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WEAK HARDY SPACES $WH^p_L(\mathbb{R}^n)$ ASSOCIATED TO OPERATORS SATISFYING k-DAVIES-GAFFNEY ESTIMATES

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Abstract. Let L be a one-to-one operator of type ω having a bounded H_∞ functional calculus and satisfying the k-Davies-Gaffney estimates with $k \in \mathbb{N}$. In this article, the authors introduce the weak Hardy space $WH^p_L(\mathbb{R}^n)$ associated to L for $p \in (0, 1]$ via the non-tangential square function S_L and establish a weak molecular characterization of $WH^p_L(\mathbb{R}^n)$. A typical example of such operators is the $2k$-order divergence form homogeneous elliptic operator $L := (-1)^k \sum_{|\alpha|=k=|\beta|} \partial^\beta(a_{\alpha,\beta}\partial^\alpha)$, where $\{a_{\alpha,\beta}\}_{|\alpha|=k=|\beta|}$ are complex bounded measurable functions. As applications, for $p \in (\frac{n}{n+k}, 1]$, the authors prove that the associated Riesz transform $\nabla^k L^{-1/2}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$ and, for all $0 < p < r \le 1$ and $\alpha = n(\frac{1}{p} - \frac{1}{r})$, the fractional power $L^{-\frac{\alpha}{2k}}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to $WH^r_L(\mathbb{R}^n)$. Also, the authors establish an interpolation theorem of $WH^p_L(\mathbb{R}^n)$ by showing that $L^2(\mathbb{R}^n) \cap WH^p_L(\mathbb{R}^n)$ for all $p \in (0, 1]$ are the intermediate spaces in the real method of interpolation between the spaces $L^2(\mathbb{R}^n) \cap H^p_L(\mathbb{R}^n)$ for different $p \in (0, 1]$. In particular, if L is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates, the authors further establish the weak atomic characterization of $WH^p_L(\mathbb{R}^n)$. Furthermore, the authors find the dual space of $WH^p_L(\mathbb{R}^n)$ for $p \in (0, 1]$, which can be defined via mean oscillations based on some subtle coverings of bounded open sets and, even when $L := -\Delta$, are also previously unknown.

1. Introduction

It is well known that Stein and Weiss [60] originally inaugurated the study of real Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ on the Euclidean space $\mathbb{R}^n$. Later, a real-variable theory of $H^p(\mathbb{R}^n)$ for $p \in (0, 1]$ was systematically developed by Fefferman and Stein in [30]. Since then, the real-variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ has found many important applications in various fields of analysis and partial differential equations; see, for example, [17, 18, 20, 34, 48, 55, 57, 58, 61].

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It is now known that $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$ when studying the boundedness of operators; for example, when $p \in (0, 1]$, the *Riesz transform* $\nabla(-\Delta)^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$, but bounded on $H^p(\mathbb{R}^n)$, where Δ is the *Laplace operator* $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and ∇ is the *gradient operator* $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ on $\mathbb{R}^n$. Moreover, when considering some weak type inequalities for some of the most important operators from harmonic analysis and partial differential equations, we are led to the more general *weak Hardy space* $WH^p(\mathbb{R}^n)$ (see, for example, [31, 51, 33, 2, 35, 56, 1, 54]). It is well known that the weak Hardy space $WH^p(\mathbb{R}^n)$ is a suitable substitute of both the weak Lebesgue space $WL^p(\mathbb{R}^n)$ and the Hardy space $H^p(\mathbb{R}^n)$ when studying the boundedness of operators in the critical case. For example, let $\delta \in (0, 1]$, T be a *δ-Calderón-Zygmund operator* and $T^*(1) = 0$, where T^* denotes the *adjoint operator* of T. It is known that T is bounded on $H^p(\mathbb{R}^n)$ for all $p \in (\frac{n}{n+\delta}, 1]$ and not bounded on $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$, but, instead of this, T is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $WH^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ (see [51, 2]). Recall that the Riesz transform $\nabla(-\Delta)^{-1/2}$ is a 1-Calderón-Zygmund operator with convolution kernel, which is smooth on $\mathbb{R}^n \times \mathbb{R}^n$ except on the diagonal points

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \ x = y\}.$$

For more related history and properties about $WH^p(\mathbb{R}^n)$, we refer to [29, 31, 51, 2, 52, 56, 1] and the references cited therein. We should point out that Fefferman, Rivière and Sagher [29] proved that the weak Hardy space $WH^p(\mathbb{R}^n)$ naturally occurs as the intermediate spaces in the real method of interpolation between the Hardy spaces $H^p(\mathbb{R}^n)$. It is easy to see that the classical Hardy spaces $H^p(\mathbb{R}^n)$ and the weak Hardy spaces $WH^p(\mathbb{R}^n)$ are essentially related to the Laplace operator Δ.

In recent years, the study of Hardy spaces and their generalizations associated to differential operators attracts a lot of attentions; see, for example, [3, 4, 6, 11, 12, 14, 15, 16, 24, 25, 26, 27, 28, 38, 39, 40, 41, 42, 44, 64] and their references. In particular, Auscher et al. [4] first introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated to L, where the *heat kernel generated by L satisfies a pointwise Poisson type upper bound*. Later, Duong and Yan [25, 26] introduced the dual space $\mathrm{BMO}_L(\mathbb{R}^n)$ and showed that the dual space of $H^1_L(\mathbb{R}^n)$ is $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$. Yan [63] further introduced the Hardy space $H^p_L(\mathbb{R}^n)$ for some $p \in (0, 1]$ but near to 1 and generalized these results to $H^p_L(\mathbb{R}^n)$ and their dual spaces. A real-variable theory of Orlicz-Hardy spaces and their dual spaces associated to L was also developed in [45, 43].

Recently, the (Orlicz-)Hardy space associated to a *one-to-one operator of type ω satisfying the k-Davies-Gaffney estimates and having a bounded H_∞ functional calculus* was introduced in [12, 23, 10, 22]. A typical example of such operators is the following *$2k$-order divergence form homogeneous elliptic operator*

$$L := (-1)^k \sum_{|\alpha|=k=|\beta|} \partial^\beta(a_{\alpha,\beta}\partial^\alpha), \tag{1.1}$$

interpreted in the usual weak sense via a sesquilinear form, with complex bounded measurable coefficients $\{a_{\alpha,\beta}\}_{|\alpha|=k=|\beta|}$ satisfying the *elliptic condition*, namely, there exist constants $0 < \lambda \le \Lambda < \infty$ such that, for all α, β with $|\alpha| = k = |\beta|$,

$\|a_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^n)} \le \Lambda$ and, for all $f \in W^{k,2}(\mathbb{R}^n)$, $\Re(L_1 f,\, f) \ge \lambda \|\nabla^k f\|^2_{L^2(\mathbb{R}^n)}$. Here and hereafter, $\Re z$ denotes the *real part* of z for all $z \in \mathbb{C}$.

Notice that, when $k = 1$, $H^p_L(\mathbb{R}^n)$ is the Hardy space associated to the *second-order divergence form elliptic operator on* $\mathbb{R}^n$ *with complex bounded measurable coefficients*, which was introduced by Hofmann and Mayboroda [39, 40], Hofmann et al. [41], and Jiang and Yang [44]. It is known that the associated Riesz transform $\nabla^k L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ for all $p \in (\frac{n}{n+k},\, 1]$ (see [12]). Unlike the classical case, in this case, $\nabla^k L^{-1/2}$ may even not have a smooth convolution kernel. Thus, the boundedness of $\nabla^k L^{-1/2}$ can not be extended to the full range of $p \in (0,\, \infty)$ as before. However, when considering the endpoint boundedness of the associated Riesz transforms, it is found that the weak Hardy space is useful. For example, it was proved in [50] that $\nabla^k L^{-1/2}$ is bounded from $H^{n/(n+k)}_L(\mathbb{R}^n)$ to the weak Hardy space $WH^{n/(n+k)}(\mathbb{R}^n)$, which may not be bounded on $H^{n/(n+k)}_L(\mathbb{R}^n)$.

Motivated by the above results, in this article, we wish to develop a real-variable theory of weak Hardy spaces associated to a class of differential operators and study their applications. More precisely, we always assume that L *is a one-to-one operator of type* ω *having a bounded* H_∞ *functional calculus and satisfying the* k*-Davies-Gaffney estimates.* For $p \in (0,\, 1]$, we introduce the weak Hardy space $WH^p_L(\mathbb{R}^n)$ associated to L via the non-tangential square function S_L and establish its weak molecular characterization. In particular, *if* L *is a nonnegative self-adjoint operator in* $L^2(\mathbb{R}^n)$ *satisfying the Davies-Gaffney estimates*, we further establish the weak atomic decomposition of $WH^p_L(\mathbb{R}^n)$. By their atomic characterizations, we easily see that $WH^p_{-\Delta}(\mathbb{R}^n)$ and the *closure* of $WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ on the quasi-norm $\|\cdot\|_{WH^p(\mathbb{R}^n)}$ coincide with equivalent quasi-norms. Let L be the $2k$-order divergence form homogeneous elliptic operator as in (1.1). As applications, we prove that, for all $p \in (n/(n+k),\, 1]$, the associated Riesz transform $\nabla^k L^{-1/2}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$; furthermore, for all $0 < p < r \le 1$ and $\alpha = n(1/p - 1/r)$, the fractional power $L^{-\alpha/(2k)}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to $WH^r_L(\mathbb{R}^n)$. We also establish a real interpolation theorem on $WH^p_L(\mathbb{R}^n)$ by showing that $L^2(\mathbb{R}^n) \cap WH^p_L(\mathbb{R}^n)$ for all $p \in (0,\, 1]$ are the the intermediate spaces in the real method of interpolation between the spaces $L^2(\mathbb{R}^n) \cap H^p_L(\mathbb{R}^n)$ for different $p \in (0,\, 1]$. Moreover, if L is nonnegative self-adjoint and satisfies the Davies-Gaffney estimates, then, for $p \in (0,\, 1]$, we give out the dual space of $WH^p_L(\mathbb{R}^n)$, which is defined via mean oscillations of distributions based on some subtle coverings of bounded open sets, and prove that the elements in $W\Lambda^\alpha_L(\mathbb{R}^n)$ can be viewed as a weak type Carleson measure of order α. We point out that, even when $L := -\Delta$, the dual spaces of $WH^p_L(\mathbb{R}^n)$ are also previously unknown, since the seminal article [31] of Fefferman and Soria on $WH^1(\mathbb{R}^n)$ was published in 1986. Our aforementioned result on dual spaces of $WH^p_L(\mathbb{R}^n)$ may give some light on this problem. In short, the results of this article round out the picture on weak Hardy spaces associated to operators satisfying k-Davies-Gaffney estimates. As in the aforementioned articles on the theory of Hardy spaces associated with operators, the achievement of all results in this article stems from subtle atomic decompositions of weak tent spaces introduced in this article. To the best of our knowledge, all results obtained in this article are new even when L is the Laplace operator.

This article is organized as follows.

In Section 2, we first present some assumptions on the operator L used throughout the whole article (see Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$, $(\mathcal{L})_3$ and $(\mathcal{L})_4$ below) and recall some basic facts concerning the k-Davies-Gaffney estimates (see Lemmas 2.3, 2.4 and 2.5 below) in Subsection 2.1. In Subsection 2.2, we introduce the weak tent space and establish its weak atomic decomposition (see Theorem 2.6 below). Later, via the Whitney decomposition lemma, we obtain another weak atomic decomposition of $WT^p(\mathbb{R}^{n+1}_+)$ (see Theorem 2.11 below), which plays an important role in the dual theory of our weak Hardy spaces.

Finally, in Section 2.3, after recalling some necessary results on the Hardy space $H^p_L(\mathbb{R}^n)$ associated to L, we introduce the weak Hardy space $WH^p_L(\mathbb{R}^n)$ associated to L (see Definition 2.12 below) and establish its weak molecular characterization (see Theorem 2.21 below). As applications, if L is the $2k$-order divergence form homogeneous elliptic operator as in (1.1), we prove the boundedness of the associated Riesz transform $\nabla^k L^{-1/2}$ and fractional power $L^{-\alpha/(2k)}$ on $WH^p_L(\mathbb{R}^n)$ (see Theorems 2.24 and 2.25 below). Moreover, when L is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates, we obtain its weak atomic characterization (see Theorem 2.15 below). Recall that, in [31, 51], for all $p \in (0, 1]$, a weak atomic decomposition of the classical weak Hardy space $WH^p(\mathbb{R}^n)$ is obtained. However, the "atoms" appeared in the weak atomic characterization of $WH^p(\mathbb{R}^n)$ in [31, 51] are essentially closer, in spirit, to the classical "$L^\infty(\mathbb{R}^n)$-atoms", while the "atoms" appeared in our weak atomic characterization of $WH^p_L(\mathbb{R}^n)$ are just $H^p_L(\mathbb{R}^n)$-atoms associated to L from [38] when $p = 1$ and from [42] when $p \in (0, 1]$ (see Theorem 2.15 below).

In order to establish the weak atomic decomposition of weak tent spaces in Theorem 2.6 below, we want to point out that we borrow some ideas from the proof of [19, Proposition 2], with some necessary adjustments by changing the formation of the norm from the original strong version to the present weak version. We also remark that this weak atomic characterization still holds true under some small modifications of the level set of the $\mathcal{A}$-functional (see (2.1) and Remark 2.10 below). An innovation of Theorem 2.6 is to establish an explicit relation between the supports of $T^p(\mathbb{R}^{n+1}_+)$-atoms and the corresponding coefficients, which plays a key role in establishing the weak atomic/molecular characterizations of weak Hardy spaces associated to L (see Theorems 2.6(ii), 2.15 and 2.21 below). Indeed, the proof of Theorem 2.15 strongly depends on Theorem 2.6 and a *superposition principle on weak type estimates* from Stein et al. [59] (see also Lemma 2.17 below). We point out that, without Theorem 2.6(ii), Theorem 2.15 seems impossible (see (2.19) and (2.20) below). The proof of Theorem 2.21 is similar to that of Theorem 2.15, but needs finer off-diagonal estimates because of the lack of the support condition of molecules.

In Section 3, we establish an interpolation theorem of $WH^p_L(\mathbb{R}^n)$ by showing that $L^2(\mathbb{R}^n) \cap WH^p_L(\mathbb{R}^n)$ for all $p \in (0, 1]$ are the intermediate spaces in the real method of interpolation between the Hardy spaces $L^2(\mathbb{R}^n) \cap H^p_L(\mathbb{R}^n)$ for different $p \in (0, 1]$ (see Theorem 3.5 below). Unlike in the classical case in [29], we prove Theorem 3.5

by using a real interpolation result on the tent space $T^p(\mathbb{R}^{n+1}_+)$ from [9] and a result on the interpolation of intersections from Krugljak et al. [47].

Section 4 is devoted to the dual theory of $WH^p_L(\mathbb{R}^n)$. Let L be nonnegative self-adjoint and satisfy the Davies-Gaffney estimates. We first introduce the notion of the *weak Lipschitz space* $W\Lambda^{\alpha}_{L,\mathcal{N}}(\mathbb{R}^n)$ via the mean oscillation over bounded open sets, then we prove that the elements in $W\Lambda^{\alpha}_{L,\mathcal{N}}(\mathbb{R}^n)$ can be viewed as some weak Carleson measures of order α (see Proposition 4.9 below) and prove that the dual space of $WH^p_L(\mathbb{R}^n)$ is $W\Lambda^{n(1/p-1)}_{L,\mathcal{N}}(\mathbb{R}^n)$ (see Theorem 4.6 below).

Recall that the dual space of the classical weak Hardy space $WH^1(\mathbb{R}^n)$ was first considered by Fefferman and Soria [31]. More precisely, for any bounded open set $\Omega \subset \mathbb{R}^n$ and function φ on $\mathbb{R}^n$, the *mean oscillation* $\mathcal{O}(\varphi, \Omega)$ of φ *over* Ω was defined by Fefferman and Soria in [31] as

$$\mathcal{O}(\varphi, \Omega) := \sup \frac{1}{|\Omega|} \sum_k \int_{Q_k} \left|\varphi(x) - \varphi_{Q_k}\right| dx,$$

where $\varphi_Q := \frac{1}{|Q|}\int_Q \varphi(x)\, dx$ for any cube Q and the supremum is taken over all collections $\{Q_k\}_k$ of subcubes of Ω with *bounded* $C(n)$*-overlap* (which means that there exists a positive constant $C(n)$ such that $\sum_k \chi_{Q_k} \le C(n)$). Let $\omega(\delta) := \sup_{|\Omega|=\delta} \mathcal{O}(\varphi, \Omega)$,

$$L^1_0(\mathbb{R}^n) := \left\{f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)\, dx = 0\right\}$$

and $\overline{L^1_0(\mathbb{R}^n)}$ be the *closure* of $L^1_0(\mathbb{R}^n)$ in the norm of the weak Hardy space $WH^1(\mathbb{R}^n)$. In [31], Fefferman and Soria proved that the dual of $\overline{L^1_0(\mathbb{R}^n)}$ is the *set* of all the functions φ satisfying

$$\|\varphi\|_* := \int_0^\infty \frac{\omega(\delta)}{\delta}\, d\delta < \infty.$$

In the present article, we show, in Theorem 4.6 below, that the dual space of $WH^p_L(\mathbb{R}^n)$ for all $p \in (0, 1]$ is $W\Lambda^{n(1/p-1)}_{L,\mathcal{N}}(\mathbb{R}^n)$, which is defined by means of a similar integral of the mean oscillation based on some smart coverings of bounded open sets (see Definitions 4.1 and 4.2 below). Here the integral mean φ_Q is replaced by some approximation of identity, and the collections of subcubes of an open set with bounded $C(n)$-overlap by another new class of sets (see Definition 4.1). In particular, when $L = -\Delta$, $W\Lambda^{n(1/p-1)}_{-\Delta,\vec{\mathcal{N}}}(\mathbb{R}^n)$ is the dual space of the space $\overline{WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)}^{\|\cdot\|_{WH^p(\mathbb{R}^n)}}$ for $p \in (0, 1]$, which seems also new. Here, the *space* $\overline{WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)}^{\|\cdot\|_{WH^p(\mathbb{R}^n)}}$ denotes the *closure* of $WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ on the quasi-norm $\|\cdot\|_{WH^p(\mathbb{R}^n)}$.

The proof of Theorem 4.6 strongly depends on a Carderón reproducing formula obtained in [39], a subtle weak atomic decomposition of the weak tent space (see Theorem 2.11), and a resolvent characterization of $WH^p_L(\mathbb{R}^n)$ (see Proposition 4.5 below) and Proposition 4.9 below.

Recall that a key ingredient to prove the duality between Hardy spaces and Lipschitz spaces is to represent the Lipschitz norm by means of a dual norm expression of some Hilbert spaces. It is known that, in the case of the classical "strong" Lipschitz space $\Lambda_L^\alpha(\mathbb{R}^n)$, this Hilbert space can be chosen to be $L^2(B)$, where B is some ball (see the proof of [41, Theorem 3.51]). Observe also that the mean oscillation appearing in the norm of the "strong" Lipschitz space $\Lambda_L^\alpha(\mathbb{R}^n)$ has the form

$$\left\{\frac{1}{|B|}\int_B \left|\left(I-e^{-r_B^2 L}\right)^M f(x)\right|^2 dx\right\}^{1/2},$$

which involves only one ball and hence the off-diagonal estimates can be applied directly. However, in the weak case, the mean oscillation involves a general bounded open set (see (4.3) below). Therefore, we can not apply off-diagonal estimates directly. To overcome this difficulty, we first introduce, in Definition 4.1, subtle coverings of bounded open sets, which stem from the proof of the weak atomic decomposition for weak tent spaces in Theorem 2.11, obtained via the Whitney-type decompositions on level sets for $\mathcal{A}$-functionals in (2.1). More precisely, we first find a sequence of balls which cover the considered open set via the Whitney-type decomposition. Then we construct the annuli sets based on a sequence of balls and consider the off-diagonal estimates on these annuli. Since the radius of the balls in the sequence are different, the off-diagonal estimates on these annuli are more complicated than those on a single ball.

As usual, we make some conventions on the notation. Throughout the whole article, we always let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ and

$$\mathbb{R}^{n+1}_+ := \{(x,\, t) :\ x\in\mathbb{R}^n,\, t\in(0,\,\infty)\}$$

. We use C to denote a *positive constant*, independent of the main parameters involved, but whose value may differ from line to line. *Constants with subscripts*, such as C_0, M_0 and α_0, do not change in different occurrences. If $f\le Cg$, we write $f\lesssim g$ and, if $f\lesssim g\lesssim f$, we then write $f\sim g$. For all $x\in\mathbb{R}^n$, $r\in(0,\infty)$ and $\alpha\in(0,\,\infty)$, let $B(x,r) := \{y\in\mathbb{R}^n : |x-y|<r\}$, $\alpha B(x,r) := B(x,\alpha r)$, $S_0(B) := B$, $S_i(B) := 2^iB\setminus(2^{i-1}B)$ and $\widetilde{S}_i(B) := 2^{i+1}B\setminus(2^{i-2}B)$ for all $i\in\mathbb{Z}_+$, where, when $i<0$, $2^iB := \emptyset$. Also, for any set $E\subset\mathbb{R}^n$, we use $E^{\complement}$ to denote the *set* $\mathbb{R}^n\setminus E$ and χ_E its *characteristic function*. For any index $q\in[1,\infty]$, we denote by q' its *conjugate index*, namely, $1/q+1/q'=1$. Let $\mathcal{S}(\mathbb{R}^n)$ be the *space of Schwartz functions* on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its *dual space*.

2. The weak Hardy space $WH^p_L(\mathbb{R}^n)$

The main purpose of this section is to introduce the weak Hardy space $WH^p_L(\mathbb{R}^n)$ and establish its weak atomic and molecular characterizations. As applications of this weak molecular characterization, we obtain the boundedness of the associated Riesz transform and fractional power on $WH^p_L(\mathbb{R}^n)$. In order to achieve this goal, we need to describe our hypotheses on the operator L throughout the whole article.

2.1. **Assumptions on L.**

In this subsection, we first survey some known results on the bounded H_∞ functional calculus. Then, after stating our assumptions on the operator L throughout the whole article, we recall some useful technical lemmas on the k-Davies-Gaffney estimates.

For $\theta \in [0, \pi)$, the *open* and *closed sectors, S_θ^0* and S_θ*, of angle* θ in the complex plane $\mathbb{C}$ are defined, respectively, by setting $S_\theta^0 := \left\{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\right\}$ and $S_\theta := \left\{z \in \mathbb{C} : |\arg z| \le \theta\right\}$. Let $\omega \in [0, \pi)$. A closed operator T on $L^2(\mathbb{R}^n)$ is said to be of *type* ω, if

(i) the *spectrum* of T, $\sigma(T)$, is contained in S_ω;

(ii) for each $\theta \in (\omega, \pi)$, there exists a nonnegative constant C such that, for all $z \in \mathbb{C} \setminus S_\theta$,

$$\left\|(T - zI)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C|z|^{-1},$$

here and hereafter, for any normed linear space $\mathcal{H}$, $\|S\|_{\mathcal{L}(\mathcal{H})}$ denotes the *operator norm* of the linear operator $S : \mathcal{H} \to \mathcal{H}$.

For $\mu \in [0, \pi)$ and $\sigma, \tau \in (0, \infty)$, let

$$H(S_\mu^0) := \{f : f \text{ is a holomorphic function on } S_\mu^0\},$$

$$H_\infty(S_\mu^0) := \left\{f \in H(S_\mu^0) : \|f\|_{L^\infty(S_\mu^0)} < \infty\right\}$$

and

$$\Psi_{\sigma,\tau}(S_\mu^0) := \Big\{f \in H(S_\mu^0) : \text{ there exists a positive constant } C \text{ such that, for all } \xi \in S_\mu^0, |f(\xi)| \le C \inf\{|\xi|^\sigma, |\xi|^{-\tau}\}\Big\}.$$

It is known that every one-to-one operator T of type ω in $L^2(\mathbb{R}^n)$ has a unique holomorphic functional calculus; see, for example, [53]. More precisely, let T be a one-to-one operator of type ω, with $\omega \in [0, \pi)$, $\mu \in (\omega, \pi)$, $\sigma, \tau \in (0, \infty)$ and $f \in \Psi_{\sigma,\tau}(S_\mu^0)$. The *function of the operator* T, $f(T)$, can be defined by the H_∞ functional calculus in the following way,

$$f(T) = \frac{1}{2\pi i}\int_\Gamma (\xi I - T)^{-1} f(\xi)\, d\xi,$$

where $\Gamma := \{re^{i\nu} : \infty > r > 0\} \cup \{re^{-i\nu} : 0 < r < \infty\}$, $\nu \in (\omega, \mu)$, is a curve consisting of two rays parameterized anti-clockwise. It is known that $f(T)$ is independent of the choice of $\nu \in (\omega, \mu)$ and the integral is absolutely convergent in $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ (see [53, 36]).

