q.\tag{3.63}$$

We show that

$$S_{k+1} - S_k \le (p_0 + 2)L_0. \tag{3.64}$$

Assume the contrary. Then

$$S_{k+1} - S_k > (p_0 + 2)L_0. \tag{3.65}$$

By (3.65) there is a natural number p such that

$$p > p_0,\ (p + 2)L_0 \le S_{k+1} - S_k < (p + 3)L_0. \tag{3.66}$$

By (3.59),

$$\rho(x_t, x_t^f) > \gamma_f/4 \text{ for all integers } t = S_k + 1, \dots, S_{k+1} - 1. \tag{3.67}$$

Let an integer

$$j \in [0, p - 1]. \tag{3.68}$$

By (3.68), (3.67), (3.66) and the property (P3),

$$\sum_{t=S_k+(j+1)L_0}^{S_k+(j+2)L_0-1} \alpha_t g_t(x_t, x_{t+1}) \ge \sum_{t=S_k+(j+1)L_0}^{S_k+(j+2)L_0-1} \alpha_t g_t(x_t^f, x_{t+1}^f) + 3\alpha_{S_k+(j+1)L_0}. \tag{3.69}$$

By (3.59), (3.65), (3.5) and (C4) there exist $\eta_1, \eta_2 \in X$ such that

$$(x_{S_k}, \eta_1) \in \Omega_{S_k},\ (\eta_1, x^f_{S_k+2}) \in \Omega_{S_k+1}, \tag{3.70}$$

$$(x^f_{S_{k+1}-2}, \eta_2) \in \Omega_{S_{k+1}-2},\ (\eta_2, x_{S_k+1}) \in \Omega_{S_{k+1}-1}.$$

By (3.70), (3.64) and (3.5) there is a program $\{\widehat{x}_t\}_{t\in I}$ such that

$$\widehat{x}_t = x_t,\ t = T_1, \dots, S_k,\ \widehat{x}_{S_k+1} = \eta_1,$$

$$\widehat{x}_t = x_t^f,\ t = S_k + 2, \dots, S_{k+1} - 2,\ \widehat{x}_{S_{k+1}-1} = \eta_2,$$

$$\widehat{x}_t = x_t \text{ for all integers } t \in I \text{ satisfying } t \ge S_{k+1}. \tag{3.71}$$

By (3.71), (3.14) (in the case of Theorem 2.2) and $(\{\alpha_i g_i\}_{i=T_1}^{\infty})$-overtaking optimality of the program $\{x_t\}_{t=T_1}^{\infty}$ (in the case of Theorem 2.5), (3.13), (3.17), (3.66), (3.10), (3.69), (3.12), (3.9) and (3.16),

$$\begin{aligned}
0 &\ge \sum_{t=T_1}^{S_{k+1}-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{S_{k+1}-1} \alpha_t g_t(\widehat{x}_t, \widehat{x}_{t+1}) \\
&= \sum_{t=S_k}^{S_{k+1}-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=S_k}^{S_{k+1}-1} \alpha_t g_t(\widehat{x}_t, \widehat{x}_{t+1}) \\
&\ge \sum_{t=S_k}^{S_{k+1}-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=S_k}^{S_{k+1}-1} \alpha_t g_t(x_t^f, x_{t+1}^f) \\
&\quad - 8(\sup\{\|g_t\| :\ t = 0, 1, \dots\}\alpha_{S_k}\lambda^{-1} \\
&\ge \sum_{t=S_k}^{S_k+L_0-1} (\alpha_t g_t(x_t, x_{t+1}) - \alpha_t g_t(x_t^f, x_{t+1}^f))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{p-1}\Big(\sum_{t=S_k+(j+1)L_0}^{S_k+(j+2)L_0-1}\alpha_t g_t(x_t,x_{t+1})-\alpha_t g_t(x_t^f,x_{t+1}^f)\Big)\\
&+\sum_{t=S_k+(p+1)L_0}^{S_{k+1}-1}(\alpha_t g_t(x_t,x_{t+1})-\alpha_t g_t(x_t^f,x_{t+1}^f))-8\alpha_{S_k}\lambda^{-1}(d_0+1)\\
&\geq -2\alpha_{S_k}\lambda^{-1}L_0\sup\{\|g_t\|:\ t=0,1,\dots\}+\sum_{j=0}^{p-1}3\alpha_{S_k+(j+1)L_0}\\
&\quad -4\alpha_{S_k}\lambda^{-1}L_0\sup\{\|g_t\|:\ t=0,1,\dots\}-8\alpha_{S_k}\lambda^{-1}(d_0+1)\\
&\geq 3\alpha_{S_k}\sum_{j=0}^{p_0-1}\lambda^{j+1}-2\alpha_{S_k}(6L_0+8)(d_0+1)\\
&\geq 3\alpha_{S_k}(p_0/2)-2\alpha_{S_k}(6L_0+8)(d_0+1)\\
&\geq \alpha_{S_k}(p_0-(12L_0+16)(d_0+1))>\alpha_{S_k},
\end{aligned}
$$

