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ERGODIC CONVERGENCE OF THE DOUBLE BACKWARD METHOD FOR MONOTONE OPERATORS

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Abstract. The double backward method for finding zeros of the sum of two maximal monotone operators is investigated. This method was initially introduced by Passty in 1979 who used an equal index to the resolvents of both operators. In this paper, we use distinct indices in order to see different roles played by the two operators in the double backward method. Under certain conditions on the indices, we prove the ergodic convergence of our method.

1. Introduction

Many problems can be formulated as finding a zero of a maximal monotone operator C in a Hilbert space H, i.e., a solution of the inclusion $0 \in Cx$. (A typical example is to find a minimizer of a convex functional.) A split problem corresponds to the situation where C is decomposed as the sum of two maximal monotone operators A and B, that is, $C = A + B$. A splitting method for solving the split inclusion

$$0 \in (A + B)x \tag{1.1}$$

means a method where information on A and B (e.g., the resolvent J_λ^A and/or J_λ^B) separately, not on the sum $A + B$, is involved.

Peaceman and Rachford [9] and Douglas and Rachford [3] initiated splitting methods for the case of linear operators. Extensions to the case of nonlinear operators were studied by Lions and Mercier [7] and Passty [8].

Assume (1.1) is consistent (i.e., solvable) and use S to denote the set of solutions of (1.1). There are mainly three iterative methods for solving the split inclusion problem (1.1). The first method, referred to as the Peaceman-Rachford method, generates a sequence $\{v^n\}$ via the recursion

$$v^{n+1} = (2J_A^\lambda - I)(2J_B^\lambda - I)v^n \tag{1.2}$$

where J_A^λ and J_B^λ are the resolvents at level $\lambda > 0$ of A and B, respectively.

When A and B are linear, the algorithm (1.2) was introduced by Peaceman-Rachford in [9]. The above form (1.2) was studied by Lions and Mercier [7]. This algorithm is however not convergent, in general, for the reason that the operator

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$(2J_A^\lambda - I)(2J_B^\lambda - I)$ is only nonexpansive (possibly a reflection in the plane) which is insufficient to guarantee convergence.

The second method, the Douglas-Rachford method, generates a sequence $\{v^n\}$ via the recursion

$$v^{n+1} = J_A^\lambda(2J_B^\lambda - I)v^n + (I - J_B^\lambda)v^n. \tag{1.3}$$

The advantage of the Douglas-Rachford method (1.3) over the the Peaceman-Rachford method (1.2) is that the former is always convergent. It has been paid much attention [2, 4, 5].

The third method, known as the double backward method and introduced by Passty [8], generates a sequence $\{x^n\}$ by the iteration process:

$$x^{n+1} = (J_B^{\lambda_{n+1}} \circ J_A^{\lambda_{n+1}})x^n, \quad n \geq 0, \tag{1.4}$$

where $\{\lambda_n\}$ is a sequence of positive numbers, and the initial guess $x^0 \in H$ is arbitrarily chosen.

Similar to the Peaceman-Rachford method (1.2), the double backward method (1.4) fails to be convergent, in general. So ergodic convergence is considered instead. This means that we use the sequence of averages, $\{z^n\}$, defined by:

$$z^n = \frac{\sum_{k=1}^n \lambda_k x^k}{\sum_{k=1}^n \lambda_k}, \quad n \geq 1. \tag{1.5}$$

Passty [8] proved the weak convergence of $\{z^n\}$ to a solution of the inclusion (1.1) under the condition that $\{\lambda_n\} \in \ell^2 \setminus \ell^1$.

In the double backward method (1.4), we notice that at each iteration an equal index applies to both operators A and B. However, the orders of A and B are different and this suggests that A and B may play different role in this algorithm, and therefore, one may consider that different indices should apply to A and B. This is indeed the problem that we will address in the current paper. The double backward method that we consider in this paper generates a sequence $\{x^n\}$ through the following iteration process:

$$x^{n+1} = (J_B^{\mu_{n+1}} \circ J_A^{\lambda_{n+1}})x^n, \quad n \geq 0, \tag{1.6}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences of positive numbers. We will study the convergence of the sequence of the averages $\{z^n\}$ where $\{x_n\}$ is defined by (1.6). We find that in order to ensure convergence of $\{z^n\}$, the role played by $\{\mu_n\}$ is that it is sufficiently close to $\{\lambda_n\}$ in the sense that the sequence of differences, $\{\lambda_n - \mu_n\}$, is summable, that is, $\{\lambda_n - \mu_n\} \in \ell^1$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let A be a (possibly multivalued) operator with domain $\mathrm{dom}(A)$ and range $\mathrm{ran}(A)$ in H. That is, $\mathrm{dom}(A) = \{x \in H : Ax \neq \emptyset\}$ and $\mathrm{ran}(A) = \{y \in H : y \in Ax,\ x \in \mathrm{dom}(A)\}$. The graph of A, $\mathrm{gph}(A)$, is the set