In what follows, we *always assume* $\omega \in [0, \pi/2)$. Then, it follows, from [36, Proposition 7.1.1], that, for every operator T of type ω in $L^2(\mathbb{R}^n)$, $-T$ generates a holomorphic C_0-semigroup $\{e^{-zT}\}_{z \in S^0_{\pi/2-\omega}}$ on the open sector $S^0_{\pi/2-\omega}$ such that, for all $z \in S^0_{\pi/2-\omega}$, $\|e^{-zT}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 1$ and, moreover, every *nonnegative self-adjoint operator is of type* 0.

Let $\Psi(S_\mu^0) := \cup_{\sigma,\tau>0}\Psi_{\sigma,\tau}(S_\mu^0)$. It is well known that the above holomorphic functional calculus defined on $\Psi(S_\mu^0)$ can be extended to $H_\infty(S_\mu^0)$ via a limit process (see [53]). Recall that, for $\mu \in (0, \pi)$, the operator T is said to have a *bounded $H_\infty(S_\mu^0)$ functional calculus* in the Hilbert space $\mathcal{H}$ if there exists a positive constant C such that, for all $\psi \in H_\infty(S_\mu^0)$, $\|\psi(T)\|_{\mathcal{L}(\mathcal{H})} \le C\|\psi\|_{L^\infty(S_\mu^0)}$, and T is said to have a *bounded* H_∞ *functional calculus* in the Hilbert space $\mathcal{H}$ if there exists $\mu \in (0, \pi)$ such that T has a bounded $H_\infty(S_\mu^0)$ functional calculus.

Throughout the whole article, we *always assume* that L satisfies the following three assumptions:

Assumption $(\mathcal{L})_1$**.** *The operator L is a one-to-one operator of type ω in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \pi/2)$.*

Assumption $(\mathcal{L})_2$**.** *The operator L has a bounded H_∞ functional calculus in $L^2(\mathbb{R}^n)$.*

Assumption $(\mathcal{L})_3$**.** *Let $k \in \mathbb{N}$. The operator L generates a holomorphic semigroup $\{e^{-tL}\}_{t>0}$ which satisfies the k-Davies-Gaffney estimates, namely, there exist positive constants C and C_1 such that, for all closed sets E and F in $\mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$ supported in E,*

$$\left\|e^{-tL}f\right\|_{L^2(F)} \le C\exp\left\{-C_1\frac{\left[\operatorname{dist}(E,\,F)\right]^{2k/(2k-1)}}{t^{1/(2k-1)}}\right\}\|f\|_{L^2(E)},$$

here and hereafter, for any $p \in (0, \infty)$, $\|f\|_{L^p(E)} := \left\{\int_E |f(x)|^p\,dx\right\}^{1/p}$ and

$$\operatorname{dist}(E,\,F) := \inf_{x\in E,\,y\in F}|x-y|$$

denotes the distance between E and F.

In many cases, we also need the following assumption, which is stronger than Assumption $(\mathcal{L})_3$.

Assumption $(\mathcal{L})_4$**.** *Let $k \in \mathbb{Z}_+$ and $(p_-(L), p_+(L))$ be the* range of exponents *$p \in [1, \infty]$ for which the holomorphic semigroup $\{e^{-tL}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$. Then, for all $p_-(L) < p \le q < p_+(L)$, $\{e^{-tL}\}_{t>0}$ satisfies the $L^p - L^q$ k-off-diagonal estimates, namely, there exist positive constants C_2 and C_3 such that, for all closed sets $E, F \subset \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ supported in E,*

$$\left\|e^{-tL}f\right\|_{L^q(F)} \le C_2 t^{\frac{n}{2k}(\frac{1}{q}-\frac{1}{p})}\exp\left\{-\frac{\left[\operatorname{dist}(E,\,F)\right]^{2k/(2k-1)}}{C_3 t^{1/(2k-1)}}\right\}\|f\|_{L^p(E)}.$$

Remark 2.1. The notion of the off-diagonal estimates (or the so called Davies-Gaffney estimates) of the semigroup $\{e^{-tL}\}_{t>0}$ are first introduced by Gaffney [32] and Davies [21], which serves as good substitutes of the Gaussian upper bound of the associated heat kernel; see also [8, 5] and related references. We point out that, when $k = 1$, the k-Davies-Gaffney estimates are the usual Davies-Gaffney estimates (or the L^2 *off-diagonal estimates* or just the *Gaffney estimates*) (see, for example, [38, 39, 40, 42, 41]).

Proposition 2.2 ([12])**.** *Let L be the $2k$-order divergence form homogeneous elliptic operator as in* (1.1)*. Then L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$, $(\mathcal{L})_3$ and $(\mathcal{L})_4$.*

In order to make this article self-contained, we list the following three technical lemmas which are needed in the proofs of our main results.

Lemma 2.3 ([12]). *Assume that the operator L defined on $L^2(\mathbb{R}^n)$ satisfies Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. Then, for all $m \in \mathbb{N}$, the family of operators, $\{(tL)^m e^{-tL}\}_{t>0}$, also satisfy the k-Davies-Gaffney estimates.*

Lemma 2.4 ([12]). *Let $\{A_t\}_{t>0}$ and $\{B_t\}_{t>0}$ be two families of linear operators satisfying the k-Davies-Gaffney estimates. Then the families of linear operators $\{A_tB_t\}_{t>0}$ also satisfy the k-Davies-Gaffney estimates.*

Lemma 2.5 ([12]). *Let $M \in \mathbb{N}$ and L be as in (1.1). Then there exists a positive constant C such that, for all closed sets E, F in $\mathbb{R}^n$ with $\operatorname{dist}(E, F) > 0$, $f \in L^2(\mathbb{R}^n)$ supported in E and $t \in (0, \infty)$,*

$$\left\|\nabla^k L^{-1/2}\left(I - e^{-tL}\right)^M f\right\|_{L^2(F)} \le C\left(\frac{t}{\left[\operatorname{dist}(E, F)\right]^{2k}}\right)^M \|f\|_{L^2(E)}$$

and

$$\left\|\nabla^k L^{-1/2}\left(tLe^{-tL}\right)^M f\right\|_{L^2(F)} \le C\left(\frac{t}{\left[\operatorname{dist}(E, F)\right]^{2k}}\right)^M \|f\|_{L^2(E)}.$$

2.2. **The weak tent spaces $WT^p(\mathbb{R}^{n+1}_+)$.** In this subsection, we introduce the weak tent space and establish its weak atomic characterization. This construction constitutes a crucial component to obtain the weak atomic or molecular characterizations of the weak Hardy space.

We first recall the notion of the *tent space*. Let F be a function on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$. For all $x \in \mathbb{R}^n$, the *$\mathcal{A}$-functional* $\mathcal{A}(F)(x)$ of F is defined by setting

$$\mathcal{A}(F)(x) := \left\{\iint_{\Gamma(x)} \left|F(y, t)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{\frac{1}{2}}, \tag{2.1}$$

where

$$\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ :\ |y - x| < t\} \tag{2.2}$$

is a *cone with vertex x*. For all $p \in (0, \infty)$, the *tent space* $T^p(\mathbb{R}^{n+1}_+)$ is defined by

$$T^p(\mathbb{R}^{n+1}_+) := \left\{F :\ \mathbb{R}^{n+1}_+ \to \mathbb{C} :\ \|F\|_{T^p(\mathbb{R}^{n+1}_+)} := \|\mathcal{A}(F)\|_{L^p(\mathbb{R}^n)} < \infty\right\}. \tag{2.3}$$

For all open sets Ω, let $\widehat{\Omega} := \mathbb{R}^{n+1}_+ \setminus \cup_{x\in\mathbb{R}^n\setminus\Omega}\Gamma(x)$ be the *tent over* Ω. For all $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, let $B := B(x_B, r_B)$ be the *ball* in $\mathbb{R}^n$. It is easy to see that $\widehat{B} = \{(y, t) \in \mathbb{R}^{n+1}_+ :\ |y - x_B| \le r_B - t\}$. For any $p \in (0, \infty)$ and ball B, a function A defined on $\mathbb{R}^{n+1}_+$ is called a *$T^p(\mathbb{R}^{n+1}_+)$-atom associated to B* if $\operatorname{supp} A \subset \widehat{B}$ and

$$\left\{\iint_{\widehat{B}} \left|A(x,t)\right|^2 \frac{dx\,dt}{t}\right\}^{\frac{1}{2}} \le |B|^{\frac{1}{2}-\frac{1}{p}}.$$

For $p \in (0, \infty)$, let $WL^p(\mathbb{R}^n)$ be the *weak Lebesgue space* with the *quasi-norm*

$$\|f\|_{WL^p(\mathbb{R}^n)} := \left[\sup_{\alpha\in(0,\infty)} \alpha^p\left|\{x \in \mathbb{R}^n :\ |f(x)| > \alpha\}\right|\right]^{1/p}.$$

The *weak tent space* $WT^p(\mathbb{R}^{n+1}_+)$ is defined to be the collection of all functions F on $\mathbb{R}^{n+1}_+$ such that its $\mathcal{A}$-functional $\mathcal{A}(F) \in WL^p(\mathbb{R}^n)$. For any $F \in WT^p(\mathbb{R}^{n+1}_+)$, define its *quasi-norm* by $\|F\|_{WT^p(\mathbb{R}^{n+1}_+)} := \|\mathcal{A}(F)\|_{WL^p(\mathbb{R}^n)}$.

For the weak tent space, we have the following weak atomic decomposition.

Theorem 2.6. *Let $p \in (0, 1]$ and $F \in WT^p(\mathbb{R}^{n+1}_+)$. Then there exists a sequence of $T^p(\mathbb{R}^{n+1}_+)$-atoms, $\{A_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$, associated, respectively, to the balls $\{B_{i,j}\}_{i\in\mathbb{Z},\,j\in\mathbb{Z}_+}$ such that*

(i) *$F = \sum_{i\in\mathbb{Z},\,j\in\mathbb{Z}_+} \lambda_{i,j} A_{i,j}$ pointwisely almost everywhere in $\mathbb{R}^{n+1}_+$, where $\lambda_{i,j} := \widetilde{C}2^i|B_{i,j}|^{1/p}$ and $\widetilde{C}$ is a positive constant independent of F;*

(ii) *there exists a positive constant C, independent of F, such that*

$$\sup_{i\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}_+} |\lambda_{i,j}|^p\Big)^{\frac{1}{p}} \le C\|F\|_{WT^p(\mathbb{R}^{n+1}_+)};$$

(iii) *for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, let $\widetilde{B}_{i,j} := \frac{1}{10\sqrt{n}} B_{i,j}$. Then, for all $i \in \mathbb{Z}$, $\{\widetilde{B}_{i,j}\}_{j\in\mathbb{Z}_+}$ are mutually disjoint.*

In order to prove this theorem, we need the following *Whitney decomposition theorem* (see, for example, [34, p. 463]).

Lemma 2.7. *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$. Then there exists a family of closed cubes $\{Q_j\}_{j\in\mathbb{Z}_+}$ such that*

(i) *$\cup_{j\in\mathbb{Z}_+} Q_j = \Omega$ and $\{Q_j\}_{j\in\mathbb{Z}_+}$ have disjoint interiors;*

(ii) *for all $j \in \mathbb{Z}_+$, $\sqrt{n}\, l_{Q_j} \le \operatorname{dist}(Q_j, \Omega^{\complement}) \le 4\sqrt{n}\, l_{Q_j}$, where l_{Q_j} denotes the length of the cube Q_j;*

(iii) *for any $j, k \in \mathbb{Z}_+$, if the boundaries of two cubes Q_j and Q_k touch, then $\frac{1}{4} \le \frac{l_{Q_j}}{l_{Q_k}} \le 4$;*

(iv) *for a given $j \in \mathbb{Z}_+$, there exist at most $12n$ different cubes Q_k that touch Q_j.*

For any fixed $\gamma \in (0, 1)$ and bounded open set Ω in $\mathbb{R}^n$ with the complementary set F, let $\Omega^*_\gamma := \{x \in \mathbb{R}^n : \ \mathcal{M}(\chi_\Omega)(x) > 1 - \gamma\}$ and $F^*_\gamma := \mathbb{R}^n \setminus \Omega^*_\gamma$, where $\mathcal{M}$ denotes the usual *Hardy-Littlewood maximal function*, namely, for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}f(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \tag{2.4}$$

where the supremum is taken over all balls containing x. We also need the following auxiliary lemma.

Lemma 2.8 ([19])**.** *Let $\alpha \in (0, \infty)$. Then there exist constant $\gamma \in (0, 1)$, sufficiently close to 1, and positive constant C such that, for any closed set F, whose complement (denoted by Ω) has finite measure, and for any non-negative function Φ on $\mathbb{R}^{n+1}_+$,*

$$\int_{\mathcal{R}(F^*_\gamma)} \Phi(y, t)t^n\, dy\, dt \le C \int_F \Big\{ \int_{\Gamma(x)} \Phi(y, t)\, dy\, dt\Big\}\, dx,$$

where $\mathcal{R}(F^*_\gamma) := \cup_{x\in F^*_\gamma}\Gamma(x)$ and $\Gamma(x)$ for $x \in \mathbb{R}^n$ is as in (2.2).

Proof of Theorem 2.6. We show this theorem in the order of (ii), (i) and (iii).

To show (ii), let $F \in WT^p(\mathbb{R}^{n+1}_+)$. For all $i \in \mathbb{Z}$, let

$$O_i := \{x \in \mathbb{R}^n : \mathcal{A}(F)(x) > 2^i\}.$$

It is easy to see that $O_{i+1} \subset O_i$. Moreover, since $F \in WT^p(\mathbb{R}^{n+1}_+)$, we readily see that $|O_i| < \infty$. For fixed $\gamma \in (0, 1)$ satisfying the same restriction as in Lemma 2.8, let

$$(O_i)^*_\gamma := \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{O_i})(x) > 1 - \gamma\}.$$

By abuse of notation, we *simply write* O_i^* instead of $(O_i)^*_\gamma$. Since O_i is open, we easily see that $O_i \subset O_i^*$ and, by the weak $(1, 1)$ boundedness of $\mathcal{M}$, we further know that there exists a positive constant $C(\gamma)$, depending on γ, such that $|O_i^*| \le C(\gamma)|O_i|$. For each O_i^*, using Lemma 2.7, we obtain a Whitney decomposition $\{Q_{i,j}\}_{j\in\mathbb{Z}_+}$ of O_i^*. Let $B_{i,j}$ be the ball having the same center as $Q_{i,j}$ with the radius $5\sqrt{n}\, l_{Q_{i,j}}$, where $l_{Q_{i,j}}$ denotes the length of $Q_{i,j}$. By Lemma 2.7(ii), we immediately see that $B_{i,j} \cap (O_i^*)^\complement \neq \emptyset$.

Now, for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, let $\Delta_{i,j} := \widehat{B}_{i,j} \cap (Q_{i,j} \times (0, \infty)) \cap (\widehat{O}_i^* \setminus \widehat{O}_{i+1}^*)$, $\lambda_{i,j} := \widetilde{C} 2^i |B_{i,j}|^{\frac{1}{p}}$ and $A_{i,j} := F\chi_{\Delta_{i,j}}/\lambda_{i,j}$, where $\widehat{B}_{i,j}$ and $\widehat{O}_i^*$ denote, respectively, the tents over $B_{i,j}$ and O_i^*, and $\widetilde{C}$ is a positive constant independent of F, which will be determined later. From the fact that $\operatorname{supp} F \subset \cup_{i\in\mathbb{Z}} \cup_{j\in\mathbb{Z}_+} \Delta_{i,j}$, it follows that

$$F(x,t) = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+} F(x,t)\chi_{\Delta_{i,j}}(x,t) = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+} \lambda_{i,j} A_{i,j}(x, t) \tag{2.5}$$

pointwisely almost every $(x, t) \in \mathbb{R}^{n+1}_+$.

Moreover, for all $i \in \mathbb{Z}$, by the definition of $\lambda_{i,j}$, Lemma 2.7(i), the fact that $|O_i^*| \le C(\gamma)|O_i|$ and the definition of O_i, we conclude that

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+} |\lambda_{i,j}|^p &\sim 2^{ip} \sum_{j\in\mathbb{Z}_+} |B_{i,j}| \sim 2^{ip} \sum_{j\in\mathbb{Z}_+} |Q_{i,j}| \lesssim 2^{ip}|O_i| \sim 2^{ip}|\{x \in \mathbb{R}^n : \mathcal{A}(F)(x) > 2^i\}| \\ &\lesssim \|\mathcal{A}(F)\|^p_{WL^p(\mathbb{R}^n)} \sim \|F\|^p_{WT^p(\mathbb{R}^{n+1}_+)},\end{aligned} \tag{2.6}$$

which immediately implies (ii).

On the other hand, for any closed set F, let $\mathcal{R}(F) := \cup_{x\in F}\Gamma(x)$. For all $(y, t) \in \Delta_{i,j}$, it follows, from the fact that $\widehat{O}_i^* = \{(y, t) : \operatorname{dist}(y, \mathbb{R}^n \setminus O_i^*) \ge t\}$ and Lemma 2.7(ii), that $t \le \operatorname{dist}(y, \mathbb{R}^n \setminus O_i^*) < r_{B_{i,j}}$. Thus, for all $H \in T^2(\mathbb{R}^{n+1}_+)$ satisfying $\|H\|_{T^2(\mathbb{R}^{n+1}_+)} \le 1$, by the fact that $\mathbb{R}^{n+1}_+ \setminus \widehat{O}_{i+1}^* = \mathcal{R}(\mathbb{R}^n \setminus O_{i+1}^*)$, the support condition of $A_{i,j}$ ($\operatorname{supp} A_{i,j} \subset \widehat{B}_{i,j}$), Lemma 2.8, Hölder's inequality and the definitions of $A_{i,j}$ and O_{i+1}, we see that

$$\begin{aligned}&\left|\left\langle A_{i,j}, H\right\rangle_{T^2(\mathbb{R}^{n+1}_+)}\right| \\ &\quad\le \int_{\mathcal{R}(\mathbb{R}^n\setminus O_{i+1}^*)} \left|A_{i,j}(y, t)H(y, t)\right| \chi_{\Delta_{i,j}}(y, t)\, \frac{dy\,dt}{t} \\ &\quad\lesssim \int_{\mathbb{R}^n\setminus O_{i+1}} \left\{ \int_0^{r_{B_{i,j}}} \int_{|y-x|<t} \left|A_{i,j}(y,t)H(y,t)\right| \chi_{\Delta_{i,j}}(y, t) \frac{dy\,dt}{t^{n+1}} \right\} dx\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&\sim \int_{2B_{i,j}\setminus O_{i+1}} \left\{ \int_0^{r_{B_{i,j}}} \int_{|y-x|<t} \left| A_{i,j}(y,t)H(y,t) \right| \chi_{\Delta_{i,j}}(y,\, t) \frac{dy\,dt}{t^{n+1}} \right\} dx \\
&\lesssim \left\{ \int_{2B_{i,j}\setminus O_{i+1}} \left| \mathcal{A}(A_{i,j})(x) \right|^2 dx \right\}^{\frac{1}{2}} \|\mathcal{A}(H)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \frac{1}{\widetilde{C}} 2^{-i} |B_{i,j}|^{-\frac{1}{p}} \left\{ \int_{2B_{i,j}\setminus O_{i+1}} \left| \mathcal{A}(F)(x) \right|^2 dx \right\}^{\frac{1}{2}} \\
&\lesssim \frac{1}{\widetilde{C}} 2^{-i} |B_{i,j}|^{-\frac{1}{p}} \left[2^{2i} |B_{i,j}| \right]^{\frac{1}{2}} \sim \frac{1}{\widetilde{C}} |B_{i,j}|^{\frac{1}{2}-\frac{1}{p}},
\end{aligned}
$$

which, together with $(T^2(\mathbb{R}^{n+1}_+))^* = T^2(\mathbb{R}^{n+1}_+)$, further implies that $\|A_{i,j}\|_{T^2(\mathbb{R}^{n+1}_+)} \lesssim \frac{1}{\widetilde{C}} |B_{i,j}|^{\frac{1}{2}-\frac{1}{p}}$. Moreover, from the definitions of $\Delta_{i,j}$ and $\widehat{O}_i^*$, and Lemma 2.7(ii), we deduce that $\Delta_{i,j} \subset \widehat{B_{i,j}}$. Thus, by choosing $\widetilde{C}$ large enough such that

$$
\|A_{i,j}\|_{T^2(\mathbb{R}^{n+1}_+)} \le |B_{i,j}|^{\frac{1}{2}-\frac{1}{p}},
$$

then $A_{i,j}$ is a $T^p(\mathbb{R}^{n+1}_+)$-atom associated to the ball $B_{i,j}$, which, combined with (2.5), implies (i).

Observe that (iii) follows readily from Lemma 2.7(vi), which completes the proof of Theorem 2.6. □

If a function $F \in WT^p(\mathbb{R}^{n+1}_+)$ also belongs to $T^2(\mathbb{R}^{n+1}_+)$, then the weak atomic decomposition obtained in Theorem 2.6 also converges in $T^2(\mathbb{R}^{n+1}_+)$, which is the conclusion of the following corollary.

Corollary 2.9. *Let $p \in (0, 1]$. For all $F \in WT^p(\mathbb{R}^{n+1}_+) \cap T^2(\mathbb{R}^{n+1}_+)$, the weak atomic decomposition $F = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+} \lambda_{i,j} A_{i,j}$ obtained in Theorem 2.6 also holds true in $T^2(\mathbb{R}^{n+1}_+)$.*

Proof. Let $F \in WT^p(\mathbb{R}^{n+1}_+) \cap T^2(\mathbb{R}^{n+1}_+)$. We use the same notation as in Theorem 2.6 and its proof. By Theorem 2.6, we see that the weak atomic decomposition $F = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+} \lambda_{i,j} A_{i,j}$ holds true pointwisely almost everywhere in $\mathbb{R}^{n+1}_+$. Thus, for all $N_1,\ N_2 \in \mathbb{N}$, from Fubini's theorem, the definitions of $A_{i,j}$ and $\lambda_{i,j}$ and the bound overlap property of $\{\Delta_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{Z}_+}$, we deduce that

$$
\begin{aligned}
\left\| \sum_{|i|\ge N_1 \text{ or } j\ge N_2} \lambda_{i,j} A_{i,j} \right\|^2_{T^2(\mathbb{R}^{n+1}_+)} &\sim \int_0^\infty \int_{\mathbb{R}^n} \left| \sum_{|i|\ge N_1 \text{ or } j\ge N_2} \lambda_{i,j} A_{i,j}(y,\, t) \right|^2 \frac{dy\,dt}{t} \\
&\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left| F(y,\, t) \chi_{\cup_{|i|\ge N_1 \text{ or } j\ge N_2} \Delta_{i,j}}(y,\, t) \right|^2 \frac{dy\,dt}{t}.
\end{aligned}
$$

By letting $N_1,\ N_2 \to \infty$ and using the condition that $F \in T^2(\mathbb{R}^{n+1}_+)$, we know that $F = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+} \lambda_{i,j} A_{i,j}$ holds true in $T^2(\mathbb{R}^{n+1}_+)$, which completes the proof of Corollary 2.9. □

Remark 2.10. Let $k \in \mathbb{Z}$, $F \in WT^p(\mathbb{R}^{n+1}_+)$, $F \neq 0$ and

$$
O_k := \{x \in \mathbb{R}^n :\ \mathcal{A}(F)(x) > 2^k\}.
$$

Since $F \in WT^p(\mathbb{R}^{n+1}_+)$, it is easy to see that, for all $k \in \mathbb{Z}$, $|O_k| < \infty$ and $|O_{k+1}| \le |O_k|$. Observe that, for some $k \in \mathbb{Z}$, $|O_{k+1}|$ may equal to $|O_k|$.