a contradiction. The contradiction we have reached proves that the following property holds:

(P6) $S_{k+1}-S_k\leq (p_0+2)L_0$ for all integers k satisfying $1\leq k$ and $k+1\leq q$.

We set

$$\tilde{S}_1=S_1. \tag{3.72}$$

In the case of Theorem 2.5 set

$$S_\infty=S_q=\infty. \tag{3.73}$$

Assume that j is a natural number and that we have already defined a finite strictly increasing sequence of integers

$$\{\tilde{S}_i\}_{i=1}^j\subset\{S_i: i \text{ is a natural number for which } i\leq q\} \tag{3.74}$$

such that for each i satisfying $1\leq i<j$,

$$\tilde{S}_{i+1}-\tilde{S}_i\in[L_1,L_2]. \tag{3.75}$$

(Clearly, for $j=1$ our assumption holds.)

If $\tilde{S}_j+L_2>S_q$, then the construction is completed. Assume that

$$\tilde{S}_j+L_2\leq S_q. \tag{3.76}$$

Then by (3.6) and (3.76),

$$\tilde{S}_j+L_1+(p_0+4)L_0\leq \tilde{S}_j+L_2\leq S_q. \tag{3.77}$$

By (3.56), (3.61) and (3.72) there is a natural number k such that

$$\tilde{S}_j+L_1\in[S_{k-1},S_k]. \tag{3.77}$$

By (3.74), (3.77) and (3.75),

$$\tilde{S}_j\leq S_{k-1},\ k\leq q. \tag{3.78}$$

We set

$$\tilde{S}_{j+1} = S_k. \tag{3.79}$$

By (3.79), (3.77), (3.6) and (P6),

$$L_1 \le \tilde{S}_{j+1} - \tilde{S}_j \le L_1 + S_k - S_{k-1} \le L_1 + (p_0 + 2)L_0 \le L_2$$

and the assertion made for j also holds for $j+1$.

In the case of Theorem 2.5 we obtain a sequence of integers $\{\tilde{S}_i\}_{i=1}^{\infty}$ such that

$$\{\tilde{S}_i\}_{i=1}^{\infty} \subset \{S_i\}_{i=1}^{\infty},\ \tilde{S}_1 = S_1 < T_1 + L_0(1+p_0),$$

$$\tilde{S}_{i+1} - \tilde{S}_i \in [L_1, L_2] \text{ for all integers } i \ge 1. \tag{3.80}$$