$$\mathrm{gph}(A) = \{(x,y) : x \in \mathrm{dom}(A),\ y \in Ax\}.$$

Definition 2.1. The operator A is said to be *monotone* if, for all $x_1, x_2 \in \mathrm{dom}(A)$ and $y_1 \in Ax_1$ and $y_2 \in Ax_2$,

$$\langle x_1 - x_2, y_1 - y_2\rangle \geq 0.$$

A monotone operator A is said to be *maximal* monotone if a point (x', y') satisfies the property

$$\langle x - x', y - y'\rangle \geq 0 \quad \forall x \in \mathrm{dom}(A),\ y \in Ax,$$

then $(x', y') \in \mathrm{gph}(A)$; i.e., $x' \in \mathrm{dom}(A)$ and $y' \in Ax'$. It is known that a monotone operator A is maximal monotone if and only if for any $\lambda > 0$, $\mathrm{ran}(I + \lambda A) = H$.

A typical example of a maximal monotone operator is the subdifferential $\partial\varphi$ of a proper lower semicontinuous convex function $\varphi : H \to \overline{\mathbb{R}}$.

It can easily be shown that if A is monotone, then for any positive number λ, the resolvent

$$J_A^\lambda := (I + \lambda A)^{-1}$$

is single-valued and nonexpansive:

$$\|J_A^\lambda x - J_A^\lambda y\| \leq \|x - y\|$$

for all $x, y \in \mathrm{dom}(J_A^\lambda) = \mathrm{ran}(I + \lambda A)$. Actually, J_A^λ is *firmly* nonexpansive:

$$\|J_A^\lambda x - J_A^\lambda y\|^2 \leq \langle x - y, J_A^\lambda x - J_A^\lambda y\rangle$$

for all $x, y \in \mathrm{dom}(J_A^\lambda) = \mathrm{ran}(I + \lambda A)$.

Moreover, it is well-known that A is maximal monotone if and only if for any $\lambda > 0$, the resolvent J_A^λ is single-valued and nonexpansive, from the entire space H into H. More properties of monotone operators can be found in [1].

3. The Double Backward Method

The double backward method of Passty [8] generates a sequence $\{x^n\}$ via the recursive manner:

$$x^{n+1} = (J_B^{\lambda_{n+1}} \circ J_A^{\lambda_{n+1}})x^n, \quad n \geq 0, \tag{3.1}$$

where $\{\lambda_n\}$ is a sequence of positive numbers, and the initial guess $x^0 \in H$ is arbitrarily chosen.

The double backward algorithm (3.1) may fail to converge even in the weak topology [6]. Therefore one turns to consider the averages:

$$z^n := \frac{\sum_{i=1}^n \lambda_i x^i}{\sum_{i=1}^n \lambda_i}, \quad n = 1, 2, \cdots. \tag{3.2}$$

Passty [8] proved the following result.

Theorem 3.1. *Assume the solution set S of the inclusion (1.1) is nonempty. Assume $\{\lambda_n\} \in \ell^2 \setminus \ell^1$. Then the sequence $\{z^n\}$ defined by the averages (3.2) converges weakly to a point in S.*

In the double backward method (3.1) it is required that the resolvents of A and B have the same index at each iteration. We next extend (3.1) to the case where we allow the resolvents of A and B to have distinct indexes at each iteration. In other words, we introduce the following double backward method with distinct indexes:

$$x^{n+1} = (J_B^{\mu_{n+1}} \circ J_A^{\lambda_{n+1}})x^n, \qquad n \geq 0, \tag{3.3}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences of positive numbers, and the initial guess $x^0 \in H$ is arbitrarily chosen. We still use the same notation z^n to denote the average of the sequence $\{x^n\}$ given by (3.3).

To prove our main result, Theorem 3.4, we need the following two lemmas.