We now construct two index sets $\mathcal{I} \subset \mathbb{Z}$ and $\widetilde{\mathcal{I}} \subset \mathbb{Z}$, which both are needed in establishing the dual theory of the weak Hardy space $WH^p_L(\mathbb{R}^n)$ in Section 4 below and have the following properties:

Case 1) If, for all $k \in \mathbb{Z}$, $|O_k| \in (0, \infty)$, in this case, we fix any $i_0 \in \mathbb{Z}$ and $\widetilde{i}_0 = 0$ and then choose $\{i_j\}_{j\in\mathbb{Z}\setminus\{0\}} \subset \mathbb{Z}$ and $\{\widetilde{i}_j\}_{j\in\mathbb{Z}\setminus\{0\}} \subset \mathbb{Z}$ such that

(i) $\mathcal{I} := \{i_j\}_{j\in\mathbb{Z}} \subset \mathbb{Z}$ and $\widetilde{\mathcal{I}} := \{\widetilde{i}_j\}_{j\in\mathbb{Z}} \subset \mathbb{Z}$ are strictly increase in j;

(ii) for all $j \in \mathbb{Z}$ and $i_j \in \mathcal{I}$, $|O_{i_{j+1}}| < |O_{i_j}|$;

(iii) for all $j \in \mathbb{N}$, $i_j \in \mathcal{I}$ and $\widetilde{i}_j \in \widetilde{\mathcal{I}}$,

$$|O_{i_j}| \in [2^{-\widetilde{i}_j-1}|O_{i_0}|,\ 2^{-\widetilde{i}_j}|O_{i_0}|)$$

and

$$|O_{i_{-j}}| \in (2^{-\widetilde{i}_{-j}-1}|O_{i_0}|,\ 2^{-\widetilde{i}_{-j}}|O_{i_0}|].$$

Case 2) If there exists $k \in \mathbb{Z}$ such that $|O_k| = 0$, in this case, let

$$i_0 := \min\{k - 1 \in \mathbb{Z} :\ |O_k| = 0\}.$$

Then choose $\{i_j\}_{j\in\mathbb{Z}\setminus\{0\}} \subset \mathbb{Z}$ and $\{\widetilde{i}_j\}_{j\in\mathbb{Z}\setminus\{0\}} \subset \mathbb{Z}$ such that

(i) for all $j \in \mathbb{Z}_+$, $i_j \in \mathcal{I}$ and $\widetilde{i}_j \in \widetilde{\mathcal{I}}$, let $i_j = i_0$ and $\widetilde{i}_j = 0$;

(ii) for all $j \in \mathbb{Z} \setminus \mathbb{Z}_+$, choose $i_j \in \mathcal{I}$ and $\widetilde{i}_j \in \widetilde{\mathcal{I}}$ satisfy that $|O_{i_{j+1}}| < |O_{i_j}|$ and

$$|O_{i_j}| \in (2^{-\widetilde{i}_j-1}|O_{i_0}|,\ 2^{-\widetilde{i}_j}|O_{i_0}|].$$

Indeed, to prove the above claim in Case 1), for any fixed $i_0 \in \mathbb{Z}$ as in Case 1), we let

$$i_1 := \min\{i \in \mathbb{Z} :\ |O_i| < |O_{i_0}|\}$$

and

$$i_{-1} := \max\{i \in \mathbb{Z} :\ |O_i| > |O_{i_0}|\}.$$

From the facts $\lim_{i\to\infty} |O_i| = 0$ and $\lim_{i\to-\infty} |O_i| = \infty$, we deduce that such i_1 and i_{-1} do exist.

Then choose $\widetilde{i}_1, \widetilde{i}_{-1} \in \mathbb{Z}$ satisfying

$$|O_{i_1}| \in [2^{-\widetilde{i}_1-1}|O_{i_0}|,\ 2^{-\widetilde{i}_1}|O_{i_0}|)$$

and

$$|O_{i_{-1}}| \in (2^{-\widetilde{i}_{-1}-1}|O_{i_0}|,\ 2^{-\widetilde{i}_{-1}}|O_{i_0}|].$$

By a simple calculation, it is easy to see that $i_{-1} < i_0 < i_1$, $\widetilde{i}_{-1} < 0 \le \widetilde{i}_1$ and $|O_{i_1}| < |O_{i_0}| < |O_{i_{-1}}|$.

Now, let

$$i_2 := \min\{i \in \mathbb{Z} :\ |O_i| < 2^{-\widetilde{i}_1-1}|O_{i_0}|\},$$

$$i_{-2} := \max\{i \in \mathbb{Z} :\ |O_i| > 2^{-\widetilde{i}_{-1}}|O_{i_0}|\}$$

and choose $\widetilde{i}_2, \widetilde{i}_{-2} \in \mathbb{Z}$ satisfying

$$|O_{i_2}| \in [2^{-\widetilde{i}_2-1}|O_{i_0}|,\ 2^{-\widetilde{i}_2}|O_{i_0}|)$$

and

$$|O_{i_{-2}}| \in (2^{-\widetilde{i}_{-2}-1}|O_{i_0}|,\, 2^{-\widetilde{i}_{-2}}|O_{i_0}|].$$

It is easy to see that $i_{-2} < i_{-1} < i_0 < i_1 < i_2$, $\widetilde{i}_{-2} < \widetilde{i}_{-1} < 0 \le \widetilde{i}_1 < \widetilde{i}_2$ and

$$|O_{i_2}| < |O_{i_1}| < |O_{i_0}| < |O_{i_{-1}}| < |O_{i_{-2}}|.$$

Continuing this process, we obtain a sequence $\{O_{i_j}\}_{j\in\mathbb{Z}}$ of strictly decreasing open sets, and sequences $\{i_j\}_{j\in\mathbb{Z}}$, $\{\widetilde{i}_j\}_{j\in\mathbb{Z}}$ of increasing numbers. Denote the index sets $\{i_j\}_{j\in\mathbb{Z}}$ and $\{\widetilde{i}_j\}_{j\in\mathbb{N}}$, respectively, by $\mathcal{I}$ and $\widetilde{\mathcal{I}}$, we conclude that $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ have the desired properties in Case 1).

We now turn to Case 2). In this case, we define the index sets $\mathcal{I} := \{i_j\}_{j\in\mathbb{Z}}$ and $\widetilde{\mathcal{I}} := \{\widetilde{i}_j\}_{j\in\mathbb{Z}}$ as follows. For all $j \in \mathbb{Z}\setminus\mathbb{Z}_+$, since $|O_{i_0}| > 0$, we choose the indices i_j and $\widetilde{i}_j$ as in Case 1). For all $j \in \mathbb{Z}_+$, let $i_j := i_0$ and $\widetilde{i}_j := 0$. Observe that, for all $j \in \mathbb{N}$, $|O_{i_j}| = 0$. By some calculations similar to those used in Case 1), we know that $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ also have the desired properties. Thus, both claims in Cases 1) and 2) hold true.

Finally, we point out that, by following the same line of the proof of Theorem 2.6, but replacing $\{O_i\}_{i\in\mathbb{Z}}$ by $\{O_i\}_{i\in\mathcal{I}}$ defined here, we also obtain a weak atomic decomposition of $T^p(\mathbb{R}^{n+1}_+)$ with the same properties. In this case, the achieved atomic decomposition is of the following form

$$F = \sum_{i\in\mathcal{I}}\sum_{j\in\mathbb{Z}_+} \lambda_{i,j}A_{i,j},$$

here and hereafter, for notional simplicity, we denote $i_j \in \mathcal{I}$ simply by $i \in \mathcal{I}$.

Now, we establish another weak atomic decomposition of $WT^p(\mathbb{R}^{n+1}_+)$ which plays an important role in establishing the dual theory of $WH^p_L(\mathbb{R}^n)$ in Section 4.

Theorem 2.11. *Let $p \in (0, 1]$. Then, for all $F \in WT^p(\mathbb{R}^{n+1}_+)$, there exist an index set $\mathcal{I} \subset \mathbb{Z}$ and $\{A_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$ of $T^p(\mathbb{R}^{n+1}_+)$-atoms associated to balls $\{B_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$ such that*

(i)

$$F = \sum_{i\in\mathcal{I}}\sum_{j\in\Lambda_i} \lambda_{i,j}A_{i,j}$$

pointwisely almost everywhere in $\mathbb{R}^{n+1}_+$, where, for all $i \in \mathcal{I}$, $\Lambda_i \subset \mathbb{Z}_+$ is an index set depending on i and, for all $j \in \Lambda_i$, $\lambda_{i,j} := \widetilde{C}2^i|B_{i,j}|^{\frac{1}{p}}$ and $\widetilde{C}$ is a positive constant independent of F;

(ii) *for all $i \in \mathcal{I}$ and $j \in \Lambda_i$, let $r_i := \inf_{j\in\Lambda_i}\{r_{B_{i,j}}\}$ and $\widetilde{B}_{i,j} := \frac{1}{10\sqrt{n}}B_{i,j}$. Then, for all $i \in \mathcal{I}$, $r_i > 0$ and $\{\widetilde{B}_{i,j}\}_{j\in\Lambda_i}$ are mutually disjoint;*

(iii) *there exists a positive constant C, depending only on n and p, such that, for all $i \in \mathcal{I}$,*

$$\left\{\sum_{j\in\Lambda_i}\left|\lambda_{i,j}\right|^p\right\}^{\frac{1}{p}} \le C\|F\|_{WT^p(\mathbb{R}^{n+1}_+)}.$$

Proof. We first prove (i) and (ii) of Theorem 2.11. Let $\mathcal{I}$ be as in Remark 2.10, $i \in \mathcal{I}$ and O_i^* be as in the proof of Theorem 2.6. Without loss of generality, we may assume that $|O_i^*| > 0$; otherwise, we neglect the set O_i^*, since we only need the atomic decomposition to hold true almost everywhere in $\mathbb{R}_+^{n+1}$.

For all $i \in \mathcal{I}$, let $\epsilon \in (0, \infty)$ such that the *open set*

$$O_{i,\epsilon}^* := O_i^* \cup \left\{x \in (O_i^*)^{\complement} : \operatorname{dist}(x, \partial O_i^*) < \epsilon\right\} \tag{2.8}$$

satisfies $|O_{i,\epsilon}^*| < 2|O_i^*|$. Let $\{Q_{i,j}\}_{j\in\mathbb{Z}_+}$ be a Whitney decomposition of $O_{i,\epsilon}^*$ as in Lemma 2.7. Assume that $\{Q_{i,j}\}_{j\in\Lambda_i}$ is the *maximal subsequence* of $\{Q_{i,j}\}_{j\in\mathbb{Z}_+}$ such that, for all $j \in \Lambda_i$, $Q_{i,j} \cap O_i^* \neq \emptyset$, where $\Lambda_i \subset \mathbb{Z}_+$. Let $l_i := \inf\{l_{Q_{i,j}}, j \in \Lambda_i\}$, we now claim that, for all $i \in \mathcal{I}$, $l_i > 0$.

Indeed, if $Q_{i,j} \cap \partial O_i^* \neq \emptyset$, then we see that, for all $y \in \partial O_i^*$,

$$\operatorname{dist}((O_{i,\epsilon}^*)^{\complement}, y) \geq \epsilon. \tag{2.9}$$

Otherwise, if $\operatorname{dist}((O_{i,\epsilon}^*)^{\complement}, y) < \epsilon$, then there exists $x \in (O_{i,\epsilon}^*)^{\complement}$ such that $d(x, y) < \epsilon$. However, from the definition of $(O_{i,\epsilon}^*)^{\complement}$, we deduce that $d(x, y) \geq \operatorname{dist}(x, \partial O_i^*) \geq \epsilon$. This derives a contradiction. Thus, (2.9) holds true, which, together with Lemma 2.7(ii), implies that

$$l_{Q_{i,j}} \geq \frac{1}{4\sqrt{n}}\operatorname{dist}(Q_{i,j}, (O_{i,\epsilon}^*)^{\complement}) \geq \frac{\epsilon}{4\sqrt{n}}. \tag{2.10}$$

If $Q_{i,j} \cap \partial O_i^* = \emptyset$ and $Q_{i,j} \cap O_i^* \neq \emptyset$, then $Q_{i,j} \subset (O_i^*)^\circ$, where $(O_i^*)^\circ$ denotes the *interior* of O_i^*. Moreover, for all $y \in Q_{i,j}$, take $z \in (O_{i,\epsilon}^*)^{\complement}$ such that

$$\operatorname{dist}(y, (O_{i,\epsilon}^*)^{\complement}) \geq \frac{1}{2}d(y, z),$$

which, together with Lemma 2.7(ii), implies that

$$\begin{aligned} l_{Q_{i,j}} \geq& \frac{1}{4\sqrt{n}}\operatorname{dist}(y, (O_{i,\epsilon}^*)^{\complement}) \geq \frac{1}{8\sqrt{n}}d(y, z) \\ \geq& \frac{1}{8\sqrt{n}}\operatorname{dist}(O_i^*, z) \geq \frac{1}{8\sqrt{n}}\epsilon. \end{aligned} \tag{2.11}$$

Thus, combined (2.10) and (2.11), we see that, for all $j \in \Lambda_i$,

$$l_{Q_{i,j}} \geq \frac{1}{8\sqrt{n}}\epsilon \text{ and hence, for } i \in \mathcal{I},\ l_i \geq \frac{1}{8\sqrt{n}}\epsilon,$$

which shows that the above claim is true.

Now, for all $i \in \mathcal{I}$ and $j \in \Lambda_i$, let $B_{i,j}$ be the ball having the same center as $Q_{i,j}$ with the radius $5\sqrt{n}\,l_{Q_{i,j}}$. With the help of the above claim and Lemma 2.7, we conclude that, for all $i \in \mathcal{I}$, the sequence $\{B_{i,j}\}_{j\in\Lambda_i}$ of balls has the following properties:

(i) $O_i^* \subset \cup_{j\in\Lambda_i} B_{i,j}$;

(ii) $r_i := \inf\{r_{B_{i,j}}, j \in \Lambda_i\} > 0$;

(iii) let $\widetilde{B}_{i,j} := \frac{1}{10\sqrt{n}}B_{i,j}$. Then $\{\widetilde{B}_{i,j}\}_{j\in\Lambda_i}$ are mutually disjoint;

(iv) there exists $M \in (0, \infty)$, independent of $i \in \mathcal{I}$, such that $\sum_{j\in\Lambda_i}|B_{i,j}| < M|O_i^*|$.

Now, for all $i \in \mathcal{I}$ and $j \in \Lambda_i$, let

$$\Delta_{i,j} := \widehat{B}_{i,j} \bigcap \left(Q_{i,j} \times (0,\infty)\right) \bigcap \left(\widehat{O}_i^* \setminus \widehat{O}_{i+1}^*\right).$$

For all $F \in WT^p(\mathbb{R}^{n+1}_+)$, recall that

$$\operatorname{supp} F \subset \bigcup_{i\in\mathcal{I}} \bigcup_{j\in\Lambda_i} \left[\widehat{B}_{i,j} \bigcap (Q_{i,j} \times (0,\infty)) \cap (\widehat{O}_i^* \setminus \widehat{O}_{i+1}^*)\right] =: \bigcup_{i\in\mathcal{I}} \bigcup_{j\in\Lambda_i} \Delta_{i,j}.$$

Moreover, let $\lambda_{i,j} := \widetilde{C} 2^i |B_{i,j}|^{\frac{1}{p}}$ and $A_{i,j} := F\chi_{\Delta_{i,j}}/\lambda_{i,j}$, where $\widehat{O}_i^*$ denotes the tent over O_i^* and $\widetilde{C}$ a positive constant independent of F, which will be determined later. By following the same line of the proof of Theorem 2.6, if we choose $\widetilde{C}$ large enough and independent of F, we then conclude that $A_{i,j}$ is a $T^p(\mathbb{R}^{n+1}_+)$-atom and, for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,

$$F(x,t) = \sum_{i\in\mathcal{I}} \sum_{j\in\Lambda_i} \lambda_{i,j} A_{i,j}(x,t)$$

pointwisely and, moreover,

$$\sup_{i\in\mathbb{Z}} \left(\sum_{j\in\Lambda_i} |\lambda_{i,j}|^p\right)^{\frac{1}{p}} \le C\|F\|_{WT^p(\mathbb{R}^{n+1}_+)},$$

which, together with the properties of $\{B_{i,j}\}_{j\in\Lambda_i}$, completes the proof of Theorem 2.11. ☐

2.3. **The weak Hardy spaces $WH^p_L(\mathbb{R}^n)$.** In this subsection, we study the weak Hardy space $WH^p_L(\mathbb{R}^n)$. First, we recall the definition of the classical weak Hardy space from [31, 51, 52, 49]. Let $p \in (0,1]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ support in the unit ball $B(0,1)$. The *weak Hardy space* $WH^p(\mathbb{R}^n)$ is defined to be the space

$$\left\{f \in \mathcal{S}'(\mathbb{R}^n):\ \|f\|_{WH^p(\mathbb{R}^n)} := \sup_{\alpha>0} \left(\alpha^p \left|\left\{x \in \mathbb{R}^n:\ \sup_{t>0} \left|\varphi_t * f(x)\right| > \alpha\right\}\right|\right)^{1/p} < \infty\right\}.$$

Now, let L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *L-adapted non-tangential square function $S_L f$* is defined by

$$S_L f(x) := \left\{\iint_{\Gamma(x)} \left|t^{2k} L e^{-t^{2k}L} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}, \tag{2.12}$$

where $\Gamma(x)$ is as in (2.2).

Let $p \in (0,1]$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{H}^p_L(\mathbb{R}^n)$ if $S_L f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{H^p_L(\mathbb{R}^n)} := \|S_L f\|_{L^p(\mathbb{R}^n)}$. The *Hardy space $H^p_L(\mathbb{R}^n)$* associated to L is then defined to be the completion of $\mathbb{H}^p_L(\mathbb{R}^n)$ with respect to the *quasi-norm* $\|\cdot\|_{H^p_L(\mathbb{R}^n)}$ (see [12]).

Now, we introduce the notion of the weak Hardy space $WH^p_L(\mathbb{R}^n)$.

Definition 2.12. Let $p \in (0,1]$ and L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{WH}^p_L(\mathbb{R}^n)$ if $S_L f$ belongs to the weak Lebesgue space $WL^p(\mathbb{R}^n)$; moreover, define $\|f\|_{WH^p_L(\mathbb{R}^n)} := \|S_L f\|_{WL^p(\mathbb{R}^n)}$. The *weak Hardy space $WH^p_L(\mathbb{R}^n)$ associated to L* is then defined to be the completion of $\mathbb{WH}^p_L(\mathbb{R}^n)$ with respect to the *quasi-norm* $\|\cdot\|_{WH^p_L(\mathbb{R}^n)}$.

Remark 2.13. We point out that, unlike the Hardy space $H^p(\mathbb{R}^n)$, with $p \in (0, 1]$, in which the Lebesgue space $L^2(\mathbb{R}^n)$ is dense (see, for example, [52, Proposition 3.2]), the space $L^2(\mathbb{R}^n)$ is not dense in the weak Hardy space $WH^p(\mathbb{R}^n)$ in the sense of Fefferman and Soria [31] (see also a very recent work of He [37]). Thus, when $L = -\Delta$, the weak Hardy space $WH^p_{-\Delta}(\mathbb{R}^n)$ defined as in Definition 2.12 coincides with the space

$$\overline{WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)}^{\|\cdot\|_{WH^p(\mathbb{R}^n)}},$$

namely, the closure of $WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ on the quasi-norm $\|\cdot\|_{WH^p(\mathbb{R}^n)}$, which is a proper subspace of $WH^p(\mathbb{R}^n)$.

Now, let T be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. It is known that T is a special case of operators L satisfying Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. We first establish the weak atomic decomposition of the weak Hardy space $WH^p_T(\mathbb{R}^n)$.

Definition 2.14 ([38, 42])**.** Let $p \in (0, 1]$, $M \in \mathbb{N}$ and $B := B(x_B, r_B)$ be a ball with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$. A function $a \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M)_T$*-atom associated to* B if the following conditions are satisfied:

(i) there exists a function b belonging to the domain of T^M, $D(T^M)$, such that $a = T^M b$;

(ii) for all $\ell \in \{0, \ldots, M\}$, $\operatorname{supp}(T^\ell b) \subset B$;

(iii) for all $\ell \in \{0, \ldots, M\}$, $\|(r_B^2 T)^\ell b\|_{L^2(\mathbb{R}^n)} \le r_B^{2M} |B|^{\frac{1}{2}-\frac{1}{p}}$.

For all $p \in (0, 1]$ and $M \in \mathbb{N}$, let $f \in L^2(\mathbb{R}^n)$ and $\{a_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{Z}_+}$ be a sequence of $(p, 2, M)_T$-atoms associated to balls $\{B_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{Z}_+}$. The equality $f = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+} \lambda_{i,j} a_{i,j}$ holding true in $L^2(\mathbb{R}^n)$ is called a *weak atomic* $(p, 2, M)_T$*-representation of* f if

(i) $\lambda_{i,j} := \widetilde{C} 2^i |B_{i,j}|^{\frac{1}{p}}$, where $\widetilde{C}$ is a positive constant independent of f;

(ii) there exists a positive constant C_2, depending only on f, n, p, M and $\widetilde{C}$, such that

$$\sup_{i\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}_+} |\lambda_{i,j}|^p\Big)^{\frac{1}{p}} \le C_2.$$

The *weak atomic Hardy space* $WH^p_{T,\mathrm{at},M}(\mathbb{R}^n)$ is defined to be the completion of the *space*

$$\mathbb{WH}^p_{T,\mathrm{at},M}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) :\ f \text{ has a weak atomic } (p, 2, M)_T\text{-representation}\}$$

with respect to the *quasi-norm*

$$\|f\|_{WH^p_{T,\mathrm{at},M}(\mathbb{R}^n)} := \inf\Big\{\sup_{i\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}_+} |\lambda_{i,j}|^p\Big)^{\frac{1}{p}}\Big\},$$

where the infimum is taken over all the weak atomic $(p, 2, M)_T$-representations of f as above.

We have the following weak atomic characterization of $WH^p_T(\mathbb{R}^n)$.

Theorem 2.15. *Let $p \in (0, 1]$ and T be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. Assume that $M \in \mathbb{N}$ satisfies $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Then $WH_T^p(\mathbb{R}^n) = WH_{T,\mathrm{at},M}^p(\mathbb{R}^n)$ with equivalent quasi-norms.*

To prove this theorem, we need to recall some notions and known results from [38].

Let $C_3 \in [1, \infty)$. Assume that $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ is even, $\operatorname{supp}\varphi \subset (-C_3^{-1}, C_3)$, $\varphi \geq 0$ and there exists a positive constant C_4 such that, for all $t \in (-\frac{1}{2C_3}, \frac{1}{2C_3})$, $\varphi(t) \geq C_4$. Let $M \in \mathbb{N}$, Φ be the *Fourier transform* of φ and $\Psi(t) := t^{2(M+1)}\Phi(t)$ for all $t \in [0, \infty)$. For T as in Theorem 2.15, all $F \in T^2(\mathbb{R}_+^{n+1})$ and $x \in \mathbb{R}^n$, define the *operator* $\Pi_{\Psi,T}(F)(x)$ by setting

$$\Pi_{\Psi,T}(F)(x) := \int_0^\infty \Psi(t\sqrt{T})(F(\cdot, t))(x)\, \frac{dt}{t}. \tag{2.13}$$

From Fubini's theorem and the quadratic estimates, it follows that $\Pi_{\Psi,T}$ is bounded from $T^2(\mathbb{R}_+^{n+1})$ to $L^2(\mathbb{R}^n)$. By using the finite speed of the propagation of the wave equation and the Paley-Wiener theorem, Hofmann et al. [38] proved the following conclusion.

Lemma 2.16 ([38])**.** *Let $p \in (0, 1]$, $M \in \mathbb{N}$ and T be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. Assume that A is a $T^p(\mathbb{R}_+^{n+1})$-atom associated to the ball B and $\Pi_{\Psi,T}$ is as in (2.13). Then there exists a positive constant $C(M)$, independent of A, such that $[C(M)]^{-1}\Pi_{\Psi,T}(A)$ is a $(p, 2, M)_T$-atom associated to the ball $2B$.*

We also need the following *superposition principle* on the weak type estimate.

Lemma 2.17 ([59])**.** *Let $p \in (0, 1)$ and $\{f_j\}_{j\in\mathbb{Z}_+}$ be a sequence of measurable functions. If $\sum_{j\in\mathbb{Z}_+} \left|\lambda_j\right|^p < \infty$ and there exists a positive constant C such that, for all $j \in \mathbb{Z}_+$ and $\alpha \in (0,\infty)$, $|\{x \in \mathbb{R}^n : \ |f_j(x)| > \alpha\}| \leq C\alpha^{-p}$, then there exists a positive constant $\widetilde{C}$, independent of $\{\lambda_j\}_{j\in\mathbb{Z}_+}$ and $\{f_j\}_{j\in\mathbb{Z}_+}$, such that, for all $\alpha \in (0,\infty)$,*

$$\left|\left\{x \in \mathbb{R}^n : \ \Big|\sum_{j\in\mathbb{Z}_+} \lambda_j f_j(x)\Big| > \alpha\right\}\right| \leq \widetilde{C}\frac{2-p}{1-p}\alpha^{-p} \sum_{j\in\mathbb{Z}_+} |\lambda_j|^p.$$

With these preparations, we now prove Theorem 2.15.

