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WEAK SHARP SOLUTIONS FOR EQUILIBRIUM PROBLEMS IN METRIC SPACES

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Abstract. In this paper, we introduce the concept of weak sharp solutions for equilibrium problems and give its characterization by using equilibrium version of Ekelend's variational principle. As a particular case, we derive the characterization for weak sharp solutions for nonsmooth variational inequalities.

1. Introduction

In 1979, Polyak [23] (see also [24]) introduced the concept of a sharp minimum, also known as, a strong isolated minimum or a strong unique local minimum, for real-valued functions under the assumption that an optimization problem has a unique solution. It became an important tool in analysis of the perturbation behaviour of certain classes of optimization problems as well as in the convergence analysis of algorithms designed to solve these problems. As a generalization of sharp minimum, Ferris [14] (see also [9]) introduced and studied the weak sharp minima for real-valued functions to include the possibility of non-unique solutions. During the last two decades the study of weak sharp minima has drawn much attention motivated by its importance in the treatment of sensitivity analysis, error bounds and convergence analysis for a wide range of optimization algorithms, see, for example, [9, 10, 11, 12, 14] and the references therein. By using Takahashi's minimization theorem [25], Daffer et al. [13] and Hamel [17], separately, studied the weak sharp minima for a class of lower semicontinuous real-valued functions in the setting of metric spaces.

In 1998, Marcotte and Zhu [20] introduced the notion of weak sharp solutions of a variational inequality. They derived the necessary and sufficient condition for a solution set to be weakly sharp. They also studied the finite convergence of iterative algorithms for solving variational inequalities whose solution set is weakly sharp. Zhou and Wang [28] re-examined the unified treatment of finite termination of a class of iterative algorithms, and showed that some results given by Marcotte and Zhu [20] remain intact even if some conditions are relaxed. Wu and Wu [26] presented several equivalent (and sufficient) conditions for weak sharp solutions of

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variational inequalities in the setting of Hilbert spaces. They gave a finite convergence result for a class of algorithms for solving variational inequalities. By using the dual gap function, Zhang et al. [27] characterized the directional derivative and subdifferential of the dual gap function. Based on these, they proposed a better understanding of the concepts of a global error bound, weak sharpness, and minimum principle sufficiency property for variational inequalities, where the operator involved is pseudo-monotone. Hu and Song [18] extended the concept of weak sharp solutions for variational inequalities from finite dimensional spaces / Hilbert spaces to reflexive, strictly convex and smooth Banach spaces. They presented its equivalent characterizations and established finite convergence of proximal point algorithm for variational inequalities in terms of the weak sharpness of the solution set.

The main objective of this paper is to study the weak sharp solutions for equilibrium problems which include several problems, namely, variational inequalities, optimization problems, saddle point problems, Nash equilibrium problems, etc, as special cases. For further details on equilibrium problems and their applications, we refer [1, 2, 4, 5, 6, 7, 8, 15, 16, 19, 21, 22] and the references therein. We use an extended form of Takahashi's minimization theorem and a gap function [21] to study weak sharp solutions for equilibrium problems in the setting of metric spaces. As a special case, we derive a characterization for weak sharp solutions for nonsmooth variational inequalities.

2. Formulations and Preliminaries

Let (X, d) be a metric space, K be a nonempty closed subset of X and $F : K \times K \to \mathbb{R}$ be a bifunction. The equilibrium problem (in short, EP) is to find $\bar{x} \in K$ such that

$$F(\bar{x}, y) \geq 0, \quad \text{for all } y \in K. \tag{2.1}$$

The set of solutions of EP (2.1) is denoted by $\overline{S}$. It includes as special cases several fundamental mathematical problems, namely, variational inequality problems, optimization problems, Nash equilibrium problem, fixed point problem, minimax inequalities, complementarity problems, etc. During the last two decades, a large number of papers on different aspects of equilibrium problems has appeared in the literature, see, for example, [1, 2, 4, 5, 6, 7, 8, 15, 16, 19, 21, 22] and the references therein.