Lemma 3.2. *Suppose $\{x^n\}$ is a bounded sequence and define a sequence $\{z^n\}$ by (3.2). Suppose in addition there exists a nonempty subset F such that*

(a) $\lim_{n\to\infty} \|x^n - z\|$ *exists for every $z \in F$, and*

(b) $\omega_w(z_n) \subset F$, *where $\omega_w(z_n)$ denotes the set of weak accumulation points of $\{z^n\}$.*

Then $\{z^n\}$ converges weakly to a point of F.

Proof. The boundedness of $\{x^n\}$ implies that of $\{z^n\}$. For $z', z'' \in F$, we have the identity

$$\langle x^n - z', z' - z''\rangle = \frac{1}{2}(\|x^n - z''\|^2 - \|x^n - z'\|^2 - \|z' - z''\|^2)$$

Now since $\lim_{n\to\infty} \|x^n - z\|$ exists for all $z \in F$, it turns out that

$$\lim_{n\to\infty}\langle z^n - z', z' - z''\rangle \text{ exists for all } z', z'' \in F. \tag{3.4}$$

Now if $z^{n_i} \rightharpoonup w'$ and $z^{m_j} \rightharpoonup w''$, then $w', w'' \in F$ by assumption (b). So from (3.4) it follows that

$$\langle w' - w'', z' - z''\rangle = 0, \quad \forall w', w'' \in \omega_w(z^n), \quad \forall z', z'' \in F. \tag{3.5}$$

However, $\omega_w(z^n) \subset F$ by assumption (b); thus we can replace z', z'' in (3.5) by w', w'', respectively to get $\|w' - w''\|^2 = 0$ and $w' = w''$. □

The proof to the next lemma can be found in [10].

Lemma 3.3. *Assume $\{\gamma_n\}$ is a sequence of nonnegative real numbers such that*

$$\gamma_{n+1} \leq (1 + \alpha_n)\gamma_n + \sigma_n, \quad n \geq 0, \tag{3.6}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are sequences of positive numbers such that

$$\sum_{n=1}^{\infty} \alpha_n < \infty, \quad \sum_{n=1}^{\infty} \sigma_n < \infty.$$

Then $\{\gamma_n\}$ *is bounded and* $\lim_{n\to\infty}\gamma_n$ *exists.*

Now we are ready to prove the main result of this paper.

Theorem 3.4. *Assume that the solution set* S *of the inclusion (1.1) is nonempty. Assume that the sequences of positive parameters,* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *satisfy the conditions:*

(i) $\{\lambda_n\}, \{\mu_n\} \in \ell^2 \setminus \ell^1$, *that is,* $\sum_{n=1}^{\infty} |\lambda_n| = \sum_{n=1}^{\infty} |\mu_n| = \infty$, $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$, $\sum_{n=1}^{\infty} \mu_n^2 < \infty$.

(ii) $\{\lambda_n - \mu_n\} \in \ell^1$, *that is,* $\sum_{n=1}^{\infty} |\lambda_n - \mu_n| < \infty$.

Then the sequence $\{z^n\}$ *defined by the averages (1.5) converges weakly to a point in* S.

Proof. Take $F = S = (A+B)^{-1}(0)$. It suffices to verify the conditions (a) and (b) in Lemma 3.2.

Putting, for each k,

$$y^k = J_A^{\lambda_{k+1}} x^k,$$

we can, by definition, rewrite x^{k+1} as

$$x^{k+1} = J_B^{\mu_{k+1}} y^k.$$

Consequently,

$$\frac{1}{\lambda_{k+1}}(x^k - y^k) \in Ay^k, \quad \frac{1}{\mu_{k+1}}(y^k - x^{k+1}) \in Bx^{k+1}.$$

Now let $u \in \operatorname{dom}(A+B)$ and take $v^1 \in Au$ and $v^2 \in Bu$. It follows from the monotonicity of A and B that

$$\left\langle \frac{1}{\lambda_{k+1}}(x^k - y^k) - v^1, y^k - u \right\rangle \geq 0, \tag{3.7}$$

$$\left\langle \frac{1}{\mu_{k+1}}(y^k - x^{k+1}) - v^2, x^{k+1} - u \right\rangle \geq 0. \tag{3.8}$$

Multiplying both sides of (3.7) and (3.8) by λ_{k+1} and μ_{k+1}, respectively, yields

$$\langle x^k - y^k - \lambda_{k+1} v^1, y^k - u\rangle \geq 0,$$
$$\langle y^k - x^{k+1} - \mu_{k+1} v^2, x^{k+1} - u\rangle \geq 0.$$