Proof of Theorem 2.15. In order to prove Theorem 2.15, it suffices to show that

$$(WH_T^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)) = \mathbb{WH}_{T,\mathrm{at},M}^p(\mathbb{R}^n)$$

with equivalent quasi-norms. We first prove the inclusion that

$$(WH_T^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)) \subset \mathbb{WH}_{T,\mathrm{at},M}^p(\mathbb{R}^n).$$

Let $f \in WH_T^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. From its definition and the quadratic estimate, it follows that $t^2Te^{-t^2T}f \in WT^p(\mathbb{R}_+^{n+1}) \cap T^2(\mathbb{R}_+^{n+1})$. By Theorem 2.6, there exist sequences $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}\subset\mathbb{C}$ and $\{A_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ of $T^p(\mathbb{R}^{n+1}_+)$-atoms associated to the balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ such that

$$t^2Te^{-t^2T}(f)=\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+}\lambda_{i,j}A_{i,j}$$

pointwisely almost everywhere in $\mathbb{R}^{n+1}_+$, $\lambda_{i,j}=\widetilde{C}2^i|B_{i,j}|^{1/p}$ and

$$\sup_{i\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}_+}|\lambda_{i,j}|^p\Big)^{\frac{1}{p}}\lesssim\Big\|t^2Te^{-t^2T}(f)\Big\|_{WT^p(\mathbb{R}^{n+1}_+)}\sim\|f\|_{WH^p(\mathbb{R}^{n+1}_+)},\tag{2.14}$$

where $\widetilde{C}$ is a positive constant independent of f. Moreover, by the bounded H_∞ functional calculus in $L^2(\mathbb{R}^n)$, Corollary 2.9 and the fact that $\Pi_{\Psi,T}$ is bounded from $T^2(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^n)$, we conclude that there exists a constant $C(\Psi)$, depending on Ψ, such that

$$\begin{aligned}f&=C(\Psi)\int_0^\infty\Psi(t\sqrt{T})(t^2Te^{-t^2T})f\,\frac{dt}{t}\\&=C(\Psi)\int_0^\infty\Psi(t\sqrt{T})\Big(\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+}\lambda_{i,j}A_{i,j}\Big)\,\frac{dt}{t}\\&=C(\Psi)\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+}\lambda_{i,j}\int_0^\infty\Psi(t\sqrt{T})(A_{i,j})\,\frac{dt}{t},\end{aligned}\tag{2.15}$$

where the above equalities hold true in $L^2(\mathbb{R}^n)$. For all $i\in\mathbb{Z}$ and $j\in\mathbb{Z}_+$, let

$$a_{i,j}:=\int_0^\infty\Psi(t\sqrt{T})(A_{i,j})\,\frac{dt}{t}.\tag{2.16}$$

By Lemma 2.16, we see that $a_{i,j}$ is a $(p,2,M)_T$-atom associated to $\{2B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ up to a harmless positive constant multiple. Thus, we conclude that f has a weak atomic $(p,\,2,\,M)_T$-representation $\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+}\lambda_{i,j}a_{i,j}$ and $f\in\mathbb{WH}^p_{T,\mathrm{at},M}(\mathbb{R}^n)$. Moreover, from (2.14), we deduce that

$$\|f\|_{WH^p_{T,\mathrm{at},M}(\mathbb{R}^n)}\lesssim\Big(\sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}_+}|\lambda_{i,j}|^p\Big)^{\frac{1}{p}}\lesssim\|f\|_{WH^p_T(\mathbb{R}^n)},$$

which immediately implies that $(WH^p_T(\mathbb{R}^n)\cap L^2(\mathbb{R}^n))\subset\mathbb{WH}^p_{T,\mathrm{at},M}(\mathbb{R}^n)$.

Now we prove the converse, namely,

$$\mathbb{WH}^p_{T,\mathrm{at},M}(\mathbb{R}^n)\subset(WH^p_T(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)).$$

Let $f\in\mathbb{WH}^p_{T,\mathrm{at},M}(\mathbb{R}^n)$. From its definition, it follows that there exists a sequence $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ of $(p,\,2,\,M)_T$-atoms associated to the balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ such that $f=\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+}\lambda_{i,j}a_{i,j}$ in $L^2(\mathbb{R}^n)$ and, for all $i\in\mathbb{Z}$,

$$\sum_{j\in\mathbb{Z}_+}|\lambda_{i,j}|^p\lesssim\|f\|^p_{WH^p_{T,\mathrm{at},\mathrm{M}}(\mathbb{R}^n)},\tag{2.17}$$

where $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}:=\{\widetilde{C}2^i|B_{i,j}|^{1/p}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ and $\widetilde{C}$ is a positive constant independent of f.

Given $\alpha \in (0, \infty)$, let $i_0 \in \mathbb{Z}$ satisfy that $2^{i_0} \le \alpha < 2^{i_0+1}$. We then see that

$$f = \sum_{i=-\infty}^{i_0} \sum_{j\in\mathbb{Z}_+} \lambda_{i,j} a_{i,j} + \sum_{i=i_0+1}^{\infty} \sum_{j\in\mathbb{Z}_+} \cdots =: f_1 + f_2$$

holds true in $L^2(\mathbb{R}^n)$.

We first estimate f_2. Let $\widetilde{B}_{i_0} := \cup_{i=i_0+1}^{\infty} \cup_{j\in\mathbb{Z}_+} 8B_{i,j}$. From the definition of $\lambda_{i,j}$ and (2.17), we deduce that

$$\begin{aligned}\left|\widetilde{B}_{i_0}\right| &\lesssim \sum_{i=i_0+1}^{\infty} \sum_{j\in\mathbb{Z}_+} \left|B_{i,j}\right| \lesssim \sum_{i=i_0+1}^{\infty} 2^{-ip} \Big(\sum_{j\in\mathbb{Z}_+} |\lambda_{i,j}|^p \Big) \\ &\lesssim \sum_{i=i_0+1}^{\infty} 2^{-ip} \|f\|^p_{WH^p_{T,\mathrm{at},\mathrm{M}}(\mathbb{R}^n)} \lesssim \frac{1}{\alpha^p} \|f\|^p_{WH^p_{T,\mathrm{at},\mathrm{M}}(\mathbb{R}^n)}.\end{aligned} \tag{2.18}$$

Now, for $q \in (0, p)$, we write

$$f_2 = \sum_{i=i_0+1}^{\infty} \sum_{j\in\mathbb{Z}_+} \left(\widetilde{C} 2^i \left|B_{i,j}\right|^{\frac{1}{q}}\right) \left(\frac{1}{\widetilde{C}} |B_{i,j}|^{\frac{1}{p}-\frac{1}{q}} a_{i,j}\right) =: \sum_{i=i_0+1}^{\infty} \sum_{j\in\mathbb{Z}_+} \widetilde{\lambda}_{i,j} \widetilde{a}_{i,j}. \tag{2.19}$$

By (2.17) and the fact that $q \in (0, p)$, we know that

$$\begin{aligned}\sum_{i=i_0+1}^{\infty} \sum_{j\in\mathbb{Z}_+} \left|\widetilde{\lambda}_{i,j}\right|^q &\sim \sum_{i=i_0+1}^{\infty} 2^{iq} \sum_{j\in\mathbb{Z}_+} \left|B_{i,j}\right| \sim \sum_{i=i_0+1}^{\infty} 2^{i(q-p)} \Big(\sum_{j\in\mathbb{Z}_+} |\lambda_{i,j}|^p \Big) \\ &\lesssim \|f\|^p_{WH^p_{T,\mathrm{at},\mathrm{M}}(\mathbb{R}^n)} \sum_{i=i_0+1}^{\infty} 2^{i(q-p)} \lesssim 2^{i_0(q-p)} \|f\|^p_{WH^p_{T,\mathrm{at},\mathrm{M}}(\mathbb{R}^n)},\end{aligned} \tag{2.20}$$

which, combined with (2.18), (2.19) and Lemma 2.17, implies that, to show that $S_T(f_2) \in WL^p(\mathbb{R}^n)$, it suffices to prove that, for all $\alpha \in (0, \infty)$, $i \in \mathbb{Z} \cap [i_0+1, \infty)$ and $j \in \mathbb{Z}_+$,

$$\left|\left\{x \in \left(8B_{i,j}\right)^{\complement} :\ S_T(\widetilde{a}_{i,j})(x) > \alpha\right\}\right| \lesssim \frac{1}{\alpha^q}. \tag{2.21}$$

Indeed, if (2.21) is true, then, for $N_1 \in \mathbb{Z} \cap [i_0+1, \infty)$ and $N_2 \in \mathbb{Z}_+$, by Chebychev's inequality, the sub-linearity of S_T, Lemma 2.17 and the $L^2(\mathbb{R}^n)$ boundedness of S_T, we conclude that

$$\begin{aligned}\left|\left\{x \in \mathbb{R}^n : S_T\Big(f_2\Big)(x) > \alpha\right\}\right| &\le \left|\left\{x \in \mathbb{R}^n : \sum_{i=i_0+1}^{N_1} \sum_{j=0}^{N_2} \widetilde{\lambda}_{i,j} S_T\Big(\widetilde{a}_{i,j}\Big)(x) > \frac{\alpha}{2}\right\}\right| \\ &\quad + \left|\left\{x \in \mathbb{R}^n : S_T\Big(\sum_{\substack{N_1+1\le i<\infty \\ \text{or } j\ge N_2+1}} \lambda_{i,j} a_{i,j}\Big)(x) > \frac{\alpha}{2}\right\}\right| \\ &\lesssim \alpha^{-q} \sum_{i=i_0+1}^{N_1} \sum_{j=0}^{N_2} \left|\widetilde{\lambda}_{i,j}\right|^q + \alpha^{-2} \Big\| \sum_{\substack{N_1+1\le i<\infty \\ \text{or } j\ge N_2+1}} \lambda_{i,j} a_{i,j} \Big\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

where

$$\sum_{\substack{N_1+1\le i<\infty\\ \text{or } j\ge N_2+1}} := \sum_{i=N_1+1}^{\infty}\sum_{j\in\mathbb{Z}_+} + \sum_{i=i_0+1}^{\infty}\sum_{j=N_2+1}^{\infty} + \sum_{i=N_1+1}^{\infty}\sum_{j=N_2+1}^{\infty}.$$

By letting N_1 and $N_2 \to \infty$, the fact that $f_2 = \sum_{i=i_0+1}^{\infty}\sum_{j\in\mathbb{Z}_+}\lambda_{i,j}a_{i,j}$ holds true in $L^2(\mathbb{R}^n)$ and (2.20), we see that

$$\left|\left\{x\in\mathbb{R}^n:\ S_T\Big(f_2\Big)(x)>\alpha\right\}\right| \lesssim \alpha^{-q}\sum_{i=i_0+1}^{\infty}\sum_{j\in\mathbb{Z}_+}\left|\widetilde{\lambda}_{i,j}\right|^q \lesssim \frac{1}{\alpha^p}\|f\|^p_{WH^p_{T,\mathrm{at},\mathrm{M}}(\mathbb{R}^n)},$$

which is desired.

To prove (2.21), from Chebyshev's inequality and Hölder's inequality, we deduce that

$$\begin{aligned}(2.22)\qquad &\left|\left\{x\in\Big(8B_{i,j}\Big)^{\complement}:\ S_T(\widetilde{a}_{i,j})(x)>\alpha\right\}\right|\\ &\lesssim \alpha^{-q}\int_{(8B_{i,j})^{\complement}}\Big[S_T(\widetilde{a}_{i,j})(x)\Big]^q\,dx\\ &\lesssim \alpha^{-q}\sum_{l=4}^{\infty}\left\{\int_{S_l(B_{i,j})}\left|S_T(\widetilde{a}_{i,j})(x)\right|^2dx\right\}^{\frac q2}\left|S_l(B_{i,j})\right|^{1-\frac q2},\end{aligned}$$

where $S_l(B_{i,j}) := 2^lB_{i,j}\setminus(2^{l-1}B_{i,j})$ for all $l\in\mathbb{N}$. For $l\ge 4$, let

$$\mathrm{I}_{l,i,j} := \left\{\int_{S_l(B_{i,j})}\left|S_T(\widetilde{a}_{i,j})(x)\right|^2dx\right\}^{\frac q2},$$

$b_{i,j} := T^{-M}a_{i,j}$ and $\widetilde{b}_{i,j} := \left|B_{i,j}\right|^{\frac1p-\frac1q}b_{i,j}$.

By Minkowski's inequality, we write $\mathrm{I}_{l,i,j}$ into

$$\begin{aligned}\mathrm{I}_{l,i,j} &\lesssim \left\{\int_{S_l(B_{i,j})}\int_0^{r_{B_{i,j}}}\int_{|y-x|<t}\left|t^2Te^{-t^2T}\widetilde{a}_{i,j}(y,t)\right|^2\frac{dy\,dt}{t^{n+1}}\,dx\right\}^{\frac q2}\\ &\quad+\left\{\int_{S_l(B_{i,j})}\int_{r_{B_{i,j}}}^{\frac{\mathrm{dist}(x,\,B_{i,j})}{4}}\int_{|y-x|<t}\cdots\frac{dy\,dt}{t^{n+1}}\,dx\right\}^{\frac q2}\\ &\quad+\left\{\int_{S_l(B_{i,j})}\int_{\frac{\mathrm{dist}(x,\,B_{i,j})}{4}}^{\infty}\int_{|y-x|<t}\cdots\frac{dy\,dt}{t^{n+1}}\,dx\right\}^{\frac q2} =: \mathrm{J}_{l,i,j}+\mathrm{K}_{l,i,j}+\mathrm{Q}_{l,i,j}.\end{aligned}$$

To estimate $\mathrm{J}_{l,i,j}$, notice that $\mathrm{dist}(S_l(B_{i,j}),\,B_{i,j}) > 2^{l-2}r_{B_{i,j}}$, when $l\ge4$. Let

$$\mathrm{E}_{l,i,j} := \left\{x\in\mathbb{R}^n:\ \mathrm{dist}(x,\,S_l(B_{i,j})) < r_{B_{i,j}}\right\}.$$

We easily see that $\mathrm{dist}(\mathrm{E}_{l,i,j},\,B_{i,j}) > 2^{l-3}r_{B_{i,j}}$, which, together with Fubini's theorem, Lemma 2.3 and Definition 2.14, implies that there exists a positive constant $\alpha_0 > \frac nq(1-\frac q2)$ such that

$$\mathrm{J}_{l,i,j} \lesssim \left\{\int_0^{r_{B_{i,j}}}\int_{\mathrm{E}_{l,i,j}}\left|t^2Te^{-t^2T}\widetilde{a}_{i,j}(y,t)\right|^2\frac{dy\,dt}{t}\right\}^{\frac q2}$$

$$
\begin{aligned}
(2.23)\qquad &\lesssim \left\{ \int_0^{r_{B_{i,j}}} \exp\left\{ -C_1 \frac{\left[\operatorname{dist}(\mathrm{E}_{l,i,j},\, B_{i,j})\right]^2}{t^2} \right\} \frac{dt}{t} \right\}^{\frac{q}{2}} \left\| \widetilde{a}_{i,j} \right\|_{L^2(B_{i,j})}^q \\
&\lesssim 2^{-lq\alpha_0} \left| B_{i,j} \right|^{\frac{q}{2}-1} \sim 2^{-l[q\alpha_0 - n(1-\frac{q}{2})]} \left| S_l(B_{i,j}) \right|^{\frac{q}{2}-1}.
\end{aligned}
$$

To estimate $\mathrm{K}_{l,i,j}$, let $\mathrm{F}_{l,i,j} := \{x \in \mathbb{R}^n :\ \operatorname{dist}(x,\, S_l(B_{i,j})) < \frac{\operatorname{dist}(x,\, B_{i,j})}{4}\}$. It is easy to see that $\operatorname{dist}(\mathrm{F}_{l,i,j},\, B_{i,j}) > 2^{l-3} r_{B_{i,j}}$. Moreover, by Fubini's theorem, Lemma 2.3 and Definition 2.14, we know that there exists a positive constant $\alpha_1 \in (\frac{n}{q}(1-\frac{q}{2}),\, 2M)$ such that

$$
\begin{aligned}
\mathrm{K}_{l,i,j} &\lesssim \left\{ \int_{r_{B_{i,j}}}^\infty \int_{\mathrm{F}_{l,i,j}} \left| t^{2(M+1)} T^{M+1} e^{-t^2 T} \widetilde{b}_{i,j}(y) \right|^2 \frac{dy\, dt}{t^{4M+1}} \right\}^{\frac{q}{2}} \\
&\lesssim \left\{ \int_{r_{B_{i,j}}}^\infty \exp\left\{ -C_1 \frac{\left[\operatorname{dist}\left(\mathrm{F}_{l,i,j},\, B_{i,j}\right)\right]}{t^2} \right\} \frac{dy\, dt}{t^{4M+1}} \right\}^{\frac{q}{2}} \left\| \widetilde{b}_{i,j} \right\|_{L^2(B_{i,j})}^q \\
(2.24)\qquad &\lesssim \left\{ \int_{r_{B_{i,j}}}^\infty \left[\frac{t^2}{2^{2l} r_{B_{i,j}}} \right]^{\alpha_1} \frac{dt}{t^{4M+1}} \right\}^{\frac{q}{2}} r_{B_{i,j}}^{2Mq} \left| B_{i,j} \right|^{\frac{q}{2}-1} \\
&\lesssim 2^{-l[q\alpha_1 - n(1-\frac{q}{2})]} \left| S_l(B_{i,j}) \right|^{\frac{q}{2}-1}.
\end{aligned}
$$

Similar to the estimates of (2.23) and (2.24), we obtain

$$
\begin{aligned}
\mathrm{Q}_{l,i,j} &\lesssim \left\{ \int_{2^{l-2} r_{B_{i,j}}}^\infty \int_{\mathbb{R}^n} \left| (t^2 T)^{M+1} e^{-t^2 T} \widetilde{b}_{i,j}(y) \right|^2 \frac{dy\, dt}{t^{4M+1}} \right\}^{\frac{q}{2}} \\
&\lesssim \left\{ \int_{2^{l-2} r_{B_{i,j}}}^\infty \frac{dt}{t^{4M+1}} \right\}^{\frac{q}{2}} \left\| \widetilde{b}_{i,j} \right\|_{L^2(B_{i,j})}^2 \lesssim 2^{-l[2qM - n(1-\frac{q}{2})]} \left| S_l(B_{i,j}) \right|^{\frac{q}{2}-1},
\end{aligned}
$$

which, together with (2.22), (2.23) and (2.24), implies that, for all $\alpha \in (0,\, \infty)$, $i \in \mathbb{Z} \cap [i_0+1,\, \infty)$ and $j \in \mathbb{Z}_+$,

$$
\begin{aligned}
\left| \left\{ x \in \left(C_0 B_{i,j} \right)^\complement :\ S_T(\widetilde{a}_{i,j})(x) > \alpha \right\} \right| &\lesssim \alpha^{-q} \sum_{l=4}^\infty \left(\mathrm{J}_{l,i,j} + \mathrm{K}_{l,i,j} + \mathrm{Q}_{l,i,j} \right) \left| S_l(B_{i,j}) \right|^{1-\frac{q}{2}} \\
&\lesssim \frac{1}{\alpha^q}.
\end{aligned}
$$

Thus, (2.21) is true.

We now turn to the estimates of f_1. For any $r \in (1,\, 2)$, let $b \in (0,\, 1)$ such that $b < 1 - \frac{p}{r}$. By Hölder's inequality, we know that

$$
S_T(f_1) \lesssim \left\{ \sum_{i=-\infty}^{i_0} 2^{ibr'} \right\}^{\frac{1}{r'}} \left\{ \sum_{i=-\infty}^{i_0} \left[S_T\left(2^{-ib} \sum_{j \in \mathbb{Z}_+} \lambda_{i,\,j} a_{i,\,j} \right) \right]^r \right\}^{\frac{1}{r}},
$$

which, together with $\alpha \sim 2^{i_0}$, Chebyshev's inequality and the $L^r(\mathbb{R}^n)$ boundedness of S_T (which can be deduced from the interpolation of $H_T^p(\mathbb{R}^n)$; see [38, Proposition 9.5]), implies that there exists a positive constant C, independent of α, f and x, such that

$$
\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n:\ S_T(f_1)(x) > \alpha\right\}\right| \\
&\quad \le \left|\left\{x \in \mathbb{R}^n:\ \sum_{i=-\infty}^{i_0} \left[S_T\Big(2^{-ib} \sum_{j\in\mathbb{Z}_+} \lambda_{i,j} a_{i,j}\Big)(x)\right]^r > C 2^{i_0(1-b)r}\right\}\right| \\
&\quad \lesssim \frac{1}{2^{i_0(1-b)r}} \sum_{i=-\infty}^{i_0} 2^{-ibr} \int_{\mathbb{R}^n} \Big|\sum_{j\in\mathbb{Z}_+} \lambda_{i,j} a_{i,j}(x)\Big|^r\,dx \\
&\quad =: \frac{1}{2^{i_0(1-b)r}} \sum_{i=-\infty}^{i_0} 2^{-ibr} \left[\mathrm{I}_i\right]^r.
\end{aligned} \tag{2.25}
$$

Now, let $g \in L^{r'}(\mathbb{R}^n)$ satisfying $\|g\|_{L^{r'}(\mathbb{R}^n)} \le 1$ such that

$$
\int_{\mathbb{R}^n} \Big|\sum_{j\in\mathbb{Z}_+} \lambda_{i,j} a_{i,j}(x)\Big|^r\,dx \sim \Big|\int_{\mathbb{R}^n} \Big[\sum_{j\in\mathbb{Z}_+} \lambda_{i,j} a_{i,j}(x)\Big]\overline{g(x)}\,dx\Big|^r.
$$

For any $k \in \mathbb{N}$, let $S_k(B_{i,j}) := 2^k B_{i,j} \setminus 2^{k-1} B_{i,j}$ and $S_0(B_{i,j}) := B_{i,j}$. Let $\widetilde{B}_{i,j} := \frac{1}{10\sqrt{n}} B_{i,j}$. By Hölder's inequality, Definition 2.19 and the definition of the Hardy-Littlewood maximal operator $\mathcal{M}$ as in (2.4), we know that

$$
\begin{aligned}
\mathrm{I}_i &\lesssim \sum_{j\in\mathbb{Z}_+} \int_{\mathbb{R}^n} \left|\lambda_{i,j} a_{i,j}(x)\overline{g(x)}\right|\,dx \\
&\lesssim \sum_{j\in\mathbb{Z}_+} \sum_{k\in\mathbb{Z}_+} \int_{S_k(B_{i,j})} \left|\lambda_{i,j} a_{i,j}(x)\overline{g(x)}\right|\,dx \\
&\lesssim \sum_{j\in\mathbb{Z}_+} \sum_{k\in\mathbb{Z}_+} 2^i |B_{i,j}|^{\frac{1}{p}} \left\{\int_{S_k(B_{i,j})} \left|a_{i,j}(x)\right|^2 dx\right\}^{\frac12} \left\{\int_{S_k(B_{i,j})} \left|g(x)\right|^2 dx\right\}^{\frac12} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} \sum_{k\in\mathbb{Z}_+} 2^i |B_{i,j}| 2^{-k\epsilon} \left\{\frac{1}{|2^k B_{i,j}|} \int_{2^k B_{i,j}} \left|g(x)\right|^2 dx\right\}^{\frac12} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} 2^i |B_{i,j}| \inf_{x\in\widetilde{B}_{i,j}} \left\{\mathcal{M}\left(\left|g\right|^2\right)(x)\right\}^{\frac12} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} 2^i |B_{i,j}| \left\{\frac{1}{|\widetilde{B}_{i,j}|} \int_{\widetilde{B}_{i,j}} \left[\mathcal{M}\left(\left|g\right|^2\right)(x)\right]^{\frac{r'}{2}} dx\right\}^{\frac{1}{r'}} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} 2^i |B_{i,j}|^{\frac{1}{r}} \left\{\int_{\widetilde{B}_{i,j}} \left[\mathcal{M}\left(\left|g\right|^2\right)(x)\right]^{\frac{r'}{2}} dx\right\}^{\frac{1}{r'}},
\end{aligned}
$$

which, together with Hölder's inequality, the uniformly bounded overlap of $\{\widetilde{B}_{i,\,j}\}_{j\in\mathbb{Z}_+}$ on j, the $L^{r'/2}(\mathbb{R}^n)$ boundedness of $\mathcal{M}$, $r<2$ and $\|g\|_{L^{r'}(\mathbb{R}^n)}\le 1$, implies that

$$
\begin{aligned}
\mathrm{I}_i &\lesssim \Big\{\sum_{j\in\mathbb{Z}_+} 2^{ir}|B_{i,\,j}|\Big\}^{\frac{1}{r}}\Big\{\sum_{j\in\mathbb{Z}_+}\int_{\widetilde{B}_{i,\,j}}\Big[\mathcal{M}\Big(|g|^2\Big)(x)\Big]^{\frac{r'}{2}}dx\Big\}^{\frac{1}{r'}} \\
&\lesssim \Big\{\sum_{j\in\mathbb{Z}_+} 2^{i(r-p)}|\lambda_{i,\,j}|^p\Big\}^{\frac{1}{r}}.
\end{aligned}
\tag{2.26}
$$

Thus, by (2.25), (2.26), $b\in(0,\,\frac{r-p}{r})$ and $2^{i_0}\sim\alpha$, we conclude that

$$
\begin{aligned}
\Big|\Big\{x\in\mathbb{R}^n:\ S_T(f_1)(x)>\alpha\Big\}\Big| &\lesssim \frac{1}{2^{i_0(1-b)r}}\sum_{i=-\infty}^{i_0}2^{-ibr}\Big[\mathrm{I}_i\Big]^r \\
&\lesssim \frac{1}{2^{i_0(1-b)r}}\sum_{i=-\infty}^{i_0}2^{-ibr}\Big\{\sum_{j\in\mathbb{Z}_+}2^{i(r-p)}|\lambda_{i,\,j}|^p\Big\} \\
&\lesssim \frac{1}{2^{i_0(1-b)r}}\sum_{i=-\infty}^{i_0}2^{i(r-p-br)}\|f\|^p_{WH^p_{T,\mathrm{mol},\epsilon,\mathrm{M}}(\mathbb{R}^n)} \\
&\lesssim \frac{1}{\alpha^p}\|f\|^p_{WH^p_{T,\mathrm{mol},\epsilon,\mathrm{M}}(\mathbb{R}^n)},
\end{aligned}
$$

which shows that $f_1\in WH^p_T(\mathbb{R}^n)$ and

$$
\|f_1\|_{WH^p_T(\mathbb{R}^n)}\lesssim\|f\|^p_{WH^p_{T,\mathrm{mol},\epsilon,\mathrm{M}}(\mathbb{R}^n)}.
$$

Combining the estimates for f_1 and f_2, we then complete the proof of Theorem 2.15. □

Remark 2.18. Observe that, in the proof of Theorem 2.15, if we use Theorem 2.11 to replace Theorem 2.6 in the argument above (2.14), then, for all $f\in WH^p_T(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, we obtain a weak atomic decomposition of f of the form $f=\sum_{i\in\mathcal{I},j\in\Lambda_i}\lambda_{i,j}a_{i,j}$ in $L^2(\mathbb{R}^n)$, where the index sets $\mathcal{I}$ and Λ_i are as in Theorem 2.11, $\{a_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$ is a sequence of $(p,\,2,\,M)_T$-atoms associated to balls $\{B_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$ and $\lambda_{i,j}:=\widetilde{C}2^i|B_{i,j}|^{\frac{1}{p}}$, with $\widetilde{C}$ being a positive constant independent of f, satisfies

$$
\sup_{i\in\mathcal{I}}\Big(\sum_{j\in\Lambda_i}|\lambda_{i,j}|^p\Big)^{\frac{1}{p}}\le C\|f\|_{WH^p_T(\mathbb{R}^n)},
$$

where C is a positive constant independent of f.