Let $h : K \times X \to \mathbb{R}$ be a bifunction. The nonsmooth variational inequality problem (in short, NVIP) is to find $\bar{x} \in K$ such that

$$h(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K. \tag{2.2}$$

The set of solutions of NVIP (2.2) is denoted by $\widehat{S}$. A comprehensive study of nonsmooth variational inequalities is given in [3].

Of course, when $F(x, y) = h(x; y - x)$ for all $x, y \in K$, then EP (2.1) coincides with NVIP (2.2).

Definition 2.1. A function $g : X \to \mathbb{R}$ is said to be a gap function for EP (2.1) if

(a) $g(x) \geq 0$ for all $x \in K$;
(b) $g(\bar{x}) = 0$ and $\bar{x} \in K$ if and only if $\bar{x} \in K$ is a solution of EP (2.1).

Mastroeni [21] studied the gap function for EP (2.1). He observed that the function

$$g(x) := \sup_{y\in K}[-F(x,y)] \tag{2.3}$$

is a gap function for EP (2.1).

A gap function [3] for NVIP (2.2) is defined by

$$\phi(x) := \sup_{y\in K}[-h(x;y-x)]. \tag{2.4}$$

Blum and Oettli [8] extended Takahashi's minimization theorem [25] for bifunction and derived the following result which provides the existence of a solution of EP (2.1).

Theorem 2.2. *Let K be a nonempty closed subset of a complete metric space (X,d), $F : K \times K \to \mathbb{R}$ be lower semicontinuous in the second argument and satisfy the following conditions:*

(i) $F(x,x) = 0$ *for all* $x \in K$*;*
(ii) $F(x,y) \le F(x,z) + F(z,y)$ *for all* $x,y,z \in K$*;*
(iii) *There exists* $\hat{x} \in K$ *such that* $\inf_{y\in K} F(\hat{x},y) > -\infty$.

Further, assume that the following extended Takahashi's condition holds:

$$\begin{cases} \text{Assume that for every } x \in K \text{ with } \inf_{y\in K} F(x,y) < 0, \\ \text{there exists } y \in K,\ y \ne x \text{ such that } F(x,y) + d(x,y) \le 0. \end{cases} \tag{2.5}$$

Then, there exists $\bar{x} \in K$ *such that* $F(\bar{x},y) \ge 0$ *for all* $y \in K$.

Theorem 2.2 is known as extended Takahashi's minimization theorem, see, for example, [2].

Let $\overline{S} := \{x \in K : F(x,y) \ge 0 \text{ for all } y \in K\}$, that is, $\overline{S}$ is the set of solutions of EP (2.1). Then, the extended Takahashi's condition (2.5) reads as

$$\text{for all } x \in K \setminus \overline{S}, \text{ there exists } y \in K,\ y \ne x \text{ such that } F(x,y) + d(x,y) \le 0.$$

The extended Ekeland's variational principle (see [1, Theorem 2.1]) states for the same class of functions as

$$\text{there exists } x \in K \text{ such that } F(x,y) + d(x,y) > 0, \quad \text{for all } y \in K,\ y \ne x.$$

We note that Theorem 2.2 is equivalent to the Ekeland's variational principle for bifunctions, known as extended Ekeland's variational principle, see, for example, [2, 22].

If we define

$$S(x) = \{y \in K : F(x,y) + d(x,y) \le 0\},$$

then the extended Takahashi's condition (2.5) can be reformulated as

$$\text{for all } x \in K \setminus \overline{S} \ :\ S(x) \ne \{x\}.$$

Remark 2.3. Since extended Ekeland's variational principle and extended Takahashi's minimization theorem are equivalent (see [2, 22]), we can say that the only points which satisfy the assertions of extended Ekeland's variational principle are the solutions of EP (2.1).

We mention the converse of above theorem.