In the case of Theorem 2.2 our construction is completed after a finite number of steps and we obtain a finite strictly increasing sequence of integers $\{\tilde{S}_i\}_{i=1}^{k}$, where k is a natural number, such that

$$\{\tilde{S}_i\}_{i=1}^{k} \subset \{S_1, \dots, S_q\},$$

$$\tilde{S}_1 = S_1 < T_1 + L_0(1+p_0),\ \tilde{S}_k > S_q - L_2 > T_2 - L_0(1+p_0) - L_2, \tag{3.81}$$

$$\tilde{S}_{i+1} - \tilde{S}_i \in [L_1, L_2] \text{ for all integers } i \text{ satisfying } 1 \le i < k. \tag{3.82}$$

In the case of Theorem 2.2, by (3.81), (3.11) and (3.7),

$$\begin{aligned} \tilde{S}_k - \tilde{S}_1 &> T_2 - T_1 - 2L_0(p_0+1) - L_2 \\ &> 2L - 2L_0(p_0+1) - L_2 \\ &> 2L_2. \end{aligned} \tag{3.83}$$

In the case of Theorem 2.5 let k be any natural such that

$$\tilde{S}_k - \tilde{S}_1 > 2L_2. \tag{3.84}$$

We apply the property (P2) to the program $\{x_t\}_{t=\tilde{S}_1}^{\tilde{S}_k}$. By (3.14) (in the case of Theorem 2.2) and $(\{\alpha_i g_i\}_{i=T_1}^{\infty})$-overtaking optimality of the program $\{x_t\}_{t=T_1}^{\infty}$ (in the case of Theorem 2.5), for each pair of integers τ_1, τ_2 satisfying

$$\tilde{S}_1 \le \tau_1 < \tau_2 \le \tilde{S}_k,$$

we have

$$\sum_{t=\tau_1}^{\tau_2-1} \alpha_t g_t(x_t, x_{t+1}) = U(\{\alpha_i g_i\}_{i=\tau_1}^{\tau_2-1}, \tau_1, \tau_2, x_{\tau_1}, x_{\tau_2}). \tag{3.85}$$

By (3.10), (3.1), (3.7), (3.12), (3.16), (3.9) and (3.8) for each pair of integers τ_1, τ_2 satisfying

$$\tilde{S}_1 \le \tau_1 < \tau_2 \le \tilde{S}_k,\ \tau_2 \le \tau_1 + 3L_2, \tag{3.86}$$

each program $\{y_t\}_{t=\tau_1}^{\tau_2}$ we have

$$\Big|\sum_{t=\tau_1}^{\tau_2-1} f_t(y_t, y_{t+1}) - \sum_{t=\tau_1}^{\tau_2-1} \alpha_{\tau_1}^{-1} \alpha_t g_t(y_t, y_{t+1})\Big|$$

$$\begin{aligned}
&\le \sum_{t=\tau_1}^{\tau_2-1} |f_t(y_t, y_{t+1}) - g_t(y_t, y_{t+1})| + \sum_{t=\tau_1}^{\tau_2-1} |g_t(y_t, y_{t+1})||1 - \alpha_{\tau_1}^{-1}\alpha_t| \\
&\le \delta(\tau_2 - \tau_1) + (\tau_2 - \tau_1)(1 + d_0)|\lambda - 1|\lambda^{-1} \qquad (3.84)\\
&\le (\tau_2 - \tau_1)\delta + 2(\tau_2 - \tau_1)(1 + d_0)|1 - \lambda| \\
&\le 3L_2\delta + 6L_2(1 + d_0)(1 - \lambda) < \delta_1/8.
\end{aligned}$$