Now, we try to establish the molecular characterization of $WH^p_L(\mathbb{R}^n)$. We first recall the notion of $(p,\,\epsilon,\,M)_L$-molecules.

Definition 2.19 ([12])**.** Let $k\in\mathbb{N}$, $p\in(0,\,1]$, $\epsilon\in(0,\,\infty)$, $M\in\mathbb{N}$ and L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. A function $m\in L^2(\mathbb{R}^n)$ is called a $(p,\,\epsilon,\,M)_L$-*molecule* if there exists a ball $B:=B(x_B,\,r_B)$, with $x_B\in\mathbb{R}^n$ and $r_B\in(0,\,\infty)$, such that

(i) for each $\ell\in\{1,\,\dots,\,M\}$, m belongs to the range of L^ℓ in $L^2(\mathbb{R}^n)$;

(ii) for all $i \in \mathbb{Z}_+$ and $\ell \in \{0, \ldots, M\}$,

$$\left\| \left(r_B^{-2k} L^{-1} \right)^{\ell} m \right\|_{L^2(S_i(B))} \le (2^i r_B)^{n(\frac{1}{2}-\frac{1}{p})} 2^{-i\epsilon}.$$

Definition 2.20. Let $f \in L^2(\mathbb{R}^n)$, $\epsilon \in (0, \infty)$, $M \in \mathbb{Z}_+$, $p \in (0, 1]$ and L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. Assume that $\{m_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ is a sequence of $(p, \epsilon, M)_L$-molecules associated to balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ and $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+} \subset \mathbb{C}$ satisfying the conditions that

(i) for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, $\lambda_{i,j} := \widetilde{C} 2^i |B_{i,j}|^{\frac{1}{p}}$, $\widetilde{C}$ is a positive constant independent of f;

(ii) there exists a positive constant C_5, depending only on f, n, p, ϵ and M, such that

$$\sup_{i\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}_+} \Big|\lambda_{i,j}\Big|^p \Big)^{\frac{1}{p}} \le C_5.$$

Then

$$f = \sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+} \lambda_{i,j} m_{i,j}$$

is called a *weak molecular* $(p, \epsilon, M)_L$*-representation* of f if $f = \sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+} \lambda_{i,j} m_{i,j}$ holds true in $L^2(\mathbb{R}^n)$. The *weak molecular Hardy space* $WH^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)$ is then defined to be the completion of the *space*

$$\mathbb{WH}^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n) := \{f :\ f \ \text{ has a weak molecular } (p, \epsilon, M)_L\text{-representation}\}$$

with respect to the *quasi-norm*

$$\|f\|_{WH^p_{L,\mathrm{mol},\epsilon,M}(\mathbb{R}^n)} := \inf \left\{ \sup_{i\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}_+} |\lambda_{i,j}|^p \Big)^{1/p} : \quad f = \sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+} \lambda_{i,j} m_{i,j} \text{ is a weak} \right.$$
$$\left. \text{molecular } (p, \epsilon, M)_L\text{-representation} \right\},$$

where the infimum is taken over all the weak molecular $(p, \epsilon, M)_L$-representations of f as above.

We also have the following weak molecular characterization of $WH^p_L(\mathbb{R}^n)$.

Theorem 2.21. *Let* $p \in (0, 1]$, $k \in \mathbb{N}$, $\epsilon \in (0, \infty)$, $M \in \mathbb{Z}_+$ *satisfy* $M > \frac{n}{2k}(\frac{1}{p} - \frac{1}{2})$ *and* L *Assumptions* $(\mathcal{L})_1$, $(\mathcal{L})_2$ *and* $(\mathcal{L})_4$. *Then* $WH^p_L(\mathbb{R}^n) = WH^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

To prove Theorem 2.21, we need the following lemma from [10, Propostion 2.13].

Lemma 2.22 ([10])**.** *Let* L *satisfy Assumptions* $(\mathcal{L})_1$, $(\mathcal{L})_2$ *and* $(\mathcal{L})_4$, *and* $(p_-(L), p_+(L))$ *be the range of exponents* $p \in [1, \infty]$ *for which the holomorphic semigroup* $\{e^{-tL}\}_{t>0}$ *is bounded on* $L^p(\mathbb{R}^n)$. *Then, for all* $p_-(L) < p \le q < p_+(L)$, S_L *is bounded on* $L^p(\mathbb{R}^n)$.

We now prove Theorem 2.21.

Proof of Theorem 2.21. To prove this theorem, it suffices to prove

$$(WH_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)) = \mathbb{WH}^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)$$

with equivalent quasi-norms. The inclusion that $(WH_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)) \subset \mathbb{WH}^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)$ follows from a similar argument to the corresponding part of the proof of Theorem 2.15. We only remark that, in this case, the operator $\Pi_{\Psi,T}$ defined in (2.13) is replaced by a new *operator* $\Pi_{L,M}$ defined by setting, for all $F \in T^2(\mathbb{R}^{n+1}_+)$ and $x \in \mathbb{R}^n$,

$$\Pi_{L,M}(F)(x) := \int_0^\infty (t^{2k}L)^M e^{-t^{2k}L}(F(\cdot,t))(x)\,\frac{dt}{t}, \tag{2.27}$$

where k is as in Assumption $(\mathcal{L})_3$. It is known, from [12, Lemma 4.2(ii)], that $\Pi_{L,M}$ maps each $T^p(\mathbb{R}^{n+1}_+)$-atom into a $(p,\,\epsilon,\,M)_L$-molecule up to a harmless positive constant multiple.

Although the proof of the inclusion $\mathbb{WH}^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n) \subset (WH_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$ is also similar to the corresponding part of Theorem 2.15, in this case, we need more careful calculations since the lack of the support condition for the molecules. Let $f \in \mathbb{WH}^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)$. From Definition 2.20, it follows that f has a weak molecular $(p,\,\epsilon,\,M)_L$-representation $f = \sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+}\lambda_{i,j}m_{i,j}$, where $\{m_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ is a sequence of $(p,\,\epsilon,\,M)_L$-molecules associated to the balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+} := \{B(x_{B_{i,j}},\,r_{B_{i,j}})\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$, $\lambda_{i,j} := \widetilde{C}2^i|B_{i,j}|^{1/p}$, with $\widetilde{C}$ being a positive constant independent of f, and $\sup_{i\in\mathbb{Z}}(\sum_{j\in\mathbb{Z}_+}|\lambda_{i,j}|^p)^{1/p} \lesssim \|f\|_{WH^p_{L,\mathrm{mol},\epsilon,\mathrm{M}}}(\mathbb{R}^n)$.

For all $\alpha \in (0,\,\infty)$, let $i_0 \in \mathbb{Z}$ satisfy $2^{i_0} \le \alpha < 2^{i_0+1}$. We write f into

$$f = \sum_{i=-\infty}^{i_0}\sum_{j\in\mathbb{Z}_+}\lambda_{i,j}m_{i,j} + \sum_{i=i_0+1}^{\infty}\sum_{j\in\mathbb{Z}_+}\cdots =: f_1 + f_2.$$

As in the proof of Theorem 2.15, we first estimate f_2. For fixed $q \in (0,\,p)$, by Definition 2.20, we write

$$f_1 = \sum_{i=-\infty}^{i_0}\sum_{j\in\mathbb{Z}_+}\left(\widetilde{C}2^i\left|B_{i,j}\right|^{\frac{1}{q}}\right)\left(\frac{1}{\widetilde{C}}\left|B_{i,j}\right|^{\frac{1}{p}-\frac{1}{q}}m_{i,j}\right) =: \sum_{i=-\infty}^{i_0}\sum_{j\in\mathbb{Z}_+}\widetilde{\lambda}_{i,j}\widetilde{m}_{i,j}. \tag{2.28}$$

Moreover, from Definition 2.20 and the fact $q < p$, we deduce that

$$\begin{aligned}\sum_{i=i_0+1}^{\infty}\sum_{j\in\mathbb{Z}_+}|\widetilde{\lambda}_{i,j}|^q &\lesssim \sum_{i=i_0+1}^{\infty}2^{iq}\sum_{j\in\mathbb{Z}_+}|B_{i,j}| \lesssim \sum_{i=i_0+1}^{\infty}2^{i(q-p)}\sum_{j\in\mathbb{Z}_+}|\lambda_{i,j}|^p\\
&\lesssim \|f\|^p_{WH^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)}\sum_{i=i_0+1}^{\infty}2^{i(q-p)}\\
&\lesssim 2^{i_0(q-p)}\|f\|^p_{WH^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)},\end{aligned}$$

which, together with Lemma 2.17, implies that, to show that $S_L(f_2) \in WL^p(\mathbb{R}^n)$, it suffices to show that, for all $\alpha \in (0,\,\infty)$, $i \in \mathbb{Z}\cap[i_0+1,\,\infty)$ and $j \in \mathbb{Z}_+$,

$$\left|\left\{x \in \mathbb{R}^n:\ S_L(\widetilde{m}_{i,j})(x) > \alpha\right\}\right| \lesssim \frac{1}{\alpha^q}. \tag{2.29}$$

To prove (2.29), by Chebyshev's inequality and Hölder's inequality, we write

$$\left|\left\{x\in\mathbb{R}^n:\ S_L(\widetilde{m}_{i,j})(x)>\alpha\right\}\right| \lesssim \sum_{l=0}^{\infty} 2^{-i_0 q}\left\|S_L(\widetilde{m}_{i,j})\right\|^q_{L^2(S_l(B_{i,j}))}\left|S_l(B_{i,j})\right|^{1-\frac{q}{2}}. \tag{2.30}$$

For $l\in\{0,\ldots,4\}$, by Fubini's theorem, the $L^2(\mathbb{R}^n)$ boundedness of S_L, Definition 2.19 and (2.28), we conclude that

$$\left\|S_L(\widetilde{m}_{i,j})\right\|_{L^2(S_l(B_{i,j}))} \lesssim \left\|S_L(\widetilde{m}_{i,j})\right\|_{L^2(\mathbb{R}^n)} \lesssim \left\|\widetilde{m}_{i,j}\right\|_{L^2(\mathbb{R}^n)} \lesssim |B_{i,j}|^{\frac12-\frac1q}. \tag{2.31}$$

For $l\ge 5$, let

$$\widetilde{\mathrm{J}}_{l,i,j} := \left\{\int_{S_l(B_{i,j})}\left[\int_0^{r_{B_{i,j}}}\int_{|y-x|<t}\left|t^{2k}Le^{-t^{2k}L}\widetilde{m}_{i,j}(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right]dx\right\}^{\frac q2},$$

$$\widetilde{\mathrm{K}}_{l,i,j} := \left\{\int_{S_l(B_{i,j})}\left[\int_{r_{B_{i,j}}}^{\mathrm{dist}(x,B_{i,j})/4}\int_{|y-x|<t}\left|(t^{2k}L)^{M+1}e^{-t^{2k}L}\right.\right.\right.$$
$$\left.\left.\left.\circ\,(L^{-M}\widetilde{m}_{i,j})(y)\right|^2\frac{dy\,dt}{t^{4kM+n+1}}\right]dx\right\}^{\frac q2}$$

and

$$\widetilde{\mathrm{Q}}_{l,i,j} := \left\{\int_{S_l(B_{i,j})}\left[\int_{\mathrm{dist}(x,B_{i,j})/4}^{\infty}\int_{|y-x|<t}\left|(t^{2k}L)^{M+1}e^{-t^{2k}L}\right.\right.\right.$$
$$\left.\left.\left.\circ\,L^{-M}\widetilde{m}_{i,j}(y)\right|^2\frac{dy\,dt}{t^{4kM+n+1}}\right]dx\right\}^{\frac q2}.$$

To estimate $\widetilde{\mathrm{J}}_{l,i,j}$, let $\widetilde{\mathrm{E}}_{l,i,j}:=\{x\in\mathbb{R}^n:\ \mathrm{dist}(x,\,S_l(B_{i,j}))<r_{B_{i,j}}\}$ and

$$\widetilde{\mathrm{G}}_{l,i,j}:=\{x\in\mathbb{R}^n:\ \mathrm{dist}(x,\,\widetilde{\mathrm{E}}_{l,i,j})<2^{l-3}r_{B_{i,j}}\}.$$

It is easy to see that $\mathrm{dist}\,(\mathbb{R}^n\setminus\widetilde{\mathrm{G}}_{l,i,j},\widetilde{\mathrm{E}}_{l,i,j})>2^{l-4}r_{B_{i,j}}$. Moreover, by Fubini's theorem, the $L^2(\mathbb{R}^n)$ boundedness of S_L, Assumption $(\mathcal{L})_3$, Definition 2.19 and (2.28), we see that there exists a positive constant $\alpha_2\in(n(\frac1q-\frac12),\,\infty)$ such that

$$\begin{aligned}
\widetilde{\mathrm{J}}_{l,i,j} &\lesssim \left\{\int_0^{r_{B_{i,j}}}\int_{\widetilde{\mathrm{E}}_{l,i,j}}\left|t^{2k}Le^{-t^{2k}L}\left[\chi_{\widetilde{\mathrm{G}}_{l,i,j}}+\chi_{\mathbb{R}^n\setminus\widetilde{\mathrm{G}}_{l,i,j}}\right]\widetilde{m}_{i,j}(y)\right|^2\frac{dy\,dt}{t}\right\}^{\frac q2}\\
&\lesssim \left|B_{i,j}\right|^{\frac qp-1}\left\|m_{i,j}\right\|^q_{L^2(\widetilde{\mathrm{G}}_{l,i,j})}+\left\|m_{i,j}\right\|^q_{L^2(\mathbb{R}^n)}\left|B_{i,j}\right|^{\frac qp-1}\\
&\quad\times\left[\int_0^{r_{B_{i,j}}}\exp\left\{-C_1\frac{[\mathrm{dist}(\mathbb{R}^n\setminus\widetilde{\mathrm{G}}_{l,i,j},\,\widetilde{\mathrm{E}}_{l,i,j})]^{2k/(2k-1)}}{t^{2k/(2k-1)}}\right\}\frac{dt}{t}\right]^{\frac q2}\\
&\lesssim \left\{2^{-l[\epsilon q-n(\frac qp-1)]}+2^{-l[q\alpha_2-n(1-\frac q2)]}\right\}\left|2^lB_{i,j}\right|^{\frac q2-1}\lesssim 2^{-l\epsilon_0}\left|2^lB_{i,j}\right|^{\frac q2-1},
\end{aligned}\tag{2.32}$$

where $\epsilon_0:=\max\{\epsilon q-n(\frac qp-1),\,q\alpha_2-n(1-\frac q2)\}$.

The estimates for $\widetilde{\mathrm{K}}_{l,i,j}$ and $\widetilde{\mathrm{Q}}_{l,i,j}$ are deduced from a way similar to that of $\widetilde{\mathrm{J}}_{l,i,j}$. We omit the details and only point out that, to obtain the needed convergence, we need $M > \frac{n}{4k}(\frac{1}{q} - \frac{1}{2})$, which, together with (2.30), (2.31) and (2.32), implies that (2.29). Thus,

$$\left|\left\{x \in \mathbb{R}^n : \ S_L(f_2)(x) > \alpha\right\}\right| \lesssim \frac{1}{\alpha^p}\|f\|^p_{WH^p_{L,\mathrm{mol},\epsilon,\mathrm{M}}(\mathbb{R}^n)}.$$

We now turn to the estimates of f_1. Let $(p_-(L),\, p_+(L))$ be the interior of the maximal interval of the exponents p such that $\{e^{-tL}\}_{t>0}$ is $L^p(\mathbb{R}^n)$ bounded. For any $r \in (p_-(L),\, 2)$, let $a \in (0,\, 1)$ such that $a < 1 - \frac{p}{r}$. By Hölder's inequality, we know that

$$S_L(f_1) \lesssim \left\{\sum_{i=-\infty}^{i_0} 2^{iar'}\right\}^{\frac{1}{r'}} \left\{\sum_{i=-\infty}^{i_0} \left[S_L\Big(2^{-ia}\sum_{j\in\mathbb{Z}_+} \lambda_{i,\,j} m_{i,\,j}\Big)\right]^r\right\}^{\frac{1}{r}},$$

which, together with $\alpha \sim 2^{i_0}$, Chebyshev's inequality and the $L^r(\mathbb{R}^n)$ boundedness of S_L (see Lemma 2.22), implies that there exists a positive constant C such that

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : \ S_L(f_1)(x) > \alpha\right\}\right| \\
&\quad\le \left|\left\{x \in \mathbb{R}^n : \sum_{i=-\infty}^{i_0} \left[S_L\Big(2^{-ia}\sum_{j\in\mathbb{Z}_+} \lambda_{i,\,j} m_{i,\,j}\Big)(x)\right]^r > C2^{i_0(1-a)r}\right\}\right| \\
&\quad\lesssim \frac{1}{2^{i_0(1-a)r}} \sum_{i=-\infty}^{i_0} 2^{-iar} \int_{\mathbb{R}^n} \Big|\sum_{j\in\mathbb{Z}_+} \lambda_{i,\,j} m_{i,\,j}(x)\Big|^r \,dx \qquad (2.33)\\
&\quad=: \frac{1}{2^{i_0(1-a)r}} \sum_{i=-\infty}^{i_0} 2^{-iar} \left[\mathrm{I}_i\right]^r.
\end{aligned}$$

The estimate for I_i is similar to that of I_i in the proof of Theorem 2.15. We also obtain

$$\mathrm{I}_i \lesssim \left\{\sum_{j\in\mathbb{Z}_+} 2^{i(r-p)} |\lambda_{i,\,j}|^p\right\}^{\frac{1}{r}}. \tag{2.34}$$

Thus, by (2.33), (2.34), $a \in (0,\, \frac{r-p}{r})$ and $2^{i_0} \sim \alpha$, we conclude that

$$\begin{aligned}
\left|\left\{x \in \mathbb{R}^n : \ S_L(f_1)(x) > \alpha\right\}\right| &\lesssim \frac{1}{2^{i_0(1-a)r}} \sum_{i=-\infty}^{i_0} 2^{-iar}\left[\mathrm{I}_i\right]^r \\
&\lesssim \frac{1}{2^{i_0(1-a)r}} \sum_{i=-\infty}^{i_0} 2^{-iar}\Big\{\sum_{j\in\mathbb{Z}_+} 2^{i(r-p)}|\lambda_{i,\,j}|^p\Big\} \\
&\lesssim \frac{1}{2^{i_0(1-a)r}} \sum_{i=-\infty}^{i_0} 2^{i(r-p-ar)} \|f\|^p_{WH^p_{L,\mathrm{mol},\epsilon,\mathrm{M}}(\mathbb{R}^n)} \\
&\lesssim \frac{1}{\alpha^p}\|f\|^p_{WH^p_{L,\mathrm{mol},\epsilon,\mathrm{M}}(\mathbb{R}^n)},
\end{aligned}$$

which shows that $f_1 \in WH^p_L(\mathbb{R}^n)$ and

$$\|f_1\|_{WH^p_L(\mathbb{R}^n)} \lesssim \|f\|^p_{WH^p_{L,\mathrm{mol},\epsilon,\mathrm{M}}(\mathbb{R}^n)}.$$

Combining the estimates of f_1 and f_2, we then complete the proof of Theorem 2.21. $\square$

Remark 2.23. (i) As was observed in the proof of Theorem 2.11, we know that Assumption $(\mathcal{L})_4$ is needed only when proving $WH^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n) \subset WH^p_L(\mathbb{R}^n)$. Thus, if L only satisfies Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$, then $WH^p_L(\mathbb{R}^n) \subset WH^p_{L,\,\mathrm{mol},\,\epsilon,\,M}(\mathbb{R}^n)$.

(ii) Let $(p_-(L),\, p_+(L))$ be the *interval of the exponents* p for which the semigroup $\{e^{-tL}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$. Assume that $q \in (p_-(L),\, p_+(L))$. Similar to the notion of $(p,\, \epsilon,\, M)_L$-molecules as in Definition 2.19, we also define the $(p,\, q,\, \epsilon,\, M)_L$-*molecule* as $m \in L^q(\mathbb{R}^n)$ belonging to the range of L^ℓ for all $\ell \in \{0,\, \dots,\, M\}$ and satisfying that there exist a ball $B := (x_B,\, r_B)$, with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\, \infty)$, and a positive constant C such that, for all $i \in \mathbb{Z}_+$,

$$\left\|\left(r_B^{2k} L\right)^{-\ell} m\right\|_{L^q(S_i(B))} \le C 2^{-i\epsilon} \left|S_i(B)\right|^{n(\frac{1}{q}-\frac{1}{p})}.$$

Moreover, the corresponding *weak molecular Hardy space* $WH^p_{L,\mathrm{mol},q,\epsilon,M}(\mathbb{R}^n)$ can be defined analogously to Definition 2.20.

Assume further that L satisfies Assumption $(\mathcal{L})_4$. By using the method similar to that used in the proofs of [44, Proposition 4.2] and [10, Theorem 2.23], we can also prove the equivalence between $WH^p_L(\mathbb{R}^n)$ and the molecular weak Hardy space $WH^p_{L,\mathrm{mol},q,\epsilon,M}(\mathbb{R}^n)$. Recall that, in [10, Proposition 2.10] (see also Proposition 2.2), it was proved that, if L is the $2k$-order divergence form homogeneous elliptic operator as in (1.1), then L satisfies Assumption $(\mathcal{L})_4$.

Let L be as in (1.1). Applying the weak molecular characterization, we now study the boundedness of the associated Riesz transform $\nabla^k L^{-1/2}$ and the fractional power $L^{-\alpha/(2k)}$ as follows.