Theorem 2.4. *Let K be a nonempty closed subset of a complete metric space (X, d), $F : K \times K \to \mathbb{R}$ be lower semicontinuous in the second argument and satisfy conditions (i)–(iii) in Theorem 2.2. If there exists a solution $\bar{x} \in K$ of EP* (2.1) *such that $F(y, \bar{x}) + d(y, \bar{x}) \leq 0$ for all $y \in K$, then F satisfies extended Takahashi's condition* (2.5)*.*

Proof. Assume that every $x \in K$ satisfies

$$\inf_{y \in K} F(x, y) < 0. \tag{2.6}$$

By hypothesis, there exists $\bar{x} \in K$ such that

$$F(\bar{x}, y) \geq 0, \text{ for all } y \in K, \tag{2.7}$$

$$\text{and } F(y, \bar{x}) + d(y, \bar{x}) \leq 0, \text{ for all } y \in K. \tag{2.8}$$

In view of (2.6), the inequality (2.8) hold only for all $y \in K$, $y \neq \bar{x}$. Hence, we get the conclusion. □

We need the following lemma to give an alternative proof of the main result of this paper, that is, Theorem 3.1.

Lemma 2.5. *Let K be a nonempty closed subset of a complete metric space (X, d), $F : K \times K \to \mathbb{R}$ be lower semicontinuous in the second argument and satisfy conditions (i)–(iii) of Theorem 2.2. Assume that (alternative form of extended Takahashi's condition) $S(x) = \{y \in K : F(x, y) + d(x, y) \leq 0\} \neq \{x\}$ for all $x \in K \setminus \overline{S}$. Then, $S(x) \cap \overline{S} \neq \emptyset$ whenever $x \notin \overline{S}$.*

Proof. For each $x \in K$, consider the restriction F_x of F on $S(x) \times S(x)$. Then, F_x is lower semicontinuous in the second argument and $\inf_{y \in S(x)} F(x, y) > -\infty$ for some $x \in S(x)$ because $S(x)$ is nonempty and closed for each $x \in K$. Thus, F_x satisfies all the conditions of extended Ekeland's variational principle (EEVP). By applying EEVP for F_x, there exists $\bar{x} \in S(x)$ such that

$$F_x(\bar{x}, y) + d(\bar{x}, y) > 0, \quad \text{for all } y \in S(x),\ y \neq \bar{x}. \tag{2.9}$$

We need to prove that

$$F(\bar{x}, y) + d(\bar{x}, y) > 0, \quad \text{for all } y \in K,\ y \neq \bar{x}. \tag{2.10}$$

Assume that (2.10) does not hold. Then, there exists $u \in K$ such that

$$F(\bar{x}, u) + d(\bar{x}, u) \leq 0. \tag{2.11}$$

Since $\bar{x} \in S(x)$, we have

$$F(x, \bar{x}) + d(x, \bar{x}) \leq 0. \tag{2.12}$$

Combining (2.11) and (2.12), we get

$$F(\bar{x}, u) + d(\bar{x}, u) + F(x, \bar{x}) + d(x, \bar{x}) \leq 0.$$

By utilizing the triangle inequality and condition (iii), we have $d(x, u) + F(x, u) \leq 0$, and hence, $u \in S(x)$. This is a contradiction because the inequality (2.9) for $y = u$ and the inequality (2.11) cannot hold simultaneously. Therefore, inequality (2.10) holds and with the help of Remark 2.3, $\bar{x}$ is a solution of EP (2.1). This is true for all $x \in K \setminus \overline{S}$, which completes the proof. □

3. Weak Sharp Solutions for Equilibrium Problems

We say that the equilibrium problem has weak sharp solutions if

$$d(x, \overline{S}) \leq g(x), \quad \text{for all } x \in K, \tag{3.1}$$

where $\overline{S}$ is the set of solutions of EP (2.1) and $d(x, \overline{S}) = \inf_{\bar{x} \in \overline{S}} d(x, \bar{x})$.