By (3.80), (3.82), (3.87) and (3.85), for each integer i satisfying $1 \le i \le k - 2$,

$$\begin{aligned}
\sum_{t=\tilde{S}_i}^{\tilde{S}_{i+2}-1} f_t(x_t, x_{t+1}) &\le \sum_{t=\tilde{S}_i}^{\tilde{S}_{i+2}-1} \alpha_{\tilde{S}_i}^{-1}\alpha_t g_t(x_t, x_{t+1}) + \delta_1/8 \\
&= U(\{\alpha_{\tilde{S}_i}^{-1}\alpha_t g_t\}_{t=\tilde{S}_i}^{\tilde{S}_{i+2}-1}, \tilde{S}_i, \tilde{S}_{i+2}, x_{\tilde{S}_i}, x_{\tilde{S}_{i+2}}) + \delta_1/8 \qquad (3.88)\\
&\le U(\{f_t\}_{t=0}^{\infty}, \tilde{S}_i, \tilde{S}_{i+2}, x_{\tilde{S}_i}, x_{\tilde{S}_{i+2}}) + \delta_1/4.
\end{aligned}$$

Let an integer j satisfy $1 \le j < k$. By (3.80) and (3.82),

$$4 \le L_1 \le \tilde{S}_{j+1} - \tilde{S}_j \le L_2, \tag{3.89}$$

$$\rho(x_{\tilde{S}_j}, x^f_{\tilde{S}_j}) \le \gamma_f/4,\ \rho(x_{\tilde{S}_{j+1}}, x^f_{\tilde{S}_{j+1}}) \le \gamma_f/4. \tag{3.90}$$

By (3.89), (3.90) and (C4) there is $\eta_1, \eta_2 \in X$ such that

$$(x_{\tilde{S}_j}, \eta_1) \in \Omega_{\tilde{S}_j},\ (\eta_1, x^f_{\tilde{S}_j+2}) \in \Omega_{\tilde{S}_j+1},$$

$$(x^f_{\tilde{S}_{j+1}-2}, \eta_2) \in \Omega_{\tilde{S}_{j+1}-2},\ (\eta_2, x_{\tilde{S}_{j+1}}) \in \Omega_{\tilde{S}_{j+1}-1}. \tag{3.91}$$

Set

$$\bar{x}_{\tilde{S}_j} = x_{\tilde{S}_j},\ \bar{x}_{\tilde{S}_j+1} = \eta_1,\ \bar{x}_{\tilde{S}_{j+1}-1} = \eta_2,\ \bar{x}_t = x^f_t,\ t = \tilde{S}_j + 2, \dots, \tilde{S}_{j+1} - 2, \tag{3.92}$$
$$\bar{x}_{\tilde{S}_{j+1}} = x_{\tilde{S}_{j+1}}.$$

By (3.92) and (3.91), $\{\bar{x}_t\}_{t=\tilde{S}_j}^{\tilde{S}_{j+1}}$ is a program. By (3.86), (3.87), (3.80), (3.82), (3.85), (3.92) and (C2),

$$\begin{aligned}
\sum_{t=\tilde{S}_j}^{\tilde{S}_{j+1}-1} f_t(x_t, x_{t+1}) &\le \sum_{t=\tilde{S}_j}^{\tilde{S}_{j+1}-1} \alpha_{\tilde{S}_j}^{-1}\alpha_t g_t(x_t, x_{t+1}) + \delta_1/8 \\
&\le \sum_{t=\tilde{S}_j}^{\tilde{S}_{j+1}-1} \alpha_{\tilde{S}_j}^{-1}\alpha_t g_t(\bar{x}_t, \bar{x}_{t+1}) + \delta_1/8 \\
&\le \sum_{t=\tilde{S}_j}^{\tilde{S}_{j+1}-1} f_t(\bar{x}_t, \bar{x}_{t+1}) + \delta_1/4 \qquad (3.93)
\end{aligned}$$

$$\leq \sum_{t=\tilde{S}_j}^{\tilde{S}_{j+1}-1} f_t(x_t^f, x_{t+1}^f) + 8(d_0+1) + 1$$
$$\leq U(\{f_i\}_{i=0}^{\infty}, \tilde{S}_j, \tilde{S}_{j+1}) + c_f + 8(d_0+1) + 1.$$