Theorem 2.24. *Let $k \in \mathbb{N}$ and L be as in* (1.1). *Then, for all $p \in (\frac{n}{n+k},\, 1]$, the Riesz transform $\nabla^k L^{-1/2}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to $WH^p(\mathbb{R}^n)$.*

Proof. Let $f \in WH^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. From Theorem 2.21, we deduce that there exist sequences $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+} \subset \mathbb{C}$ and $\{m_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$ of $(p,\, \epsilon,\, M)_L$-molecules such that

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}_+} \lambda_{i,j} m_{i,j}$$

in $L^2(\mathbb{R}^n)$ and

$$\sup_{i\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}_+} \left|\lambda_{i,j}\right|^p \Big)^{1/p} \sim \|f\|_{WH^p_L(\mathbb{R}^n)}.$$

By the proof of [12, Theorem 6.2], we know that, for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, $\nabla^k L^{-1/2}(m_{i,j})$ is a classical $H^p(\mathbb{R}^n)$-molecule up to a harmless positive constant multiple. From this and Remark 2.13, together with Theorem 2.21 in the case $L = -\Delta$, it follows that $\nabla^k L^{-1/2} f \in WH^p(\mathbb{R}^n)$ and

$$\left\|\nabla^k L^{-1/2}(f)\right\|_{WH^p(\mathbb{R}^n)} \lesssim \sup_{i\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}_+} \left|\lambda_{i,j}\right|^p \Big)^{1/p} \sim \|f\|_{WH^p_L(\mathbb{R}^n)},$$

which, together with a density argument, then completes the proof of Theorem 2.24. □

Theorem 2.25. *Let $k \in \mathbb{N}$ and L be as in (1.1). Then, for all $0 < p < r \le 1$ and $\alpha = n(\frac{1}{p} - \frac{1}{r})$, the fractional power $L^{-\alpha/(2k)}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to $WH^r_L(\mathbb{R}^n)$.*

Proof. Similar to the proof of [44, Theorems 7.2 and 7.3], we know that $L^{-\alpha/(2k)}$ maps each $(p, \epsilon, M)_L$-*molecule* to a $(r, q, \epsilon, M)_L$-*molecule* with $\alpha = n(\frac{1}{2} - \frac{1}{q})$, up to a harmless positive constant multiple. This, together with the fact that $L^{-\alpha/(2k)}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ (see [10, Lemma 3.10]) and Remark 2.23, then finishes the proof of Theorem 2.25. □

Remark 2.26. (i) The boundedness of the Riesz transform $\nabla^k L^{-1/2}$, for $k \in \mathbb{N}$ and L as in (1.1), on the Hardy space $H^p_L(\mathbb{R}^n)$ associated to L is known. Indeed, it was proved in [12, 13] that, for all $p \in (\frac{n}{n+k}, 1]$, $\nabla^k L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ and, at the endpoint case $p = \frac{n}{n+k}$, $\nabla^k L^{-1/2}$ is bounded from $H^{n/(n+k)}_L(\mathbb{R}^n)$ to the classical weak Hardy space $WH^{n/(n+k)}(\mathbb{R}^n)$.

(ii) The boundedness of the fractional power $L^{-\alpha/(2k)}$, for $k \in \mathbb{N}$ and L as in (1.1), on the Hardy space $H^p_L(\mathbb{R}^n)$ associated to L is also well known. Indeed, it was proved that, for all $0 < p < r \le 1$ and $\alpha = n(\frac{1}{p} - \frac{1}{r})$, $L^{-\alpha/(2k)}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $H^r_L(\mathbb{R}^n)$ (see [10, 41, 44]).

3. The real interpolation of intersections

In this section, we establish a real interpolation theorem on the weak Hardy spaces $WH^p_L(\mathbb{R}^n)$ by showing that $L^2(\mathbb{R}^n) \cap WH^p_L(\mathbb{R}^n)$ is an intermediate space between the spaces $L^2(\mathbb{R}^n) \cap H^p_L(\mathbb{R}^n)$ for different $p \in (0, 1]$. To this end, we first recall some basic results on the real interpolation (see [7, 62] for more details).

Let (X_0, X_1) be a *quasi-normed couple*, namely, X_0 and X_1 are two quasi-normed spaces which are linearly and continuously imbedded in some Hausdorff topological vector space X. Recall that, for any $f \in X_0 + X_1$ and $t \in (0, \infty)$, *Peetre's K-functional* $K(t, f; X_0, X_1)$ is defined by setting,

$$K(t, f; X_0, X_1) := \inf\left\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : \ f = f_0 + f_1, f_0 \in X_0, \ f_1 \in X_1\right\}.$$

Then, for all $\theta \in (0, 1)$ and $q \in [1, \infty]$, the *real interpolation space* $(X_0, X_1)_{\theta, q}$ is defined to be all $f \in X_0 + X_1$ such that, for $q \in [1, \infty)$,

$$\|f\|_{\theta, q} := \left\{\int_0^\infty \left[t^{-\theta} K(t, f; X_0, X_1)\right]^q \frac{dt}{t}\right\}^{1/q} < \infty \tag{3.1}$$

and

$$\|f\|_{\theta, \infty} := \sup_{(x, t) \in \mathbb{R}^{n+1}_+} \left[t^{-\theta} K(t, f; X_0, X_1)\right] < \infty. \tag{3.2}$$

Definition 3.1. Let X be a quasi-Banach space whose elements are measurable functions. The space X is said to have the *lattice property* if, for any $g \in X$ and any measurable function f satisfying $|f| \le |g|$, then $f \in X$.

Krugljak et al. [47] proved the following interesting result on the problem of interpolation of intersections in the case of Banach spaces.

Proposition 3.2 ([47])**.** *Let X_0, X_1 and X be quasi-Banach spaces whose elements are measurable functions and have the lattice property. Then, for all $\theta \in (0, 1)$ and $q \in [1, \infty]$,*

$$X \cap (X_0, X_1)_{\theta, q} = (X \cap X_0, X \cap X_1)_{\theta, q},$$

where, for any two quasi-Banach spaces Y and Y_1, the quasi-norm in $Y \cap Y_1$ is just the restriction of the quasi-norm from Y_1.

Proof. Recall that Proposition 3.2 in the case of Banach spaces was proved in [47]. To make it still be valid in the case of quasi-Banach spaces, we also give a proof based on some ideas from [47] with some details.

The inclusion that

$$(X \cap X_0, X \cap X_1)_{\theta, q} \subset X \cap (X_0, X_1)_{\theta, q}$$

follows immediately from the fact that, for all $f \in (X \cap X_0, X \cap X_1)_{\theta, q}$,

$$K(t, f; X_0, X_1) \le K(t, f; X \cap X_0, X \cap X_1)$$

and (3.1).

We now turn to the proof of the converse inclusion. Let $f \in X \cap (X_0, X_1)_{\theta, q}$. By (3.1), it suffices to show that, for all $t \in (0, \infty)$,

$$K(t, f; X \cap X_0, X \cap X_1) \lesssim K(t, f; X_0, X_1). \tag{3.3}$$

To prove (3.3), we make the claim: For any $g \in X \cap (X_0, X_1)_{\theta, q}$ and $t \in (0, \infty)$,

$$K(t, g; X_0, X_1) = \inf \Big\{ \|g_0\|_{X_0} + t\|g_1\|_{X_1} : \ g = g_0 + g_1, \ g_0 \in X_0, \\ |g_0| \le |g|, \, g_1 \in X_1, \, |g_1| \le |g| \Big\}. \tag{3.4}$$

Indeed, let $g = \widetilde{g}_0 + \widetilde{g}_1$ be any decomposition of g satisfying $\widetilde{g}_0 \in X_0$ and $\widetilde{g}_1 \in X_1$. If $|\widetilde{g}_0| > |g|$, then using the lattice property of X_0, we know $g \in X_0$. Thus, $g = g + 0$ is also a decomposition of g with $g \in X_0$ and $0 \in X_1$. Moreover, for all $t \in (0, \infty)$,

$$\|g\|_{X_0} + t\|0\|_{X_1} < \|\widetilde{g}_0\|_{X_0} + t\|\widetilde{g}_1\|_{X_1},$$

which, together with the definition of Peetre's K-functional, shows (3.4) is true. This immediately proves the above claim.

We now continue the proof of (3.3) by estimating $K(t, f; X_0, X_1)$. Let $f = f_0 + f_1$ be any decomposition of f satisfying $f_0 \in X_0$ and $f_1 \in X_1$. By the above claim, we may assume that $|f_0| \le |f|$ and $|f_1| \le |f|$, which, together with the lattice property of X, implies that $f_0 \in X_0 \cap X$ and $f_1 \in X_1 \cap X$. This, together with the definition of Peetre's K-functional, shows (3.3) holds. Thus, $(X \cap X_0, X \cap X_1)_{\theta, q} \subset X \cap (X_0, X_1)_{\theta, q}$, which completes the proof of Proposition 3.2. □

Remark 3.3. Let X_0, X_1 and X be some quasi-Banach spaces. The *problem of interpolation of intersections* asks the question that, under which conditions, does we have the equality

$$X \cap (X_0, X_1)_{\theta, q} = (X \cap X_0, X \cap X_1)_{\theta, q}.$$

The answer for the above problem is still unknown in the general case (see [47] and the references cited therein for more details). Proposition 3.2 shows that, if X_0, X_1 and X consist of measurable functions and have the lattice property, then the problem of interpolation of intersections for X_1, X_2 and X has a positive answer.

Remark 3.4. For all $p \in (0, \infty)$, let $T^p(\mathbb{R}^{n+1}_+)$ be the tent space as in (2.3). Observe that, for all $0 < p_0 < p_1 < \infty$, $T^{p_0}(\mathbb{R}^{n+1}_+)$, $T^{p_1}(\mathbb{R}^{n+1}_+)$ and $T^2(\mathbb{R}^{n+1}_+)$ have the lattice properties. Thus, by the real interpolation of $T^p(\mathbb{R}^{n+1}_+)$ (see [19] in the case $p \in [1, \infty)$ and [9] in the case $p \in (0, 1)$) and Proposition 3.2, we conclude that, for all $0 < p_0 < p_1 < \infty$, $\theta \in (0, 1)$ and $q \in [1, \infty]$,

$$\begin{aligned}&\left(T^2(\mathbb{R}^{n+1}_+) \cap T^{p_0}(\mathbb{R}^{n+1}_+),\, T^2(\mathbb{R}^{n+1}_+) \cap T^{p_1}(\mathbb{R}^{n+1}_+)\right)_{\theta,\, q}\\ &\quad= T^2(\mathbb{R}^{n+1}_+) \cap (T^{p_0}(\mathbb{R}^{n+1}_+),\, T^{p_1}(\mathbb{R}^{n+1}_+))_{\theta,\, q},\end{aligned}$$

where $p \in (p_0, p_1)$ satisfies $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. In particular, if $q = \infty$, then

$$(T^{p_0}(\mathbb{R}^{n+1}_+),\, T^{p_1}(\mathbb{R}^{n+1}_+))_{\theta,\, \infty} = WT^p(\mathbb{R}^{n+1}_+).$$

Now, let L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_4$, and $H^p_L(\mathbb{R}^n)$ be the Hardy space associated to L defined as in Section 2.3. Our main result of this section is as follows.

Theorem 3.5. *Let $0 < p_0 < p_1 \le 1$, $\theta \in (0, 1)$ and L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_4$. Then*

$$\left(L^2(\mathbb{R}^n) \cap H^{p_0}_L(\mathbb{R}^n),\, L^2(\mathbb{R}^n) \cap H^{p_1}_L(\mathbb{R}^n)\right)_{\theta,\, \infty} = L^2(\mathbb{R}^n) \cap WH^p_L(\mathbb{R}^n),$$

where $p \in (0, 1]$ satisfies $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Remark 3.6. Recall that, in [29], Fefferman et al. showed that, for all $0 < p_0 < p_1 \le 1$, $\theta \in (0, 1)$ and $p \in (0, 1]$ satisfies $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$\left(H^{p_0}(\mathbb{R}^n),\, H^{p_1}(\mathbb{R}^n)\right)_{\theta,\, \infty} = WH^p(\mathbb{R}^n). \tag{3.5}$$

They proved (3.5) by only considering the Schwartz functions in $S(\mathbb{R}^n)$. However, as was pointed out in [31, 37], $S(\mathbb{R}^n)$ may not dense in $WH^p(\mathbb{R}^n)$. Thus, there is a gap in their proof of (3.5). Indeed, Fefferman et al. [29] proved

$$\left(S(\mathbb{R}^n) \cap H^{p_0}(\mathbb{R}^n),\, S(\mathbb{R}^n) \cap H^{p_1}(\mathbb{R}^n)\right)_{\theta,\, \infty} = S(\mathbb{R}^n) \cap WH^p(\mathbb{R}^n).$$

Thus, Theorem 3.5 is a generalization of this result.

To prove Theorem 3.5, we need the following lemma.

Lemma 3.7. *Let $p \in (0, 1]$ and L satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_4$. Then, for all $M \in \mathbb{N}$ satisfying $M > \frac{n}{2k}(\frac{1}{p} - \frac{1}{2})$,*

$$\Pi_{L,\, M}\left(T^2(\mathbb{R}^{n+1}_+) \cap WT^p(\mathbb{R}^{n+1}_+)\right) = L^2(\mathbb{R}^n) \cap WH^p_L(\mathbb{R}^n),$$

where $\Pi_{L,\, M}$ is the operator defined as in (2.27).

Proof. We first prove

$$L^2(\mathbb{R}^n)\cap WH^p_L(\mathbb{R}^n)\subset \Pi_{L,\,M}\Big(T^2(\mathbb{R}^{n+1}_+)\cap WT^p(\mathbb{R}^{n+1}_+)\Big). \tag{3.6}$$

Indeed, for any $f\in L^2(\mathbb{R}^n)\cap WH^p_L(\mathbb{R}^n)$, by the bounded H_∞-functional calculus, we know that

$$f=\Pi_{L,\,M}\circ Q_{t,\,L}(f), \tag{3.7}$$

where $Q_{t,\,L}:=t^{2k}Le^{-t^{2k}L}$. Moreover, by the definition of $WH^p_L(\mathbb{R}^n)$ and k-Davies-Gaffney estimates, we know that $Q_{t,\,L}(f)\in T^2(\mathbb{R}^{n+1}_+)\cap WT^p(\mathbb{R}^{n+1}_+)$. This, together with (3.7), implies that $f\in \Pi_{L,\,M}(T^2(\mathbb{R}^{n+1}_+)\cap WT^p(\mathbb{R}^{n+1}_+))$. Thus, (3.6) is true.

We now prove

$$\Pi_{L,\,M}\Big(T^2(\mathbb{R}^{n+1}_+)\cap WT^p(\mathbb{R}^{n+1}_+)\Big)\subset L^2(\mathbb{R}^n)\cap WH^p_L(\mathbb{R}^n). \tag{3.8}$$

Indeed, for any $g\in \Pi_{L,\,M}\Big(T^2(\mathbb{R}^{n+1}_+)\cap WT^p(\mathbb{R}^{n+1}_+)\Big)$, we know that there exists

$$G\in T^2(\mathbb{R}^{n+1}_+)\cap WT^p(\mathbb{R}^{n+1}_+)$$

such that $g=\Pi_{L,\,M}(G)$. Using the weak atomic decomposition of $WT^p(\mathbb{R}^{n+1}_+)$ (see Theorem 2.6), we know that there exist $\{\lambda_{i,\,j}\}_{i\in\mathbb{Z},\,j\in\mathbb{Z}_+}\subset\mathbb{C}$ and $\{A_{i,\,j}\}_{i\in\mathbb{Z},\,j\in\mathbb{Z}_+}$ of $T^p(\mathbb{R}^{n+1}_+)$-atoms such that

$$G=\sum_{i\in\mathbb{Z},\,j\in\mathbb{Z}_+}\lambda_{i,\,j}A_{i,\,j}$$

holds in $T^2(\mathbb{R}^{n+1}_+)$ and almost everywhere in $\mathbb{R}^{n+1}_+$. Moreover,

$$\sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{Z}_+}|\lambda_{i,\,j}|^p\Big\}^{\frac{1}{p}}\sim\|G\|_{WT^p(\mathbb{R}^{n+1}_+)}. \tag{3.9}$$

Using the boundedness of $\Pi_{L,\,M}$ from $T^2(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^n)$ (which can be deduced from the quadratic estimates, since L has a bounded H_∞ functional calculus), we know that

$$g=\sum_{i\in\mathbb{Z},\,j\in\mathbb{Z}_+}\lambda_{i,\,j}\Pi_{L,\,M}(A_{i,\,j})$$

in $L^2(\mathbb{R}^n)$, where, by the argument below (2.27), we know that $\Pi_{L,\,M}(A_{i,\,j})$ is a $(p,\,\epsilon,\,M)_L$-molecule up to a harmless positive constant multiple. This, together with Theorem 2.21, implies that $g\in L^2(\mathbb{R}^n)\cap WH^p_L(\mathbb{R}^n)$ and hence (3.8) is true, which completes the proof of Lemma 3.7. □

We now turn to the proof of Theorem 3.5.

Proof of Theorem 3.5. Let $\Pi_{L,\,M}$ be defined as in (2.27) and $Q_{t,\,L}:=t^{2k}Le^{-t^{2k}L}$. By the bounded H_∞-functional calculus, we know that

$$f=\Pi_{L,\,M}\circ Q_{t,\,L}(f), \tag{3.10}$$

which implies that $(L^2(\mathbb{R}^n)\cap H^{p_0}_L(\mathbb{R}^n),\,L^2(\mathbb{R}^n)\cap H^{p_1}_L(\mathbb{R}^n))$ is a retract of

$$(T^2(\mathbb{R}^{n+1}_+)\cap T^{p_0}(\mathbb{R}^{n+1}_+),\,T^2(\mathbb{R}^{n+1}_+)\cap T^{p_1}(\mathbb{R}^{n+1}_+)),$$

namely, there exist two linear bounded operators

$$Q_{t,\,L}:\ L^2(\mathbb{R}^n)\cap H_L^{p_i}(\mathbb{R}^n)\to T^2(\mathbb{R}^{n+1}_+)\cap T^{p_i}(\mathbb{R}^{n+1}_+)$$

and

$$\Pi_{L,\,M}:\ T^2(\mathbb{R}^{n+1}_+)\cap T^{p_i}(\mathbb{R}^{n+1}_+)\to L^2(\mathbb{R}^n)\cap H_L^{p_i}(\mathbb{R}^n)$$

such that $\Pi_{L,\,M}\circ Q_{t,\,L}=I$ on each $L^2(\mathbb{R}^n)\cap H_L^{p_i}(\mathbb{R}^n)$, where $i\in\{0,\,1\}$. Thus, by [46, Lemma 7.11], we see that

$$\begin{aligned}&(L^2(\mathbb{R}^n)\cap H_L^{p_0}(\mathbb{R}^n),\,L^2(\mathbb{R}^n)\cap H_L^{p_1}(\mathbb{R}^n))_{\theta,\,\infty}\\&\quad=\Pi_{L,\,M}\Big((T^2(\mathbb{R}^{n+1}_+)\cap T^{p_0}(\mathbb{R}^{n+1}_+),\,T^2(\mathbb{R}^{n+1}_+)\cap T^{p_0}(\mathbb{R}^{n+1}_+))_{\theta,\,\infty}\Big),\end{aligned}$$

which, together with Lemma 3.7 and Remark 3.4, implies that

$$\begin{aligned}&(L^2(\mathbb{R}^n)\cap H_L^{p_0}(\mathbb{R}^n),\,L^2(\mathbb{R}^n)\cap H_L^{p_1}(\mathbb{R}^n))_{\theta,\,\infty}\\&\quad=\Pi_{L,\,M}\Big((T^2(\mathbb{R}^{n+1}_+)\cap T^{p_0}(\mathbb{R}^{n+1}_+),\,T^2(\mathbb{R}^{n+1}_+)\cap T^{p_0}(\mathbb{R}^{n+1}_+))_{\theta,\,\infty}\Big)\\&\quad=\Pi_{L,\,M}\Big(T^2(\mathbb{R}^{n+1}_+)\cap\Big(T^{p_0}(\mathbb{R}^{n+1}_+),\,T^{p_1}(\mathbb{R}^{n+1}_+)\Big)_{\theta,\,\infty}\Big)\\&\quad=\Pi_{L,\,M}\Big(T^2(\mathbb{R}^{n+1}_+)\cap WT^p(\mathbb{R}^{n+1}_+)\Big)=L^2(\mathbb{R}^n)\cap WH_L^p(\mathbb{R}^n).\end{aligned}$$

This finishes the proof of Theorem 3.5. □

4. The dual space of $WH_L^p(\mathbb{R}^n)$

In this section, letting L be nonnegative self-adjoint and satisfy the Davies-Gaffney estimates, we study the dual space of $WH_L^p(\mathbb{R}^n)$. It turns out that the dual of $WH_L^p(\mathbb{R}^n)$ is some weak Lipschitz space, which can be defined via the mean oscillation over some bounded open sets.

Before giving the definition of weak Lipschitz spaces, we first introduce a class of coverings of all bounded open sets, which is motivated by the subtle covering, appearing in the proof of Theorem 2.11, of the level sets of $\mathcal{A}$-functionals, obtained via the Whitney decomposition lemma.

Definition 4.1. Let Ω be a bounded open set in $\mathbb{R}^n$ and Λ an index set. A family $\vec{B}:=\{B_j\}_{j\in\Lambda}$ of open balls is said to be in the *class* $\mathcal{W}_\Omega$ if
(i) $\Omega\subset\cup_{j\in\Lambda}B_j$;
(ii) $r:=\inf_{j\in\Lambda}\{r_{B_j}\}>0$;
(iii) letting $\widetilde{B}_j:=\frac{1}{20\sqrt{n}}B_j$, then $\{\widetilde{B}_j\}_{j\in\Lambda}$ are mutually disjoint;
(iv) there exists a positive constant M_0 such that $\sum_{j\in\Lambda}|B_j|<2M_0|\Omega|$.

From the argument below (2.8), it follows that, for any bounded open set Ω with $|\Omega|\in(0,\,\infty)$, $\mathcal{W}_\Omega\neq\emptyset$.

Now, let $\alpha\in[0,\,\infty)$, $\epsilon\in(0,\infty)$, $M\in\mathbb{Z}_+$ satisfy $M>\frac12(\alpha+\frac n2)$ and L be nonnegative self-adjoint in $L^2(\mathbb{R}^n)$ and satisfy the Davies-Gaffney estimates. The *space* $\mathcal{M}^{\epsilon,M}_{\alpha,L}(\mathbb{R}^n)$ is defined to be the space of all functions u in $L^2(\mathbb{R}^n)$ satisfying the following two conditions:
(i) for all $\ell\in\{0,\dots,\,M\}$, $L^{-\ell}u\in L^2(\mathbb{R}^n)$;

(ii) letting Q_0 be the *unit cube* with its center at the origin, then

$$\|u\|_{\mathcal{M}^{\epsilon,M}_{\alpha,L}(\mathbb{R}^n)} := \sup_{j\in\mathbb{Z}_+}\left\{2^{-j(\frac{n}{2}+\epsilon+\alpha)}\sum_{\ell=0}^{M}\left\|L^{-\ell}u\right\|_{L^2(S_j(Q_0))}\right\}<\infty. \tag{4.1}$$

Let $\mathcal{M}^{M,*}_{\alpha,L}(\mathbb{R}^n) := \cap_{\epsilon\in(0,\infty)}(\mathcal{M}^{\epsilon,M}_{\alpha,L}(\mathbb{R}^n))^*$. For all $r\in(0,\infty)$, let

$$\mathcal{A}_r := \left(I-e^{-r^2L}\right)^M. \tag{4.2}$$

For any $f\in\mathcal{M}^{M,*}_{\alpha,L}(\mathbb{R}^n)$, bounded open set Ω and $\mathcal{N}\in(n(\frac{1}{p}-\frac{1}{2}),\infty)$, let $\mathcal{O}_\mathcal{N}(f,\Omega)$ be the *mean oscillation of f over* Ω defined by

$$\mathcal{O}_\mathcal{N}(f,\Omega) := \sup_{i\in\mathbb{Z}_+}\sup_{\vec{B}\in\mathcal{W}_\Omega}\left[2^{-i\mathcal{N}}\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\left|\mathcal{A}_rf(x)\right|^2dx\right\}^{\frac{1}{2}}\right], \tag{4.3}$$

where $r:=\inf_{j\in\Lambda}\{r_{B_j}\}$ and, for $i\in\mathbb{N}$,

$$S_i(\vec{B}) := \left(\bigcup_{j\in\Lambda}2^iB_j\right)\setminus\left(\bigcup_{j\in\Lambda}2^{i-1}B_j\right) \tag{4.4}$$

and $S_0(\vec{B}) := \cup_{j\in\Lambda}B_j$. By the Davies-Gaffney estimates, we know that the above integral is well defined.