Theorem 3.1. *Let K be a nonempty closed subset of a complete metric space (X, d), $F : K \times K \to \mathbb{R}$ be lower semicontinuous in the second argument and satisfy the following conditions:*

(i) *$F(x, x) = 0$ for all $x \in K$;*
(ii) *$F(x, y) \leq F(x, z) + F(z, y)$ for all $x, y, z \in K$;*
(iii) *There exists $\hat{x} \in K$ such that $\inf_{y \in K} F(\hat{x}, y) > -\infty$.*

Assume that for every $x \in K$ with $\inf_{y \in K} F(x, y) < 0$, there exists $y \in K$, $y \neq x$ such that $F(x, y) + d(x, y) \leq 0$. Then, the EP (2.1) has weak sharp solutions.

Proof. For all $x \in K$, define

$$S(x) = \{y \in K : F(x, y) + d(x, y) \leq 0\}.$$

Then, by lower semicontinuity of F in the second argument, $S(x)$ is closed for all $x \in K$. By Theorem 2.2, $\overline{S}$ is nonempty. Clearly, $S(x) \neq \emptyset$ as $x \in S(x)$.

For all $y \in S(x)$, $F(x, y) \leq 0$. Indeed, for all $y \in S(x)$, we have

$$F(x, y) + d(x, y) \leq 0 \;\Leftrightarrow\; 0 \leq d(x, y) \leq -F(x, y) \;\Leftrightarrow\; F(x, y) \leq 0.$$

Suppose to the contrary that there exists $x_0 \in K$ such that

$$d(x_0, \overline{S}) > g(x_0). \tag{3.2}$$

Then, $x_0 \notin \overline{S}$. Indeed, if $x_0 \in \overline{S}$, then $d(x_0, \overline{S}) = \inf_{y \in \overline{S}} d(x_0, y) = 0$, and so, $g(x_0) < 0$ which contradicts the fact that $g(x) \geq 0$ for all $x \in K$ because g is a gap function.

For all $y \in S(x_0)$, $d(y, \overline{S}) > g(y)$. Indeed, take $y \in S(x_0)$ and $z \in \overline{S}$, then $d(x_0, y) \leq -F(x_0, y)$. Therefore,

$$d(x_0, z) \leq d(x_0, y) + d(y, z) \leq -F(x_0, y) + d(y, z),$$

that is, $d(x_0, z) \leq d(y, z) - F(x_0, y)$. Taking inf over $\overline{S}$ both the sides, we obtain

$$\inf_{z \in \overline{S}} d(x_0, z) \leq \inf_{z \in \overline{S}} d(y, z) - F(x_0, y),$$

that is, $d(x_0, \overline{S}) \leq d(y, \overline{S}) - F(x_0, y)$. By (3.2), we have

$$g(x_0) < d(y, \overline{S}) - F(x_0, y). \tag{3.3}$$

By condition (ii), for all $v \in K$, we have

$$F(x_0, v) \leq F(x_0, y) + F(y, v) \quad \Leftrightarrow \quad -F(y, v) \leq -F(x_0, v) + F(x_0, y).$$

Taking sup over K both the sides, we get

$$\sup_{v\in K}[-F(y,v)] \leq \sup_{v\in K}[-F(x_0,v)] + F(x_0,y).$$

This implies that

$$g(y) \leq g(x_0) + F(x_0,y). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain $g(y) < d(y,\overline{S})$ for all $y \in S(x_0)$.