By (3.88) which holds for all integers i satisfying $1 \leq i \leq k-2$, (3.93) which holds for all integers j satisfying $1 \leq j < k$, (3.10), (3.8), (3.83), (3.84), (3.80), (3.82) and the property (P2) applied with the program $\{x_t\}_{t=\tilde{S}_1}^{\tilde{S}_k}$,

$$\rho(x_t, x_t^f) \leq \epsilon \text{ for all integers } t \in [\tilde{S}_1 + L_2, \tilde{S}_k - L_2]. \tag{3.94}$$

In the case of Theorem 2.2 it follows from (3.94), (3.81) and (3.7) that

$$\rho(x_t, x_t^f) \leq \epsilon \text{ for all integers } t \in [T_1 + L, T_2 - L].$$

In the case of Theorem 2.5 since k is any sufficiently large natural number, it follows from (3.94), (3.80) and (3.7) that

$$\rho(x_t, x_t^f) \leq \epsilon \text{ for all integers } t \geq T_1 + L.$$

Theorems 2.2 and 2.5 are proved. □

4. Proof of Theorem 2.4

Fix

$$d_0 > \sup\{\|f_i\| : i = 0, 1, \dots\}. \tag{4.1}$$

In the proof we use the following auxiliary result.

Lemma 4.1. *Let $\gamma > 0$. Then there is a natural number n_0 such that for each pair of integers $T_1 \geq 0$, $T_2 > T_1 + n_0$, each integer $S \in [T_1 + n_0, T_2 - 1]$ and each program $\{x_t\}_{t=T_1}^{T_2}$ such that*

$$x_{T_1} \text{ is } (\{f_i\}_{i=0}^{\infty}, T_1, M) - \text{good}, \tag{4.2}$$

$$\sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) = U(\{\alpha_t g_t\}_{t=0}^{\infty}, T_1, T_2, x_{T_1}) \tag{4.3}$$

the following inequality holds:

$$\sum_{t=T_1}^{S-1} \alpha_t g_t(x_t, x_{t+1}) \leq U(\{\alpha_t g_t\}_{t=0}^{\infty}, T_1, S, x_{T_1}) + \gamma.$$

Proof. By (2.6) there is a natural number

$$n_0 > 4L + 4 \tag{4.4}$$

such that for all integers $t > n_0 - L - 4$

$$\alpha_t \leq \gamma (8L+8)^{-1} (d_0+1)^{-1}. \tag{4.5}$$

Assume that integers $T_1 \geq 0$, $T_2 > T_1 + n_0$, an integer $S \in [T_1 + n_0, T_2 - 1]$ and that a program $\{x_t\}_{t=T_1}^{T_2}$ satisfies (4.2) and (4.3). Clearly, there is a program $\{\tilde{x}_t\}_{t=T_1}^{S}$ such that

$$\tilde{x}_{T_1} = x_{T_1}, \ \sum_{t=T_1}^{S-1} \alpha_t g_t(\tilde{x}_t, \tilde{x}_{t+1}) = U(\{\alpha_i g_i\}_{i=0}^{\infty}, T_1, S, x_{T_1}). \tag{4.6}$$

By the choice of δ and L, (4.4), (2.5), (2.1), (2.8), (4.2), (4.3), (4.6), (4.7) and Theorem 2.2,

$$\rho(x_t, x_t^f) \leq \gamma_f/4, \ t = T_1 + L, \dots, T_2 - L, \tag{4.7}$$

$$\rho(\tilde{x}_t, x_t^f) \leq \gamma_f/4, \ t = T_1 + L, \dots, S - L. \tag{4.8}$$

By (4.7), (4.8) and (C4) there is $\xi \in X$ such that

$$(\tilde{x}_{S-L-1}, \xi) \in \Omega_{S-L-1}, \ (\xi, x_{S-L+1}) \in \Omega_{S-L}. \tag{4.9}$$

Define

$$y_t = \tilde{x}_t, \ t = T_1, \dots, S - L - 1, \ y_{S-L} = \xi, \ y_t = x_t, \ t = S - L + 1, \dots, T_2. \tag{4.10}$$