For all $\delta\in(0,\infty)$, define

$$\omega_\mathcal{N}(\delta) := \sup_{|\Omega|=\delta}\mathcal{O}_\mathcal{N}(f,\Omega). \tag{4.5}$$

From its definition, it follows that ω is a decreasing function on $(0,\infty)$. Indeed, assume that $f\in\mathcal{M}^{M,*}_{\alpha,L}(\mathbb{R}^n)$, $\delta\in(0,\infty)$ and Ω is an open set satisfying $|\Omega|=\delta$. For all $\vec{B}\in\mathcal{W}_\Omega$, we know, from Definition 4.1, that there exists a positive constant $\widetilde{C}\in(1,\infty)$ such that $\sum_{j\in\Lambda}|B_j|<\frac{2M_0}{\widetilde{C}}|\Omega|$. This implies that, for all open sets $\widetilde{\Omega}\subset\Omega$ satisfying that $|\widetilde{\Omega}|=:\widetilde{\delta}\in[\frac{\delta}{\widetilde{C}},\delta)$,

$$\sum_{j\in\Lambda}|B_j|<2M_0|\widetilde{\Omega}|,$$

which immediately shows that $\vec{B}\in\mathcal{W}_{\widetilde{\Omega}}$. Thus, from its definition, it follows that

$$\mathcal{O}_\mathcal{N}(f,\Omega)\le\mathcal{O}_\mathcal{N}(f,\widetilde{\Omega})$$

and hence $\omega_\mathcal{N}(\delta)\le\omega_\mathcal{N}(\widetilde{\delta})$ for all $\widetilde{\delta}\in[\frac{\delta}{\widetilde{C}},\delta)$. This implies that ω is decreasing.

Now, we introduce the notion of the weak Lipschitz space associated to L.

Definition 4.2. Let $\alpha\in[0,\infty)$, $\epsilon\in(0,\infty)$, $M\in\mathbb{Z}_+$ satisfy $M>\frac{1}{2}(\alpha+\frac{n}{2})$, and L be nonnegative self-adjoint and satisfy the Davies-Gaffney estimates. The *weak Lipschitz space* $W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)$ is defined to be the space of all functions $f\in\mathcal{M}^{M,*}_{\alpha,L}(\mathbb{R}^n)$ such that

$$\|f\|_{W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)} := \int_0^\infty\frac{\omega_\mathcal{N}(\delta)}{\delta}\,d\delta<\infty,$$

where $\omega_\mathcal{N}(\delta)$ for $\delta\in(0,\infty)$ is as in (4.5) and $\mathcal{N}\in(n(\frac{1}{p}-\frac{1}{2}),\infty)$.

We also introduce the notion of the resolvent weak Lipschitz space. To this end, we need another class of open sets as follows, which is a slight variant of the class $\mathcal{W}_\Omega$.

Definition 4.3. Let Ω be a bounded open set in $\mathbb{R}^n$ and Λ an index set. A family $\vec{B} := \{B_j\}_{j\in\Lambda}$ of open sets is said to be in the *class* $\widetilde{\mathcal{W}}_\Omega$ if
(i) $\Omega \subset \cup_{j\in\Lambda} B_j$;
(ii) $r := \inf_{j\in\Lambda}\{r_{B_j}\} > 0$;
(iii) letting $\widetilde{B}_j := \frac{1}{10\sqrt{n}} B_j$, then $\{\widetilde{B}_j\}_{j\in\Lambda}$ are mutually disjoint;
(iv) there exists a positive constant M_0 such that $\sum_{j\in\Lambda}|B_j| < 2M_0|\Omega|$.

It is easy to see that, for any bounded open set Ω, $\widetilde{\mathcal{W}}_\Omega \subset \mathcal{W}_\Omega$. For all $\alpha \in [0, \infty)$, and $M \in \mathbb{N}$ satisfying $M > \frac{1}{2}(\alpha + \frac{n}{2})$ and $r \in (0, \infty)$, let

$$\mathcal{B}_r := \left[I - \left(I + r^2 L\right)^{-1}\right]^M. \tag{4.6}$$

Assume that $f \in \mathcal{M}^{M,*}_{\alpha,L}(\mathbb{R}^n)$ and $\delta \in (0, \infty)$. Let

$$\mathcal{O}_{\mathrm{res},\widetilde{\mathcal{N}}_\Omega}(f, \Omega) := \sup_{i\in\mathbb{Z}_+} \sup_{\vec{B}\in\widetilde{\mathcal{W}}_\Omega} \left[2^{-i\mathcal{N}}\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}} \int_{S_i(\vec{B})} \left|\mathcal{B}_r f(x)\right|^2 dx\right\}^{\frac{1}{2}}\right],$$

where $\mathcal{N} \in (n(\frac{1}{p} - \frac{1}{2}), \infty)$, $r := \inf_{j\in\Lambda}\{r_{B_j}\}$ and $S_i(\vec{B})$ is as in (4.4). Let also, for any $\delta \in (0, \infty)$,

$$\omega_{\mathrm{res},\mathcal{N}}(\delta) := \sup_{|\Omega|=\delta} \mathcal{O}_{\mathrm{res},\mathcal{N}}(f, \Omega). \tag{4.7}$$

Definition 4.4. Let $\alpha \in [0, \infty)$, $\epsilon \in (0,\infty)$, $M \in \mathbb{Z}_+$ satisfy $M > \frac{1}{2}(\alpha + \frac{n}{2})$, $\mathcal{N} \in (n(\frac{1}{p}-\frac{1}{2}), \infty)$, and L be nonnegative self-adjoint and satisfy the Davies-Gaffney estimates. The *resolvent weak Lipschitz space* $W\Lambda^\alpha_{L,\mathrm{res},\mathcal{N}}(\mathbb{R}^n)$ is then defined to be the space of all functions $f \in \mathcal{M}^{M,*}_{\alpha,L}(\mathbb{R}^n)$ satisfying

$$\|f\|_{W\Lambda^\alpha_{L,\mathrm{res},\mathcal{N}}(\mathbb{R}^n)} := \int_0^\infty \frac{\omega_{\mathrm{res},\mathcal{N}}(\delta)}{\delta}\, d\delta < \infty.$$

We have the following relationship between the weak Lipschitz space and the resolvent weak Lipschitz space.

Proposition 4.5. *Let $\alpha \in [0, \infty)$ and L be nonnegative self-adjoint and satisfy the Davies-Gaffney estimates. Let $\mathcal{N} \in (n(\frac{1}{p} - \frac{1}{2}), \infty)$. Then $W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n) \subset W\Lambda^\alpha_{L,\mathrm{res},\mathcal{N}}(\mathbb{R}^n)$.*

Proof. We prove this proposition by showing that, for all $f \in W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)$,

$$\left\|f\right\|_{W\Lambda^\alpha_{L,\mathrm{res},\mathcal{N}}(\mathbb{R}^n)} \lesssim \left\|f\right\|_{W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)}.$$

By an argument similar to that used in the proof of [41, (3.42)], we see that

$$f = 2^M\left[r^{-2}\int_r^{2^{1/2}r} s\left(I - e^{-s^2L}\right)^M ds + \sum_{\ell=1}^M \binom{M}{\ell} r^{-2}L^{-1}e^{-\ell r^2 L}\right.$$

$$
\begin{aligned}
&\circ\Big(I-e^{-r^2L}\Big)\Big(\sum_{i=0}^{\ell-1}e^{-ir^2L}\Big)\Big]^M f\\
=&2^M\Big[r^{-2}\int_r^{2^{1/2}r}s\Big(I-e^{-s^2L}\Big)^M\,ds\Big]^M f\\
&+2^M\binom{M}{1}^{\ell_1}\cdots\binom{M}{M-1}^{\ell_{M-1}}\sum_{\substack{\ell_0+\cdots+\ell_M=M\\ \ell_0<M,\ell_M<M}}\binom{M}{\ell_0,\ldots,\ell_M}\\
&\times\Big[r^{-2}\int_r^{2^{1/2}r^2}s\Big(I-e^{-s^2L}\Big)^M\,ds\Big]^{\ell_0}\\
&\times\cdots\times\Big[r^{-2}L^{-1}e^{-\ell r^2L}\Big(I-e^{-r^2L}\Big)\Big(\sum_{i=0}^{M-1}e^{-ir^2L}\Big)\Big]^{\ell_M}f\\
&+2^M\Big[r^{-2}L^{-1}e^{-\ell r^2L}\Big(I-e^{-r^2L}\Big)\Big(\sum_{i=0}^{M-1}e^{-ir^2L}\Big)\Big]^M f\\
=:&\mathrm{A}_0f+\sum_{\substack{\ell_0+\cdots+\ell_M=M\\ \ell_0<M,\ell_M<M}}\mathrm{A}_{\ell_0,\ldots,\ell_M}f+\mathrm{A}_Mf,
\end{aligned}
$$

where $\binom{M}{\ell}$ denotes the binomial coefficients.

For all $i\in\mathbb{Z}_+$, $\vec{B}\in\widetilde{\mathcal{W}}_\Omega$ and $\mathcal{N}\in(n(\frac{1}{p}-\frac{1}{2}),\,\infty)$, we first estimate

$$
\mathrm{D}:=2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\Big|\mathcal{B}_r\mathrm{A}_{\ell_0,\ldots,\ell_M}f(x)\Big|^2\,dx\Big\}^{\frac{1}{2}}.
$$

Let

$$
\widetilde{A}_{\ell_0,\ldots,\ell_M}:=\Big(r^2L\Big)^{\ell_1+\cdots+\ell_M}\mathrm{A}_{\ell_0,\ldots,\ell_M}\Big(r^{-2}\int_r^{2^{1/2}r}s\mathcal{A}_s\,ds\Big)^{-1}.
$$

From the functional calculus and the fact that $\{e^{-tL}\}_{t>0}$ satisfies the Davies-Gaffney estimates, we deduce that $\mathcal{B}_r$ satisfies the Davies-Gaffney estimates with $t\sim r^2$, namely, there exists a positive constant C_1 such that, for all closed sets E and F in $\mathbb{R}^n$, $t\in(0,\,\infty)$ and $f\in L^2(\mathbb{R}^n)$ supported in E,

$$
\Big\|\mathcal{B}_rf\Big\|_{L^2(F)}\lesssim\exp\left\{-C_1\frac{\big[\mathrm{dist}(E,\,F)\big]^2}{r^2}\right\}\|f\|_{L^2(E)}.
$$

Moreover, from Lemmas 2.3 and 2.4, it follows that $\widetilde{A}_{\ell_0,\ldots,\ell_M}$ also satisfies the Davies-Gaffney estimates with $t\sim r^2$. Similarly, $(r^{-2}L^{-1})^{\ell_1+\cdots+\ell_M}\mathcal{B}_r$ also satisfies the Davies-Gaffney estimates with $t\sim r^2$.

Now, let

$$
F_s:=\frac{1}{r^2}\int_r^{2^{1/2}r}s\Big(I-e^{-s^2L}\Big)^M f\,ds. \tag{4.8}
$$

By Minkowski's inequality, the Davies-Gaffney estimates and Hölder's inequality, we know that there exists a positive constant $\alpha_3 \in (0, \infty)$ such that

$$\begin{aligned}
\mathrm{D} &\lesssim 2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\Big|\Big[\Big(r^{-2}L^{-1}\Big)^{\ell_1+\cdots+\ell_M}\mathcal{B}_r\Big]\widetilde{\mathrm{A}}_{\ell_0,\ldots,\ell_M}F_s(x)\Big|^2\,dx\Big\}^{\frac12}\\
&\lesssim 2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\Big[\sum_{\ell\in\mathbb{Z}_+}\Big\{\int_{S_i(\vec{B})}\Big|\Big[\Big(r^{-2}L^{-1}\Big)^{\ell_1+\cdots+\ell_M}\mathcal{B}_r\Big]\\
&\qquad\times\widetilde{\mathrm{A}}_{\ell_0,\ldots,\ell_M}\Big(\chi_{S_\ell(\vec{B})}F_s\Big)(x)\Big|^2\,dx\Big\}^{1/2}\Big]^2\Big\}^{\frac12}\\
&\lesssim 2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\Big\{\sum_{\ell=i-1}^{i+1}\Big[\int_{S_\ell(\vec{B})}\Big|F_s(x)\Big|^2\,dx\Big]^{1/2}\\
&\qquad+\sum_{\substack{\ell\in\mathbb{Z}_+\\ \ell\ge i+2\text{ or }\ell\le i-2}}2^{-\ell(\mathcal{N}+\alpha_3)}\Big[\int_{S_\ell(\vec{B})}\Big|F_s(x)\Big|^2\,dx\Big]^{1/2}\Big\}^2\Big\}^{\frac12}\\
&\lesssim \Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\Big[\sum_{\ell=i-1}^{i+1}2^{-2\ell\mathcal{N}}+\sum_{\substack{\ell\in\mathbb{Z}_+\\ \ell\ge i+2\text{ or }\ell\le i-2}}2^{-2\ell(\mathcal{N}+\alpha_3)}\Big]\int_{S_\ell(\vec{B})}\Big|F_s(x)\Big|^2\,dx\Big\}^{\frac12}\\
&\lesssim \sup_{i\in\mathbb{Z}_+}\left[2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\Big|F_s(x)\Big|^2\,dx\Big\}^{\frac12}\right]=:\widetilde{\mathrm{D}}.
\end{aligned}$$

We now estimate $\widetilde{\mathrm{D}}$. By (4.8), Minkowski's integral inequality and the mean value theorem for integrals, we see that there exists a positive constant $\widetilde{r}\in(r,\,2^{\frac12}r)$ such that

$$\begin{aligned}
\widetilde{\mathrm{D}} &\lesssim \sup_{i\in\mathbb{Z}_+}\left[2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\Big|\frac{1}{r^2}\int_r^{2^{1/2}r}s\Big(I-e^{-s^2L}\Big)^M f(x)\,ds\Big|^2\,dx\Big\}^{\frac12}\right]\\
&\lesssim \sup_{i\in\mathbb{Z}_+}\left[2^{-i\mathcal{N}}\frac{1}{r^2}\int_r^{2^{1/2}r}s\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\Big|\Big(I-e^{-s^2L}\Big)^M f(x)\Big|^2\,dx\Big\}^{\frac12}ds\right]\\
&\sim \sup_{i\in\mathbb{Z}_+}\left[2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\Big|\Big(I-e^{-\widetilde{r}^2L}\Big)^M f(x)\Big|^2\,dx\Big\}^{\frac12}\right].
\end{aligned}\tag{4.9}$$

Moreover, since $\vec{B}\in\widetilde{W_\Omega}$, if let $C_0\in(1,\,2^{1/2})$ satisfying $\widetilde{r}=C_0r$, then it is easy to see that $C_0\vec{B}:=\{C_0B_j\}_{j\in\Lambda}\in W_\Omega$ and

$$r_{C_0\vec{B}}:=\inf_{j\in\Lambda}\Big\{r_{C_0B_j}\Big\}=C_0r=\widetilde{r},$$

which, together with (4.9) and the fact that $S_i(C_0\vec{B})\subset\cup_{j=i-1}^{i+1}S_j(C_0\vec{B})$, implies that

$$\widetilde{\mathrm{D}}\lesssim\sup_{i\in\mathbb{Z}_+}\left[2^{-i\mathcal{N}}\Big\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\sum_{j=i-1}^{i+1}\int_{S_j(\vec{B})}\Big|\Big(I-e^{-\widetilde{r}^2L}\Big)^M f(x)\Big|^2\,dx\Big\}^{\frac12}\right]$$

$$\lesssim \sup_{i\in\mathbb{Z}_+} \sup_{\vec{B}\in W_\Omega} \left[2^{-i\mathcal{N}}\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\left|\left(I-e^{-r^2L}\right)^M f(x)\right|^2 dx\right\}^{\frac{1}{2}}\right],$$

where $r := \inf\limits_{j\in\Lambda}\{r_{B_j}\}$. This, together with (4.3), implies that

$$\mathrm{D} \lesssim \mathcal{O}_{\mathcal{N}}(f,\,\Omega).$$

Thus, for all $\ell_0 + \cdots + \ell_M = M$ satisfying $\ell_0 < M$ and $\ell_M < M$, we have

$$\mathcal{O}_{\mathrm{res},\mathcal{N}_\Omega}(\mathrm{A}_{\ell_0,\ldots,\ell_M} f,\,\Omega) = \sup_{i\in\mathbb{Z}} \sup_{\vec{B}\in\widetilde{W}_\Omega} \mathrm{D} \lesssim \mathcal{O}_{\mathcal{N}}(f,\,\Omega).$$

This implies that $\|\mathrm{A}_{\ell_0,\ldots,\ell_M} f\|_{W\Lambda^\alpha_{L,\mathrm{res},\mathcal{N}}(\mathbb{R}^n)} \lesssim \|f\|_{W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)}$. Similarly, we also have

$$\|\mathrm{A}_0 f\|_{W\Lambda^\alpha_{L,\mathrm{res},\mathcal{N}}(\mathbb{R}^n)} \lesssim \|f\|_{W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)}$$

and $\|\mathrm{A}_M f\|_{W\Lambda^\alpha_{L,\mathrm{res},\mathcal{N}}(\mathbb{R}^n)} \lesssim \|f\|_{W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)}$, which completes the proof of Proposition 4.5. □

Now we state the main result of this section.

Theorem 4.6. *Let $p \in (0,\,1]$ and L be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. Then, for all $\mathcal{N} \in (n(\frac{1}{p}-\frac{1}{2}),\,\infty)$, $(WH^p_L(\mathbb{R}^n))^* = W\Lambda^{n(\frac{1}{p}-1)}_{L,\mathcal{N}}(\mathbb{R}^n)$.*

To prove Theorem 4.6, we first introduce the following notion of weak Carleson measures.

Definition 4.7. Let Ω be a bounded open set in $\mathbb{R}^n$ and Λ an index set. A family $\vec{B} := \{B_j\}_{j\in\Lambda}$ of open sets is said to be in the *class* $\widetilde{\widetilde{\mathcal{W}}}_\Omega$ if

(i) $\Omega \subset \cup_{j\in\Lambda} B_j$;

(ii) $r := \inf_{j\in\Lambda}\{r_{B_j}\} > 0$;

(iii) letting $\widetilde{B}_j := \frac{1}{10\sqrt{n}} B_j$, then $\{\widetilde{B}_j\}_{j\in\Lambda}$ are mutually disjoint;

(iv) there exists a positive constant M_0 such that $\sum_{j\in\Lambda}|B_j| < 2M_0|\Omega|$.

(v) letting Q_j for any $j \in \Lambda$ be the closed cube having the same center as B_j with the length $\frac{r_{B_j}}{5\sqrt{n}}$, then $\{Q_j\}_{j\in\Lambda}$ are mutually disjoint and $\{2Q_j\}_{j\in\Lambda}$ have bounded overlap. Moreover, $\Omega \subset \cup_{j\in\Lambda} Q_i$.

From the argument below (2.8), it follows that, for any bounded open set Ω with $|\Omega| \in (0,\,\infty)$, $\widetilde{\widetilde{\mathcal{W}}}_\Omega \neq \emptyset$. Moreover, $\widetilde{\widetilde{\mathcal{W}}}_\Omega \subset \widetilde{\mathcal{W}}_\Omega \subset \mathcal{W}_\Omega$.

Now, let $\alpha \in [0,\,\infty)$ and μ be a positive measure on $\mathbb{R}^{n+1}_+$. For all bounded open sets Ω, let

$$\mathcal{C}_\alpha(\mu,\,\Omega) := \sup_{\vec{B}\in\widetilde{\widetilde{\mathcal{W}}}_\Omega} \left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\mu\Big(\bigcup_{j\in\Lambda}\widehat{B}_j\cap(Q_j\times(0,\,\infty))\Big)\right\}^{\frac{1}{2}},$$

where $\widetilde{\widetilde{\mathcal{W}}}_\Omega$ is as in Definition 4.7.

For all $\delta \in (0,\,\infty)$, let

$$\omega_{\mathcal{C}}(\delta) := \sup_{|\Omega|=\delta} \mathcal{C}_\alpha(\mu,\,\Omega). \tag{4.10}$$

Then μ is called a *weak Carleson measure of order* α, denoted by $\mathcal{C}_\alpha$, if

$$\|\mu\|_{\mathcal{C}_\alpha} := \int_0^\infty \frac{\omega_{\mathcal{C}}(\delta)}{\delta}\,d\delta < \infty.$$

Recall that, if L is nonnegative self-adjoint and satisfies the Davies-Gaffney estimates, then L satisfies the *finite speed propagation property for solutions of the corresponding wave equation*, namely, there exists a positive constant C_0 such that

$$(\cos(t\sqrt{L})f_1, f_2) = 0 \tag{4.11}$$

for all closed sets U_1 and U_2, $0 < C_0 t < \operatorname{dist}(U_1, U_2)$, $f_1 \in L^2(U_1)$ and $f_2 \in L^2(U_2)$ (see, for example, [38]). Moreover, we have the following conclusion, which can be deduced from [38, Lemma 3.5] with a slight modification, the details being omitted.

Lemma 4.8 ([38])**.** *Let L be nonnegative self-adjoint and satisfy the Davies-Gaffney estimates. Assume that $\varphi \in C_0^\infty(\mathbb{R}^n)$ is even, $\operatorname{supp}\varphi \subset (-C_1^{-1}, C_1^{-1})$, where $C_1 \in (0, 1)$ is a positive constant. Let Φ be the Fourier transform of φ. Then, for all $k \in \mathbb{Z}_+$ and $t \in (0, \infty)$, the kernel $K_{(t^2L)^k\Phi(t\sqrt{L})}(x, y)$ of the operator $(t^2L)^k\Phi(t\sqrt{L})$ satisfies*

$$\operatorname{supp} K_{(t^2L)^k\Phi(t\sqrt{L})}(x, y) \subset \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n :\ d(x, y) \le \frac{C_0}{C_1}t\right\}. \tag{4.12}$$

Proposition 4.9. *Let $\alpha \in [0, \infty)$, $M \in \mathbb{N}$ satisfy that $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, $\mathcal{N} \in (n(\frac{1}{p} - \frac{1}{2}), \infty)$, and L be a nonnegative self-adjoint operator satisfying the Davies-Gaffney estimates. Assume that $\varphi \in C_0^\infty(\mathbb{R}^n)$ is even, $\operatorname{supp}\varphi \subset (-C_1^{-1}, C_1^{-1})$ satisfies $\frac{C_0}{C_1} \in (0, \frac{1}{10\sqrt{n}})$, where C_0 is as in* (4.11)*. Let Φ be the Fourier transform of φ. Then, for all $f \in W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)$,*

$$\mu_f(x, t) := \left|\left(t^2L\right)^M \Phi(t\sqrt{L})f(x)\right|^2 \frac{dx\,dt}{t}, \quad \forall\ (x, t) \in \mathbb{R}^{n+1}_+,$$

is a weak Carleson measure of order α. Moreover, there exists a positive constant C such that, for all $f \in W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)$,

$$\|\mu_f\|_{\mathcal{C}_\alpha} \le C\|f\|_{W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)}.$$

Proof. Let $f \in W\Lambda^\alpha_{L,\mathcal{N}}(\mathbb{R}^n)$. For all bounded open sets Ω and $\vec{B} = \{B_j\}_{j\in\Lambda} \in \widetilde{\widetilde{\mathcal{W}}}_\Omega$, by the definition of weak Carleson measures, we need to estimate

$$\mathrm{I} := \left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\mu_f\Big(\bigcup_{j\in\Lambda}\widehat{B}_j \cap (Q_j \times (0, \infty))\Big)\right\}^{\frac{1}{2}}.$$

To this end, let $r := \inf_{j\in\Lambda}\{r_{B_j}\}$ and $\mathcal{B}_r$ be as in (4.6). By Minkowski's inequality, we write

$$\begin{aligned}
\mathrm{I} &\sim \left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\sum_{j\in\Lambda}\iint_{\widehat{B}_j\cap(Q_j\times(0,\infty))}\left|\left(t^2L\right)^M\Phi(t\sqrt{L})f(x)\right|^2\frac{dx\,dt}{t}\right\}^{\frac{1}{2}}\\
&\lesssim \left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\sum_{j\in\Lambda}\iint_{\widehat{B}_j\cap(Q_j\times(0,\infty))}\left|\left(t^2L\right)^M\Phi(t\sqrt{L})\mathcal{B}_rf(x)\right|^2\frac{dx\,dt}{t}\right\}^{\frac{1}{2}}
\end{aligned}$$

$$+\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\sum_{j\in\Lambda}\iint_{\widehat{B}_j\cap(Q_j\times(0,\infty))}\left|\left(t^2L\right)^M\Phi(t\sqrt{L})\left(I-\mathcal{B}_r\right)f(x)\right|^2\frac{dx\,dt}{t}\right\}^{\frac{1}{2}}$$
$$=:\mathrm{I}_1+\mathrm{I}_2.$$

To estimate I_1, by Minkowski's inequality again, we obtain

$$\mathrm{I}_1\lesssim\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\sum_{j\in\Lambda}\left[\sum_{i\in\mathbb{Z}_+}\left\{\iint_{\widehat{B}_j\cap(Q_j\times(0,\infty))}\left|\left(t^2L\right)^M\Phi(t\sqrt{L})\left(\chi_{S_i(Q_j)}\mathcal{B}_rf\right)(x)\right|^2\right.\right.\right.$$
$$\left.\left.\left.\times\frac{dx\,dt}{t}\right\}^{1/2}\right]^2\right\}^{\frac{1}{2}}.$$

Now, let $\widetilde{S}_i(B_j):=2^{i+1}B_j\setminus 2^{i-2}B_j$. By Lemma 4.8 with $\frac{C_0}{C_1}\in(0,\frac{1}{10\sqrt{n}})$, $t\in(0,r_{B_{i,j}})$ and $Q_j=\frac{1}{5\sqrt{n}}B_j$, we know that $\operatorname{supp}\{(t^2L)^M\Phi(t\sqrt{L})(\chi_{S_i(Q_j)}\mathcal{B}_rf)\}\subset\widetilde{S}_i(Q_j)$, which, together with Definitions 4.7 and 4.3, and the quadratic estimates, implies that

$$\mathrm{I}_1\lesssim\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\right.$$
$$\left.\times\sum_{j\in\Lambda}\left[\sum_{i=0}^{1}\left\{\iint_{Q_j\cap(0,r_{B_j})}\left|\left(t^2L\right)^M\Phi(t\sqrt{L})\left(\chi_{S_i(Q_j)}\mathcal{B}_rf\right)(x)\right|^2\frac{dx\,dt}{t}\right\}^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}}$$
$$\lesssim\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\sum_{j\in\Lambda}\sum_{i=0}^{1}\int_{\mathbb{R}^n}\left[\left\{\int_0^\infty\left|\left(t^2L\right)^M\Phi(t\sqrt{L})\left(\chi_{S_i(Q_j)}\mathcal{B}_rf\right)(x)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right]^2dx\right\}^{\frac{1}{2}}$$
$$\lesssim\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\sum_{j\in\Lambda}\int_{2Q_j}\left|\mathcal{B}_rf(x)\right|^2dx\right\}^{\frac{1}{2}}\lesssim\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{\cup_{j\in\Lambda}2Q_j}\left|\mathcal{B}_rf(x)\right|^2dx\right\}^{\frac{1}{2}}$$
$$\lesssim\sup_{i\in\mathbb{Z}}\sup_{\vec{B}\in\widetilde{W}_\Omega}\left[2^{-i\mathcal{N}}\left\{\frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}}\int_{S_i(\vec{B})}\left|\mathcal{B}_rf(x)\right|^2dx\right\}^{\frac{1}{2}}\right]\lesssim\mathcal{O}_{\mathrm{res},\mathcal{N}}(f,\Omega),$$

which is desired.