In other words we get

$$\langle x^k - y^k, y^k - u\rangle \geq \lambda_{k+1}\langle v^1, y^k - u\rangle, \tag{3.9}$$

$$\langle y^k - x^{k+1}, x^{k+1} - u\rangle \geq \mu_{k+1}\langle v^2, x^{k+1} - u\rangle. \tag{3.10}$$

It turns out that

$$\begin{aligned} \|x^k - u\|^2 - \|y^k - u\|^2 &= \|x^k - y^k\|^2 + 2\langle x^k - y^k, y^k - u\rangle \\ &\geq \|x^k - y^k\|^2 + 2\lambda_{k+1}\langle v^1, y^k - u\rangle \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\|y^k - u\|^2 - \|x^{k+1} - u\|^2 &= \|y^k - x^{k+1}\|^2 + 2\langle y^k - x^{k+1}, x^{k+1} - u\rangle \\ &\geq \|y^k - x^{k+1}\|^2 + 2\mu_{k+1}\langle v^2, x^{k+1} - u\rangle. \end{aligned} \tag{3.12}$$

Adding up (3.11) and (3.12) yields, for each $k \geq 0$,

$$\begin{aligned}\|x^k - u\|^2 - \|x^{k+1} - u\|^2 \geq{}& \|x^k - y^k\|^2 + \|y^k - x^{k+1}\|^2 \\ &+ 2\lambda_{k+1}\langle v^1, y^k - u\rangle + 2\mu_{k+1}\langle v^2, x^{k+1} - u\rangle \\ ={}& \|x^k - y^k\|^2 + \|y^k - x^{k+1}\|^2 + 2\lambda_{k+1}\langle v^1, y^k - x^{k+1}\rangle \\ &+ 2\langle \lambda_{k+1}v^1 + \mu_{k+1}v^2, x^{k+1} - u\rangle. \end{aligned} \tag{3.13}$$

Since

$$2\lambda_{k+1}\langle v^1, y^k - x^{k+1}\rangle \geq -\lambda_{k+1}^2\|v^1\|^2 - \|y^k - x^{k+1}\|^2.$$

Substituting it into (3.13), we get

$$\begin{aligned}\|x^k - u\|^2 - \|x^{k+1} - u\|^2 &\geq \|x^k - y^k\|^2 - \lambda_{k+1}^2\|v^1\|^2 + 2\langle \lambda_{k+1}v^1 + \mu_{k+1}v^2, x^{k+1} - u\rangle \\ &\geq 2\langle \lambda_{k+1}v^1 + \mu_{k+1}v^2, x^{k+1} - u\rangle - \lambda_{k+1}^2\|v^1\|^2 \\ &= 2\lambda_{k+1}\langle v^1 + v^2, x^{k+1} - u\rangle \\ &\quad + 2(\mu_{k+1} - \lambda_{k+1})\langle v^2, x^{k+1} - u\rangle - \lambda_{k+1}^2\|v^1\|^2. \end{aligned} \tag{3.14}$$

Putting

$$v = v^1 + v^2 \in (A+B)u, \quad s_n = \sum_{i=1}^n \lambda_i$$

and summing up (3.14) from $k = 0$ to $n - 1$, we obtain

$$\begin{aligned}\frac{1}{s_n}(\|x^0 - u\|^2 - \|x^n - u\|^2) \geq{}& 2\langle v, z^n - u\rangle - \frac{1}{s_n}\left(\sum_{i=1}^n \lambda_i^2\right)\|v^1\|^2 \\ &+ \frac{1}{s_n}\sum_{k=1}^n (\mu_k - \lambda_k)\langle v^2, x^{k+1} - u\rangle. \end{aligned} \tag{3.15}$$

If we take $u \in S = (A+B)^{-1}0$, then we may take $v^1 \in Au$ and $v^2 \in Bu$ such that $v = v^1 + v^2 = 0$. Consequently, (3.14) is reduced to

$$\|x^{k+1} - u\|^2 \leq \|x^k - u\|^2 + 2(\lambda_{k+1} - \mu_{k+1})\langle v^2, x^{k+1} - u\rangle + \lambda_{k+1}^2\|v^1\|^2 \tag{3.16}$$

for all $u \in S$. Since

$$2|(\lambda_{k+1} - \mu_{k+1})\langle v^2, x^{k+1} - u\rangle| \leq |\lambda_{k+1} - \mu_{k+1}|(\|v^2\|^2 + \|x^{k+1} - u\|^2),$$

we get from (3.16) that

$$(1 - |\lambda_{k+1} - \mu_{k+1}|)\|x^{k+1} - u\|^2 \leq \|x^k - u\|^2 + |\lambda_{k+1} - \mu_{k+1}|\|v^2\|^2 + \lambda_{k+1}^2\|v^1\|^2.$$