By (4.9) and (4.10), $\{y_t\}_{t=T_1}^{T_2}$ is a program. In view of (4.5), (4.3), (4.10), (4.6), (2.5) and (4.1),

$$\begin{aligned}
0 &\geq \sum_{t=T_1}^{T_2-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{T_2-1} \alpha_t g_t(y_t, y_{t+1}) \\
&= \sum_{t=T_1}^{S-L} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{S-L} \alpha_t g_t(y_t, y_{t+1}) \\
&\geq \sum_{t=T_1}^{S-L-2} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{S-L-2} \alpha_t g_t(\tilde{x}_t, \tilde{x}_{t+1}) - (2\alpha_{S-L-1} + 2\alpha_{S-L})(d_0 + 1) \\
&\geq \sum_{t=T_1}^{S-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{S-1} \alpha_t g_t(\tilde{x}_t, \tilde{x}_{t+1}) - 4(d_0 + 1)\Big(\sum_{t=S-L-1}^{S-1} \alpha_t\Big) \\
&\geq \sum_{t=T_1}^{S-1} \alpha_t g_t(x_t, x_{t+1}) - \sum_{t=T_1}^{S-1} \alpha_t g_t(\tilde{x}_t, \tilde{x}_{t+1}) \\
&\quad - 4(d_0 + 1)((L + 1)\gamma(8L + 8)^{-1}(d_0 + 1)^{-1}
\end{aligned}$$

and

$$\sum_{t=T_1}^{S-1} \alpha_t g_t(x_t, x_{t+1}) \leq U(\{\alpha_t g_t\}_{t=0}^{\infty}, T_1, S, x_{T_1}) + \gamma.$$

Lemma 4.1 is proved. □

Completion of the proof of Theorem 2.4

Let an integer $S \geq 0$ and $z \in X$ be an $(\{f_i\}_{i=0}^{\infty}, S, M)$-good point. For each integer $T > S$ there is a program $\{x_t^{(z,S,T)}\}_{t=S}^{T}$ such that

$$x_S^{(z,S,T)} = z,\ \sum_{t=S}^{T-1} \alpha_t g_t(x_t^{(z,S,T)}, x_{t+1}^{(z,S,T)}) = U(\{\alpha_t g_t\}_{t=0}^{\infty}, S, T, z). \tag{4.11}$$

Clearly there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^{\infty}$ such that $T_1 > S$ and that for any integer $t \geq S$ there exists

$$x_t^{(z,S)} = \lim_{j\to\infty} x_t^{(z,S,T_j)}. \tag{4.12}$$

Clearly, $\{x_t^{(z,S)}\}_{t=S}^{\infty}$ is a program and

$$x_S^{(z,S)} = z, \tag{4.13}$$

Let $\gamma > 0$. By Lemma 4.1 there is a natural number n_0 such that the following property holds:

(P7) For each pair integer $\tilde{S} \geq 0$, each integer $T \geq \tilde{S} + n_0$, each integer $Q \in [\tilde{S} + n_0, T-1]$ and each program $\{x_t\}_{t=\tilde{S}}^{T}$ such that $x_{\tilde{S}}$ is $(\{f_i\}_{i=0}^{\infty}, \tilde{S}, M)$-good and that

$$\sum_{t=\tilde{S}}^{T-1} \alpha_t g_t(x_t, x_{t+1}) = U(\{\alpha_t g_t\}_{t=0}^{\infty}, \tilde{S}, T, x_{\tilde{S}})$$

the following inequality holds:

$$\sum_{t=\tilde{S}}^{Q-1} \alpha_t g_t(x_t, x_{t+1}) \leq U(\{\alpha_t g_t\}_{t=0}^{\infty}, \tilde{S}, Q, x_{\tilde{S}}) + \gamma.$$