Since $x_0 \notin \overline{S}$, there exists $y \in K$ such that $F(x_0,y) < 0$, and so, $\inf_{y\in K} F(x_0,y) < 0$. By hypothesis, there exists $x_1 \in K$ such that $x_1 \neq x_0$ and $F(x_0,x_1)+d(x_0,x_1) \leq 0$, that is, $x_1 \in S(x_0)$ with $x_1 \neq x_0$. Since $g(x_1) < d(x_1,\overline{S})$, then clearly $x_1 \notin \overline{S}$ and $F(x_0,x_1) < 0$ because $-F(x_0,x_1) \geq d(x_0,x_1) > 0$ since $x_0 \neq x_1$. We can again show as above that $g(y) < d(y,\overline{S})$ for all $y \in S(x_1)$ and $S(x_1) \cap \overline{S} = \emptyset$. In addition, we choose x_1 such that

$$F(x_0,x_1) = \inf\{F(x_0,x) : x \in S(x_0)\},$$

where inf exists since K is a closed subset of a complete metric space X, $S(x_0)$ is closed and F is lower semicontinuous in the second argument. Continuing in this way, we generate a sequence $\{x_n\}$ with the following properties:

- there exists $x_i \neq x_{i-1}$ for all $i = 1,2,\ldots,n$.
- $x_i \in S(x_{i-1})$ for all $i = 1,2,\ldots,n$.
- $F(x_{i-1},x_i) < 0$ for all $i = 1,2,\ldots,n$. Indeed, since $x_i \in S(x_{i-1})$, we have $d(x_{i-1},x_i)+F(x_{i-1},x_i) \leq 0$. This implies that $-F(x_{i-1},x_i) \geq d(x_{i-1},x_i) > 0$ as $x_{i-1} \neq x_i$. Thus, $F(x_{i-1},x_i) < 0$.
- $F(x_{i-1},x_i) = \inf\{F(x_{i-1},x) : x \in S(x_{i-1})\}$ for all $i = 1,2,\ldots,n$.
- $S(x_i) \cap \overline{S} = \emptyset$ for all $i = 1,2,\ldots,n$.
- $g(y) < d(y,\overline{S})$ for all $y \in \bigcup_{i=1}^n S(x_i)$.

Since $x_n \notin \overline{S}$, we can choose $x_{n+1} \in S(x_n)$, $x_{n+1} \neq x_n$ with $F(x_n,x_{n+1}) = \inf\{F(x_n,x) : x \in S(x_n)\}$. As above, we also have

$$x_{n+1} \notin \overline{S}, \quad F(x_n,x_{n+1}) < 0, \text{ and} \tag{3.5}$$

$$g(y) < d(y,\overline{S}), \quad \text{for all } y \in S(x_{n+1}). \tag{3.6}$$

To see this, let $y \in S(x_{n+1})$ and $\bar{x} \in \overline{S}$. Then,

$$d(x_{n+1},\bar{x}) \leq d(x_{n+1},y) + d(y,\bar{x}) \leq -F(x_{n+1},y) + d(y,\bar{x}).$$

Taking sup over $\overline{S}$, we obtain

$$d(x_{n+1},\overline{S}) \leq -F(x_{n+1},y) + d(y,\overline{S}).$$

Since $-F(x_{n+1},y) < g(x_{n+1})$, we have $d(y,\overline{S}) - F(x_{n+1},y) > -F(x_{n+1},x)$, and therefore,

$$\begin{aligned} d(y,\overline{S}) &> F(x_{n+1},y) - F(x_{n+1},x) \\ &\geq F(x_{n+1},y) - F(x_{n+1},y) - F(y,x). \end{aligned}$$

Thus, $d(y,\overline{S}) > g(y)$. Since as above

$$\begin{aligned} d(x_{n+1},\overline{S}) &> F(x_n,x_{n+1}) - F(x_n,x) \\ &\geq F(x_n,x_{n+1}) - F(x_n,x_{n+1}) - F(x_n,x), \end{aligned}$$

we have $d(x_{n+1}, \overline{S}) > \sup_{x\in K}[-F(x_n, x)] = g(x_{n+1})$, and hence, $S(x_{n+1}) \cap \overline{S} = \emptyset$.