This results is of the form:

$$\|x^{k+1} - u\|^2 \leq (1 + \alpha_k)\|x^k - u\|^2 + \sigma_k, \tag{3.17}$$

where

$$\alpha_k = \frac{|\lambda_{k+1} - \mu_{k+1}|}{1 - |\lambda_{k+1} - \mu_{k+1}|}, \quad \sigma_k = \frac{|\lambda_{k+1} - \mu_{k+1}| \|v^2\|^2 + \lambda_{k+1}^2 \|v^1\|^2}{1 - |\lambda_{k+1} - \mu_{k+1}|}.$$

By the assumptions (i) and (ii), we can easily see that

$$\sum_{k+1}^{\infty} \alpha_k < \infty, \quad \sum_{k+1}^{\infty} \sigma_k < \infty.$$

Therefore, Lemma 3.3 is applicable to (3.17) and we conclude that $\{x^n\}$ is bounded and

$$\lim_{n\to\infty} \|x^n - u\| \qquad \text{exists for each } u \in S. \tag{3.18}$$

Now since $\{x^n\}$ is bounded, we can return to (3.15) to get the estimate:

$$\frac{1}{s_n}(\|x^0 - u\|^2 - \|x^n - u\|^2) \geq 2\langle v, z^n - u\rangle - \frac{c}{s_n}\left(\sum_{i=1}^{n} \lambda_i^2 + \sum_{k=1}^{n} |\mu_k - \lambda_k|\right), \tag{3.19}$$

where c is a constant such that $c \geq \max\{\|v^1\|^2, \|v^2\| \|x^{k+1} - u\|\}$ for all k.

Since $\{\lambda_n\} \in \ell^2$ and $\{\lambda_n - \mu_n\} \in \ell^1$, we immediately get by virtue of (3.19) that

$$\limsup_{n\to\infty} \langle v, z^n - u\rangle \leq 0, \quad (u, v) \in \mathrm{gph}(A + B). \tag{3.20}$$

Relation (3.20) guarantees that if $\hat{z}$ is a weak accumulation point of $\{z^n\}$, then

$$\langle v, u - \hat{z}\rangle \geq 0, \quad \forall (u, v) \in \mathrm{gph}(A + B).$$

Hence, the maximality of $A + B$ implies that $(\hat{z}, 0) \in \mathrm{gph}(A + B)$; that is, $0 \in (A + B)\hat{z}$, or $\hat{z} \in S = F$. This together with the fact (3.18) indicates that Lemma 3.2 is applicable and we conclude that $\{z^n\}$ converges weakly to a point in F. $\square$

We can have strong convergence of $\{z^n\}$ under additional conditions on either the solution set of (1.1) or the monotonicity of B.

Theorem 3.5. *Let A and B be maximal monotone such that $A+B$ is also maximal monotone and (1.1) has a solution. Assume $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the conditions (i) and (ii) of Theorem 3.4. Then the sequence $\{z^n\}$ converges in norm if one of the following two conditions is satisfied:*

(i) *B is strongly monotone; i.e., there is $\beta > 0$ such that*

$$\langle Bx - By, x - y\rangle \geq \beta \|x - y\|^2 \quad \forall x, y \in \mathrm{dom}(B);$$

(ii) *the solution set $S = (A + B)^{-1}(0)$ has a nonempty interior.*

Proof. (i) Since now $A + B$ is strongly monotone, the inclusion (1.1) has a unique solution u. Let $v^1 \in Au$ and $v^2 \in Bu$ satisfy $v^1 + v^2 = 0$. Also, the proof for Theorem 3 can be refined. For instance, (3.8) can be refined as

$$\left\langle \frac{1}{\mu_{n+1}}(y^n - x^{n+1}) - v^2, x^{n+1} - u \right\rangle \geq \beta \|x^{n+1} - u\|^2 \tag{3.21}$$

and (3.14) is refined to

$$\|x^k - u\|^2 - \|x^{k+1} - u\|^2 \le 2(\mu_{k+1} - \lambda_{k+1})\langle v^2, x^{k+1} - u\rangle \\ - \lambda_{k+1}^2\|v^1\|^2 + 2\beta\mu_{k+1}\|x^{k+1} - u\|^2, \tag{3.22}$$

were $v^1 + v^2 = 0$.