Let $T \geq S + n_0$ be an integer and j be a natural number such that $T_j > T$. By (P7) (with $\tilde{S} = S$, $T = T_j$, $Q = T$) and (4.11),

$$\sum_{t=S}^{T-1} \alpha_t g_t(x_t^{(z,S,T_j)}, x_{t+1}^{(z,S,T_j)}) \leq U(\{\alpha_t g_t\}_{t=0}^{\infty}, S, T, z) + \gamma.$$

Together with (4.12) this implies that

$$\sum_{t=S}^{T-1} \alpha_t g_t(x_t^{(z,S)}, x_{t+1}^{(z,S)}) \leq U(\{\alpha_t g_t\}_{t=0}^{\infty}, S, T, z) + \gamma.$$

Theorem 2.4 is proved. □

References

[1] B. D. O. Anderson and J. B. Moore, *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[2] S. M. Aseev, K. O. Besov and A. V. Kryazhimskii, *Infinite-horizon optimal control problems in economics*, Russ. Math. Surv. **67** (2012), 195–253.

[3] S. M. Aseev and A. V. Kryazhimskii, *The Pontryagin Maximum principle and transversality conditions for a class of optimal control problems with infinite time horizons*, SIAM J. Control Optim. **43** (2004), 1094–1119.

[4] S. Aubry and P. Y. Le Daeron, *The discrete Frenkel-Kontorova model and its extensions I*, Physica D **8** (1983), 381–422.

[5] J. Baumeister, A. Leitao and G. N. Silva, *On the value function for nonautonomous optimal control problem with infinite horizon*, Systems Control Lett. **56** (2007), 188–196.
[6] J. Blot, *Infinite-horizon Pontryagin principles without invertibility*, J. Nonlinear Convex Anal. **10** (2009), 177–189.
[7] J. Blot and P. Cartigny, *Optimality in infinite-horizon variational problems under sign conditions*, J. Optim. Theory Appl. **106** (2000), 411–419.
[8] J. Blot and N. Hayek, *Sufficient conditions for infinite-horizon calculus of variations problems*, ESAIM Control Optim. Calc. Var. **5** (2000), 279–292.
[9] P. Cartigny and P. Michel, *On a sufficient transversality condition for infinite horizon optimal control problems*, Automatica J. IFAC **39** (2003), 1007–1010.
[10] D. Gale, *On optimal development in a multi-sector economy*, Review of Economic Studies **34** (1967), 1–18.
[11] N. Hayek, *Infinite horizon multiobjective optimal control problems in the discrete time case*, Optimization **60** (2011), 509–529.
[12] H. Jasso-Fuentes and O. Hernandez-Lerma, *Characterizations of overtaking optimality for controlled diffusion processes*, Appl. Math. Optim. **57** (2008), 349–369.
[13] A. Leizarowitz, *Infinite horizon autonomous systems with unbounded cost*, Appl. Math. and Opt. **13** (1985), 19–43.
[14] A. Leizarowitz and V. J. Mizel, *One dimensional infinite horizon variational problems arising in continuum mechanics*, Arch. Rational Mech. Anal. **106** (1989), 161–194.
[15] V. Lykina, S. Pickenhain and M. Wagner, *Different interpretations of the improper integral objective in an infinite horizon control problem*, J. Math. Anal. Appl. **340** (2008), 498–510.
[16] A. B. Malinowska, N. Martins and D. F. M. Torres, *Transversality conditions for infinite horizon variational problems on time scales*, Optim. Lett. **5** (2011), 41–53.
[17] M. Marcus and A. J. Zaslavski, *The structure of extremals of a class of second order variational problems*, Ann. Inst. H. Poincare, Anal. non lineare **16** (1999), 593–629.
[18] N. Martins and D. F. M. Torres, *Necessary optimality conditions for higher-order infinite horizon variational problems on time scales*, J. Optim. Theory Appl. **155** (2012), 453–476.
[19] L. W. McKenzie, *Turnpike theory*, Econometrica **44** (1976), 841–866.
[20] B. S. Mordukhovich, *Optimal control and feedback design of state-constrained parabolic systems in uncertainly conditions*, Appl. Analysis **90** (2011), 1075–1109.
[21] B. S. Mordukhovich and I. Shvartsman, *Optimization and feedback control of constrained parabolic systems under uncertain perturbations*, in Optimal Control, Stabilization and Nonsmooth Analysis, Lecture Notes Control Inform. Sci., Springer, New York, 2004, pp. 121–132.
[22] E. Ocana and P. Cartigny, *Explicit solutions for singular infinite horizon calculus of variations*, SIAM J. Control Optim. **50** (2012), 2573–2587.
[23] E. Ocana, P. Cartigny and P. Loisel, *Singular infinite horizon calculus of variations. Applications to fisheries management*, J. Nonlinear Convex Anal. **10** (2009), 157–176.
[24] S. Pickenhain, V. Lykina and M. Wagner, *On the lower semicontinuity of functionals involving Lebesgue or improper Riemann integrals in infinite horizon optimal control problems*, Control Cybernet. **37** (2008), 451–468.
[25] T. Prieto-Rumeau and O. Hernandez-Lerma, *Bias and overtaking equilibria for zero-sum continuous-time Markov games*, Math. Methods Oper. Res. **61** (2005), 437–454.
[26] A. M. Rubinov, *Economic dynamics*, J. Soviet Math. **26** (1984), 1975–2012.
[27] P. A. Samuelson, *A catenary turnpike theorem involving consumption and the golden rule*, American Economic Review **55** (1965), 486–496.
[28] A. J. Zaslavski, *Ground states in Frenkel-Kontorova model*, Math. USSR Izvestiya **29** (1987), 323–354.
[29] A. J. Zaslavski, *Optimal programs on infinite horizon 1, 2*, SIAM Journal on Control and Optimization **33** (1995), 1643–1686.
[30] A. J. Zaslavski, *Turnpike theorem for nonautonomous infinite dimensional discrete-time control systems*, Optimization **48** (2000), 69–92.
[31] A. J. Zaslavski, *The turnpike property of discrete-time control problems arising in economic dynamics*, Discrete and Continuous Dynamical Systems, B **5** (2005), 861–880.