So, the sequence $\{x_n\}$ consisting different elements and $F(x_n, x_{n+1}) < 0$. Since

$$d(x_{n+k}, x_n) \le \sum_{i=1}^{k} d(x_{n+i}, x_{n+i-1}) \le \sum_{i=1}^{k} -F(x_{n+i}, x_{n+i-1}) \le F(x_n, x_{n+k}),$$

and $F(x_n, x_{n+k})$ monotonically decreasing to some point, $\{x_n\}$ is a Cauchy sequence in a closed subset K of a complete metric space X, so we can assume that x_n converges to some point $x \in K$. We show that $x \in \bigcap_{i=0}^{\infty} S(x_i)$. To prove it, we show that for every n, $x_n \in \bigcap_{i=0}^{n-1} S(x_i)$.

Since

$$\begin{aligned} d(x_{n-k}, x_n) &\le \sum_{j=0}^{k-1} d(x_{n-k+j}, x_{n-k+j+1}) \\ &\le \sum_{j=0}^{k-1} -F(x_{n-k+j+1}, x_{n-k+j}) \\ &= -F(x_{n-k}, x_n), \end{aligned}$$

we have $x_n \in S(x_{n-k})$ for all $k = 1, 2, \dots n$ (recall that $x_i \notin \overline{S}$). Therefore, $x_n \in \bigcap_{i=0}^{n-1} S(x_i)$, and hence, $x_k \in \bigcap_{i=0}^{n-1} S(x_i)$ for all $k \ge n$. Since $\bigcap_{i=0}^{n-1} S(x_i)$ is a closed set, $x \in \bigcap_{i=0}^{\infty} S(x_i)$. Thus, $x \in S(x_n)$, and $x \ne x_n$, and therefore, $F(x_n, x) < -d(x_n, x) < 0$ which contradicts the fact that $F(x_n, y) \ge 0$ for all $y \in S(x_n)$. □

Inspired by Hamel [17], we give the alternative proof of Theorem 3.1.

Alternative Proof of Theorem 3.1. By Lemma 2.5, for each $x \in K \setminus \overline{S}$, we find $z \in S(x) \cap \overline{S}$ (depending on x). Then, $F(x, z) + d(x, z) \le 0$. Since $d(x, \overline{S}) \le d(x, z)$, we have

$$F(x, z) + d(x, \overline{S}) \le F(x, z) + d(x, z) \le 0, \quad \text{for this } z \in S(x) \cap \overline{S}.$$

Since for each $x \in K \setminus \overline{S}$, we find $z \in S(x) \cap \overline{S}$, we have

$$F(x, z) + d(x, \overline{S}) \le 0, \quad \text{for all } x \in K \text{ and } z \in \overline{S}.$$

Therefore, $d(x, \overline{S}) \le \sup_{z\in\overline{S}} [-F(x, z)] = g(x)$ for all $x \in K$. Hence, EP (2.1) has weak sharp solutions. □

Remark 3.2. We would like to mention that the first proof of Theorem 3.1 is constructive and does not dependent on Lemma 2.5. While the alternative proof of Theorem 3.1 is analytical and based on Lemma 2.5.

We say that the nonsmooth variational inequality problem (NVIP) (2.2) has weak sharp solutions if

$$d(x, \widehat{S}) \le \phi(x), \quad \text{for all } x \in K,$$

where $\widehat{S}$ is the set of solutions of NVIP (2.2).

By considering $F(x, y) = h(x; y - x)$, from Theorem 3.1, we can derive the existence result for weak sharp solutions of NVIP (2.2).

References

[1] A. Amini-Harandi, Q. H. Ansari and A. P. Farajzadeh, *Existence of equilibria in complete metric spaces*, Taiwanese J. Math. **16** (2012), 777–785.

[2] Q. H. Ansari, *Ekeland's variational principle and its extensions with applications, in Fixed Point Theory and Applications*, S. Almezel, Q. H. Ansari and M. A. Khamsi (eds), Springer International Publishing Switzerland, 2014, pp. 65–99.

[3] Q. H. Ansari, C. S. Lalitha and M. Mehta, *Generalized Convexity, Nonsmooth Variational Inequalities and Nonsmooth Optimization*, Taylor & Francis, CRC Press, England, 2013.