Since $\{x^k\}$ is bounded, we can reduce (3.22) to the following relation

$$2\beta\mu_{k+1}\|x^{k+1} - u\|^2 \le \|x^k - u\|^2 - \|x^{k+1} - u\|^2 \\ + \alpha(|\mu_{k+1} - \lambda_{k+1}| + \lambda_{k+1}^2), \tag{3.23}$$

where $\alpha > 0$ is a constant.

By the conditions (i) and (ii) of Theorem 3.4 together with the fact that $\lim_{n\to\infty}\|x^n - u\|$ exists, we can immediately conclude from (3.23) that

$$\sum_{n=1}^{\infty} \mu_n\|x^n - u\|^2 < \infty.$$

Since $\{\mu_n\} \notin \ell^1$, we must have

$$\lim_{n\to\infty}\|x^n - u\|^2 = \liminf_{n\to\infty}\|x^n - u\|^2 = 0.$$

(ii) Assume $\mathrm{int}(S) \ne \emptyset$, where $S = (A+B)^{-1}(0)$ is the solution set of (1.1). Take $z_0 \in \mathrm{int}(S)$. Then we have a $\delta > 0$ such that

- $z \in S$ whenever $\|z - z_0\| \le \delta$;
- $\|v\| \le M$ whenever $\|z - z_0\| \le \delta$ and $v \in (A + B)z$, where $M > 0$ is a constant.

Now since the closed ball $B(z_0, \delta) \subset S \subset \mathrm{dom}(A) \cap \mathrm{dom}(B)$, the inequality (3.13) in the proof of Theorem 3.4 becomes

$$\begin{aligned} \|x^n - z\|^2 - \|x^{n+1} - z\|^2 \ \ge\ & \|x^n - y^n\|^2 + \|y^n - x^{n+1}\|^2 \\ & +2\lambda_{n+1}\langle v^1, y^n - x^{n+1}\rangle \\ & +2\langle \lambda_{k+1}v^1 + \mu_{k+1}v^2, x^{n+1} - z\rangle \\ \ge\ & \|x^n - y^n\|^2 - \lambda_{n+1}^2\|v^1\|^2 \\ & +2\langle \lambda_{k+1}v^1 + \mu_{k+1}v^2, x^{n+1} - z\rangle, \end{aligned} \tag{3.24}$$

where $\|z - z_0\| \le \delta$, $v^1 \in Az$ and $v^2 \in Bz$ (note that we may choose $v^1 \in Az$ and $v^2 \in Bz$ such that $v^1 + v^2 = 0$ for $z \in S$).

Let $z = z_0 + \delta w$, where $\|w\| \le 1$. Noticing

$$\begin{aligned} \|x^n - z\|^2 - \|x^{n+1} - z\|^2 &= \|(x^n - z_0) - \delta w\|^2 - \|(x^{n+1} - z_0) - \delta w\|^2 \\ &= \|x^n - z_0\|^2 - \|x^{n+1} - z_0\|^2 - 2\delta\langle x^n - x^{n+1}, w\rangle, \end{aligned}$$

we get from (3.24) that

$$\|x^n - z_0\|^2 - \|x^{n+1} - z_0\|^2 \ge 2\delta\langle x^n - x^{n+1}, w\rangle - \lambda_{n+1}^2\|v^1\|^2 \\ + 2(\mu_{k+1} - \lambda_{k+1})\langle v^2, x^{k+1} - z\rangle. \tag{3.25}$$

It turns out that, as $\{x^n\}$ is bounded, we can find a constant $\gamma > 0$ such that

$$\langle x^n - x^{n+1}, w\rangle \le \frac{1}{2\delta}[\|x^n - z\|^2 - \|x^{n+1} - z\|^2$$

$$+ \gamma(|\mu_{k+1} - \lambda_{k+1}| + \lambda_{k+1}^2)]$$

for all $\|w\| \leq 1$. Consequently, we get

$$\begin{aligned} \|x^n - x^{n+1}\| \leq \frac{1}{2\delta}[&\|x^n - z\|^2 - \|x^{n+1} - z\|^2 \\ &+ \gamma(|\mu_{k+1} - \lambda_{k+1}| + \lambda_{k+1}^2)] \end{aligned} \tag{3.26}$$

By the conditions (i) and (ii) of Theorem 3.4, it follows from (3.26) that

$$\sum_{n=1}^{\infty} \|x^n - x^{n+1}\| < \infty$$

and this suffices to guarantee that $\{x^n\}$ is a norm-Cauchy sequence, hence strongly convergent. □

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