[32] A. J. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Springer, New York, 2006.

[33] A. J. Zaslavski, *Turnpike results for a discrete-time optimal control system arising in economic dynamics*, Nonlinear Analysis **67** (2007), 2024–2049.

[34] A. J. Zaslavski, *Two turnpike results for a discrete-time optimal control system*, Nonlinear Analysis **71** (2009), 902–909.

[35] A. J. Zaslavski, *Stability of a turnpike phenomenon for a discrete-time optimal control systems*, Journal of Optimization Theory and Applications **145** (2010), 597–612.

[36] A. J. Zaslavski, *Existence and structure of solutions for a class of optimal control systems with discounting arising in economic dynamics*, Nonlinear Analysis: Real World Applications **13** (2012), 1749–1760.

[37] A. J. Zaslavski, *Necessary and sufficient conditions for turnpike properties of solutions of optimal control systems arising in economic dynamics*, Dynamics of Continuous, Discrete and Impulsive Systems, Ser. B Appl. Algorithms **20** (2013), 391–420.

[38] A. J. Zaslavski, *Stability of a turnpike phenomenon for approximate solutions of nonautonomous discrete-time optimal control systems*, Nonlinear Analysis **100** (2014), 1–22.

[39] A. J. Zaslavski and A. Leizarowitz, *Optimal solutions of linear control systems with nonperiodic integrands*, Mathematics of Operations Research **22** (1997), 726–746.

Manuscript received January 19, 2014
revised January 24, 2014

A. J. Zaslavski
Department of Mathematics, Technion-Israel Institute of Technology, 32000, Haifa, Israel
E-mail address: ajzasl@tx.technion.ac.il