[4] Q. H. Ansari, N. C. Wong and J.-C. Yao, *The existence of nonlinear inequalities*, Appl. Math. Lett. **12** (19990, 89–92.

[5] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl. **90** (1969), 31–43.

[6] M. Bianchi and S. Schaible, *Equilibrium problems under generalized convexity and generalized monotonicity*, J. Global Optim. **30** (2004), 121–134.

[7] M. Bianchi, G. Kassay and R. Pini, *Existence of equilibria via Ekeland's principle*, J. Math. Anal. Appl. **305** (2005), 502–512.

[8] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[9] J. V. Burke and M. C. Ferris, *Weak sharp minima in mathematical programming*, SIAM J. Control Optim. **31** (1993), 1340–1359.

[10] J. V. Burke and S. Deng, *Weak sharp minima revisited, part I: Basic theory*, Control Cybern. **31** (2002), 439–469.

[11] J. V. Burke and S. Deng, *Weak sharp minima revisited, part II: Application to linear regularity and error bounds*, Math. Program., Ser. B, **104** (2005), 235–261.

[12] J. V. Burke and S. Deng, *Weak sharp minima revisited, part III: Error bounds for differentiable convex inclusions*, Math. Program., Ser. B, **116** (2009), 37–56.

[13] P. Z. Daffer, H. Kaneko and W. Li, *Variational principle and fixed points*, SIMAA **4** (2002), 129–136.

[14] M.C. Ferris, Weak Sharp Minima and Penalty Functions in Mathematical Programming, Ph.D. Thesis, University of Cambridge, 1988.

[15] F. Flores-Bazán, *Existence theorems for generalized noncoercive equilibrium problems: The quasi-convex case*, SIAM J. Optim. **11** (2000), 675–690.

[16] F. Flores-Bazán, *Existence theory for finite-dimensional pseudomonotone equilibrium problems*, Acta Appl. Math. **77** (2003), 249–297.

[17] A. Hamel, *Remarks to an equivalent formulation of Ekeland's variational principle*, Optimization **31** (1994), 233–238.

[18] Y. H. Hu and W. Song, *Weak sharp solutions for variational inequalities in Banach spaces*, J. Math. Anal. Appl. **374** (2011), 118–132.

[19] A. N. Isuem, G. Kassay and W. Sosa, *On certain condiitons for the existence of solutions of equilibrium problems*, Math. Program. Ser. B. **116** (2009), 259–273.

[20] P. Marcotte and D.L. Zhu, *Weak sharp solutions of variational inequalities*, SIAM J. Optim. **9** (1998), 179–189.

[21] G. Mastroeni, *Gap functions for equilibrium problems*, J. Global Optim. **27** (2003), 411–426.

[22] W. Oettli and M. Théra, *Equivalents of Ekeland's principle*, Bull. Austral. Math. Soc. **48** (1983), 385–392.

[23] B. T. Polyak, Sharp minima, Institute of control sciences lecture notes, Moscow, USSR, 1979. Presented at the IIASA workshop on generalized Lagrangians and their applications, IIASA, Laxenburg, Austria, 1979.

[24] B. T. Polyak, *Introduction to Optimization*, Optimization Software, Inc., Publications Division, New York, 1987.

[25] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings*, In Fixed Point Theory and Applications, J.-B. Baillon and M. Théra (eds), Pitman Research Notes in Mathematics, 252, Longman, Harlow, 1991, pp. 397–406.

[26] Z. L. Wu and S. Y. Wu, *Weak sharp solutions of variational inequalities in Hilbert spaces*, SIAM J. Optim. **14** (2004), 1011–1027.
[27] J. Zhang, C. Wan and N. Xiu, *The dual gap function for variational inequalities*, Appl. Math. Optim. **48** (2003), 129–148.
[28] J. Zhou and C. Wang, *A note on finite termination of iterative algorithms in mathematical programming*, Oper. Res. Lett. **36** (2008), 715–717.

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