The estimate of I_2 is similar to that for I_1. In this case, we need the following operator equality that, for any $r\in(0,\infty)$,

$$\left(I-\left[I-\left(I+r^2L\right)^{-1}\right]^M\right)\left[I-\left(I+r^2L\right)^{-1}\right]^{-M}=\sum_{\ell=1}^{M}\frac{M!}{(M-\ell)!\ell!}\left(r^2L\right)^{-\ell},$$

the details being omitted.

By combining the estimates for I_1 and I_2, we conclude that

$$\mathcal{C}_\alpha(\mu_f,\Omega)\lesssim\mathcal{O}_{\mathrm{res},\mathcal{N}}(f,\Omega),$$

which, together with Proposition 4.5, shows that μ_f is a weak Carleson measure of order α. This finishes the proof of Proposition 4.9. □

We now turn to the proof of Theorem 4.6.

Proof of Theorem 4.6. We first prove that

$$W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset (WH_L^p(\mathbb{R}^n))^*.$$

For any $\epsilon \in (0, \infty)$ and $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, let $\mathcal{M}^{\epsilon,M}_{n(\frac{1}{p}-1),L}(\mathbb{R}^n)$ be the space defined as in (4.1). For all $g \in W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}(\mathbb{R}^n)$, since

$$g \in \bigcap_{\epsilon\in(0,\,\infty)} (\mathcal{M}^{\epsilon,M}_{n(\frac{1}{p}-1),L}(\mathbb{R}^n))^*,$$

from the fact that all $(p,\, 2,\, M)_L$-atoms a belong to $\cup_{\epsilon\in(0,\,\infty)}\mathcal{M}^{\epsilon,M}_{n(\frac{1}{p}-1),L}(\mathbb{R}^n)$, it follows that, for any $(p,\, 2,\, M)_L$-atom a, $\langle g,\, a\rangle$ is well defined. Moreover, for any $f \in WH_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, by Remark 2.18, we see that f has a weak atomic $(p,\, 2,\, M)_L$-representation $\sum_{i\in\mathcal{I},j\in\Lambda_i} \widetilde{\lambda}_{i,j} a_{i,j}$ such that $\{a_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$ is a sequence of $(p,\, 2,\, M)_T$-atoms associated to the balls $\{B_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$ and $\{\widetilde{\lambda}_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i} = \{\widetilde{C}2^i|B_{i,j}|^{\frac{1}{p}}\}_{i\in\mathcal{I},j\in\Lambda_i}$, with $\widetilde{C}$ being a positive constant independent of f, satisfies

$$\sup_{i\in\mathcal{I}}\Big(\sum_{j\in\Lambda_i}|\widetilde{\lambda}_{i,j}|^p\Big)^{1/p} \lesssim \|f\|_{WH_L^p(\mathbb{R}^n)}. \tag{4.13}$$

By the definition of $WH_L^p(\mathbb{R}^n)$, we further know that $t^2Le^{-t^2L}f \in WT^p(\mathbb{R}^{n+1}_+) \cap T^2(\mathbb{R}^{n+1}_+)$. This, together with Theorem 2.11 and the proof of Theorem 2.15 (see (2.15) and (2.16)), implies that there exist sequences $\{A_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$ of $T^p(\mathbb{R}^{n+1}_+)$-atoms, associated with balls $\{B_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i}$, and $\{\lambda_{i,j}\}_{i\in\mathcal{I},j\in\Lambda_i} \subset \mathbb{C}$ satisfying $\lambda_{i,j} = \widetilde{\widetilde{C}}2^i|B_{i,j}|^{\frac{1}{p}}$, with $\widetilde{\widetilde{C}}$ being a positive constant independent of f, such that

$$t^2Le^{-t^2L}f = \sum_{i\in\mathcal{I}}\sum_{j\in\Lambda_i}\lambda_{i,j}A_{i,j}$$

holds true pointwisely almost everywhere in $\mathbb{R}^{n+1}_+$ and in $T^2(\mathbb{R}^{n+1}_+)$, where $\mathcal{I}$ and Λ_i are as in Theorem 2.11. By (4.13) and the definitions of $\lambda_{i,\,j}$ and $\widetilde{\lambda}_{i,\,j}$, we further obtain

$$\sup_{i\in\mathcal{I}}\Big(\sum_{j\in\mathbb{Z}_+}|\lambda_{i,j}|^p\Big)^{\frac{1}{p}} \lesssim \|f\|_{WH_L^p(\mathbb{R}^n)}. \tag{4.14}$$

Now, for all $i \in \mathcal{I}$, let $\Omega_i := \cup_{j\in\Lambda_i}B_{i,j}$. From the proof of Theorem 2.11 (see the argument below (2.11)), it follows that $\{B_{i,\,j}\}_{j\in\Lambda_i} \in \widetilde{\mathcal{W}}_{\Omega_i}$. Moreover, by comparing the quasi-norms between $W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ and $\Lambda_L^{n(\frac{1}{p}-1)}(\mathbb{R}^n)$, we see that $W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset \Lambda_L^{n(\frac{1}{p}-1)}(\mathbb{R}^n)$, where $\Lambda_L^{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ denotes the *"strong" Lipschitz space associated to* L defined as in [41, (1.26)]. This, together with the Calderón reproducing formula [39, Lemma 8.4], Theorem 2.6, Remark 2.10, Theorem 2.11 and Hölder's inequality, implies that

$$\Big|\langle g,\, f\rangle\Big| \sim \Big|\iint_{\mathbb{R}^{n+1}_+}\Big(t^2L\Big)^M\Phi(t\sqrt{L})g(x)\overline{t^2Le^{-t^2L}f(x)}\,\frac{dx\,dt}{t}\Big|$$

$$\begin{aligned}
&\lesssim \iint_{\mathbb{R}^{n+1}_+} \left|\left(t^2L\right)^M \Phi(t\sqrt{L})g(x)\left[\sum_{i\in\mathcal{I}}\sum_{j\in\Lambda_i} 2^i|B_{i,j}|^{\frac{1}{p}}\overline{A_{i,j}(x,\,t)}\right]\right| \frac{dx\,dt}{t}\\
&\lesssim \sum_{i\in\mathcal{I}} 2^i \sum_{j\in\Lambda_i} \left|B_{i,j}\right|^{\frac{1}{p}} \left[\iint_{\widehat{B}_{i,j}\cap(\operatorname{supp} A_{i,\,j})} \left|G_L(t,x)\right|^2 \frac{dx\,dt}{t}\right]^{\frac12}\\
&\quad\times \left[\iint_{\widehat{B}_{i,j}} \left|A_{i,j}(x,t)\right|^2 \frac{dx\,dt}{t}\right]^{\frac12}\\
&=: \mathrm{I},
\end{aligned}$$

where, for all $t\in(0,\,\infty)$ and $x\in\mathbb{R}^n$, we let

$$G_L(t,x) := (t^2L)^M\Phi(t\sqrt{L})g(x).$$

To estimate I, by Proposition 4.9, Hölder's inequality, the definition of $\lambda_{i,\,j}$ and (4.14), we conclude that

$$\begin{aligned}
\mathrm{I} &\lesssim \sum_{i\in\mathcal{I}} 2^i \left|\Omega_i\right|^{\frac1p-\frac12} \left\{\sum_{j\in\Lambda_i} \left|B_{i,j}\right|^{\frac12} \left[\frac{1}{|\Omega_i|^{\frac2p-1}} \iint_{\widehat{B}_{i,j}\cap(\operatorname{supp} A_{i,\,j})} \left|G_L(t,x)\right|^2 \frac{dx\,dt}{t}\right]^{\frac12}\right\}\\
&\lesssim \sum_{i\in\mathcal{I}} 2^i \left|\Omega_i\right|^{\frac1p-\frac12} \left[\sum_{j\in\Lambda_i} \left|B_{i,j}\right|\right]^{\frac12} \left[\sum_{j\in\Lambda_i} \frac{1}{|\Omega_i|^{\frac2p-1}} \iint_{\widehat{B}_{i,j}\cap(\operatorname{supp} A_{i,\,j})} \left|G_L(t,x)\right|^2 \frac{dx\,dt}{t}\right]^{\frac12}\\
&\lesssim \sum_{i\in\mathcal{I}} 2^i |\Omega_i|^{\frac1p} \left[\frac{1}{|\Omega_i|^{\frac2p-1}} \iint_{\cup_{j\in\Lambda_i}\widehat{B}_{i,j}\cap(\operatorname{supp} A_{i,\,j})} \left|G_L(t,x)\right|^2 \frac{dx\,dt}{t}\right]^{\frac12}\\
&\lesssim \|f\|_{WH^p_L(\mathbb{R}^n)} \sum_{i\in\mathcal{I}} \omega_{\mathcal{C}}(|\Omega_i|).
\end{aligned}$$

From this, together with Remark 2.10, the fact that $\omega_{\mathcal{C}} \lesssim \omega_{\mathcal{N}}$ and the decreasing property of $\omega_{\mathcal{N}}$, where $\omega_{\mathcal{N}}$ and $\omega_{\mathcal{C}}$ are, respectively, as in (4.5) and (4.10), we know that

$$\begin{aligned}
\mathrm{I} &\lesssim \|f\|_{WH^p_L(\mathbb{R}^n)} \sum_{\widetilde{i}_j\in\widetilde{\mathcal{I}}} \omega_{\mathcal{N}}\left(2^{-\widetilde{i}_j}|\Omega_{i_0}|\right)\\
&\lesssim \|f\|_{WH^p_L(\mathbb{R}^n)} \sum_{i\in\mathbb{Z}} \omega_{\mathcal{N}}(2^{-i}|\Omega_{i_0}|) \lesssim \|f\|_{WH^p_L(\mathbb{R}^n)} \int_0^\infty \frac{\omega_{\mathcal{N}}(\delta)}{\delta}\,d\delta\\
&\sim \|f\|_{WH^p_L(\mathbb{R}^n)} \|g\|_{W\Lambda^{n(\frac1p-1)}_{L,\mathcal{N}}(\mathbb{R}^n)},
\end{aligned}$$

where both $\widetilde{i}_j$ and i_0 are as in Remark 2.10, which, combined with a density argument, implies that

$$W\Lambda^{n(\frac1p-1)}_{L,\mathcal{N}}(\mathbb{R}^n) \subset \left(WH^p_L(\mathbb{R}^n)\right)^*.$$

Now, we prove the inclusion that $(WH^p_L(\mathbb{R}^n))^* \subset W\Lambda^{n(\frac1p-1)}_{L,\mathcal{N}}(\mathbb{R}^n)$.

Let $g\in(WH^p_L(\mathbb{R}^n))^*$. For all $(p,\,\epsilon,\,M)_L$-molecules m, from the fact that $\|m\|_{WH^p_L(\mathbb{R}^n)} \lesssim 1$, it follows that $|g(m)|\lesssim 1$. By this and the fact that, for all $\epsilon\in(0,\,\infty)$, $M\in\mathbb{N}$ and $m_0\in\mathcal{M}^{\epsilon,M}_{\alpha,L}(\mathbb{R}^n)$, m_0 is a $(p,\,\epsilon,\,M)$-molecule, we conclude that $g\in\mathcal{M}^{M,*}_{\alpha,L}(\mathbb{R}^n)$.

Now, we prove that $\|g\|_{W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}(\mathbb{R}^n)} \lesssim \|g\|_{(WH_L^p(\mathbb{R}^n))^*}$. By Definition 4.2, (4.3) and (4.5), we first write

$$
\begin{aligned}
\int_0^\infty \frac{\omega_{\mathcal{N}}(\delta)}{\delta}\,d\delta &\lesssim \sum_{l\in\mathbb{Z}} \omega_{\mathcal{N}}(2^{-l}) \\
&\lesssim \sum_{l\in\mathbb{Z}} \sup_{|\Omega|=2^{-l}} \sup_{i\in\mathbb{Z}_+} \sup_{\vec{B}\in\mathcal{W}_\Omega} 2^{-i\mathcal{N}} \Big[\frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{S_i(\vec{B})} \Big|\mathcal{A}_r g(x)\Big|^2\,dx\Big]^{\frac{1}{2}} \qquad (4.15)\\
&=: \sum_{l\in\mathbb{Z}} \sup_{|\Omega|=2^{-l}} \sup_{i\in\mathbb{Z}_+} \sup_{\vec{B}\in\mathcal{W}_\Omega} \mathrm{A}_l,
\end{aligned}
$$

where $\mathcal{A}_r$ and $S_i(\vec{B})$ are, respectively, as in (4.2) and (4.4).

To estimate A_l, from the dual norm of $L^2(\mathbb{R}^n)$, we deduce that there exists $\varphi_l \in L^2(\mathbb{R}^n)$ satisfying $\|\varphi_l\|_{L^2(\mathbb{R}^n)} \le 1$ such that

$$
\mathrm{A}_l \sim \Big|\Big\langle g,\, 2^{-i\mathcal{N}} \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{2}}} \mathcal{A}_r\Big(\chi_{S_i(\vec{B})}\varphi_l\Big)\Big\rangle_{L^2(\mathbb{R}^n)}\Big| =: \Big|\Big\langle g,\, f_l\Big\rangle_{L^2(\mathbb{R}^n)}\Big|. \tag{4.16}
$$

We now estimate the $WH_L^p(\mathbb{R}^n)$ quasi-norm of f_l. For all $\alpha \in (0,\infty)$, by Chebyshev's inequality, Hölder's inequality, the definition of S_L and the fact that, for all $\ell \in \mathbb{Z}_+$, $|S_\ell(\vec{B})| \lesssim 2^{\ell n}|\Omega| \sim 2^{\ell n}2^{-l}$, we obtain

(4.17)

$$
\begin{aligned}
&\alpha^p\Big|\Big\{x\in\mathbb{R}^n:\ S_L(f_l)(x) > \alpha\Big\}\Big| \\
&\quad\lesssim \int_{\mathbb{R}^n} \Big|S_L(f_l)(x)\Big|^p\,dx \lesssim \sum_{\ell\in\mathbb{Z}_+} \Big\{\int_{S_\ell(\vec{B})} \Big|S_L(f_l)(x)\Big|^2\,dx\Big\}^{\frac{p}{2}} |S_\ell(\vec{B})|^{1-\frac{p}{2}} \\
&\quad\lesssim \sum_{\ell\in\mathbb{Z}_+} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} \Big\{\Big(\int_{S_\ell(\vec{B})} \Big[\int_0^r + \int_r^{2^{\ell-3}r} + \int_{2^{\ell-3}r}^\infty\Big] \int_{\{y\in\mathbb{R}^n:\ |y-x|<t\}} \\
&\qquad\times \Big|t^2Le^{-t^2L}\mathcal{A}_r\Big(\chi_{S_i(\vec{B})}\varphi_l\Big)(y)\Big|^2 \frac{dy\,dt\,dx}{t^{n+1}}\Big)^{\frac{1}{2}}\Big\}^p =: \mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_3.
\end{aligned}
$$

We first estimate Q_1. For all $\ell \in \mathbb{Z}_+$, let

$$
\mathrm{E}_\ell := \Big\{x \in \mathbb{R}^n : \mathrm{dist}(S_\ell(\vec{B}),\, x) < r\Big\}. \tag{4.18}
$$

It is easy to see that, for all $\ell > i+1$,

$$
\frac{r}{\mathrm{dist}(E_\ell,\, S_i(\vec{B}))} \lesssim 2^{-\ell},
$$

where r is as in Definition 4.1(ii). Thus, by Fubini's theorem, the quadratic estimates and the Davies-Gaffney estimates, we know that there exists a positive constant α_6, independent of i, l and ℓ, such that

$$
\mathrm{Q}_1 \lesssim \sum_{\ell\in\mathbb{Z}_+} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} \Big\{\Big[\int_0^r \int_{\mathrm{E}_\ell} \Big|t^2Le^{-t^2L}\mathcal{A}_r\Big(\chi_{S_i(\vec{B})}\varphi_l\Big)(y)\Big|^2 \frac{dy\,dt}{t}\Big]^{\frac{1}{2}}\Big\}^p
$$

$$
\begin{aligned}
&\lesssim \sum_{\ell=0}^{i+1} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} \|\varphi_l\|^p_{L^2(S_i(\vec{B}))} \\
&\quad + \sum_{\ell=i+2}^{\infty} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} \Big[\int_0^r \exp\Big\{ -\frac{[\operatorname{dist}(E_l,\, S_i(\vec{B}))]^2}{t^2} \Big\} \frac{dt}{t} \Big]^{\frac{p}{2}} \|\varphi_l\|^p_{L^2(S_i(\vec{B}))} \\
&\lesssim \sum_{\ell=0}^{i+1} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} \|\varphi_l\|^p_{L^2(S_i(\vec{B}))} \\
&\quad + \sum_{\ell=i+2}^{\infty} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} 2^{-\ell(\mathcal{N}+\alpha_6)} \|\varphi_l\|^p_{L^2(S_i(\vec{B}))} \lesssim 1.
\end{aligned}
$$

To estimate Q_2, for all $\ell \in \mathbb{Z}_+$, let

$$
F_\ell := \Big\{ x \in \mathbb{R}^n :\ \operatorname{dist}(x,\, S_\ell(\vec{B})) < 2^{\ell-3} r \Big\}.
$$

By Fubini's theorem, the Davies-Gaffney estimates and the quadratic estimates, we know that there exists a positive constant α_7, independent of i, l and ℓ, such that

$$
\begin{aligned}
\mathrm{Q}_2 &\lesssim \sum_{\ell\in\mathbb{Z}_+} 2^{\ell n(1-\frac{p}{2})} 2^{-i\mathcal{N}p} \Big\{ \Big[\int_r^{2^{\ell-3}r} \int_{F_\ell} \Big| t^2 L e^{-t^2 L} \mathcal{A}_r \Big(\chi_{S_i(\vec{B})} \varphi_l \Big)(y) \Big|^2 \frac{dy\,dt}{t} \Big]^{\frac{1}{2}} \Big\}^p \\
&\lesssim \sum_{\ell\in\mathbb{Z}_+} 2^{\ell n(1-\frac{p}{2})} 2^{-i\mathcal{N}p} \Big\{ \Big[\int_r^{\infty} \int_{F_\ell} \Big| \Big(t^2 L \Big)^{M+1} e^{-t^2 L} \Big(r^2 L \Big)^{-M} \mathcal{A}_r \Big(\chi_{S_i(\vec{B})} \varphi_l \Big)(y) \Big|^2 \\
&\quad \times \frac{dy\,dt}{t^{4M+1}} \Big]^{\frac{1}{2}} r^{2M} \Big\}^p \\
&\lesssim \sum_{\ell=0}^{i+1} 2^{\ell n(1-\frac{p}{2})} 2^{-i\mathcal{N}p} \Big\{ \Big[\int_r^{\infty} \frac{dt}{t^{4M+1}} \Big]^{\frac{1}{2}} r^{2M} \|\chi_{S_i(\vec{B})} \varphi_l\|_{L^2(\mathbb{R}^n)} \Big\}^p \\
&\quad + \sum_{\ell=i+2}^{\infty} 2^{\ell n(1-\frac{p}{2})} 2^{-i\mathcal{N}p} \Big\{ \Big[\int_r^{\infty} \frac{dt}{t^{4M+1}} \Big]^{\frac{1}{2}} r^{2M} \exp\Big\{ -C \frac{[\operatorname{dist}(F_\ell,\, S_i(\vec{B}))]^2}{t^2} \Big\} \\
&\quad \times \|\chi_{S_i(\vec{B})} \varphi_l\|_{L^2(\mathbb{R}^n)} \Big\}^p \\
&\lesssim \sum_{\ell=0}^{i+1} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} \|\chi_{S_i(\vec{B})} \varphi_l\|^p_{L^2(\mathbb{R}^n)} \\
&\quad + \sum_{\ell=0}^{i+1} 2^{-i\mathcal{N}p} 2^{\ell n(1-\frac{p}{2})} 2^{-\ell(\mathcal{N}+\alpha_7)} \|\chi_{S_i(\vec{B})} \varphi_l\|^p_{L^2(\mathbb{R}^n)} \\
&\lesssim 1.
\end{aligned}
$$

Similar to the estimates of Q_2, we also obtain $\mathrm{Q}_3 \lesssim 1$. Thus, by (4.17) and the estimates of Q_1, Q_2 and Q_3, we see that

$$
\alpha^p \Big| \Big\{ x \in \mathbb{R}^n :\ S_L(f_l)(x) > \alpha \Big\} \Big| \lesssim 1,
$$

where the implicit positive constant is independent of l. Thus, $\{f_l\}_{l\in\mathbb{Z}}$ are uniformly bounded in $WH^p_L(\mathbb{R}^n)$, which implies that, for all $N\in\mathbb{N}$, $\sum_{l=-N}^N f_l\in WH^p_L(\mathbb{R}^n)$. By choosing $\widetilde{\alpha}\in(0,\infty)$ satisfying

$$\Big\|\sum_{l=-N}^N f_l\Big\|^p_{WH^p_L(\mathbb{R}^n)}\sim\widetilde{\alpha}^p\Big|\Big\{x\in\mathbb{R}^n:\ S_L\Big(\sum_{l=-N}^N f\Big)(x)>\widetilde{\alpha}\Big\}\Big|$$

and $l_0\in\mathbb{N}$ satisfying $2^{l_0}\le\widetilde{\alpha}^p<2^{l_0+1}$, we write

$$\sum_{l=-N}^N f_l=\sum_{l=-N}^{l_0} f_l+\sum_{l=l_0+1}^N\cdots=:f_1+f_2.$$

Here, without loss of generality, we may assume that $l_0\le N$; otherwise, we only need to estimate f_1.

To estimate f_1, let $q\in(p,2)$. By Chebyshev's inequality, Minkowski's inequality, Hölder's inequality and the definition of f_l, we know that

$$\begin{aligned}
&\Big|\Big\{x\in\mathbb{R}^n:\ S_L(f_1)(x)>\widetilde{\alpha}\Big\}\Big|\\
&\lesssim\widetilde{\alpha}^{-q}\Big(\Big[\int_{\mathbb{R}^n}\Big|S_L(f_1)(x)\Big|^q\,dx\Big]^{\frac1q}\Big)^q\lesssim\widetilde{\alpha}^{-q}\Big(\sum_{l=-N}^{l_0}\Big[\int_{\mathbb{R}^n}\Big|S_L(f_l)(x)\Big|^q\,dx\Big]^{\frac1q}\Big)^q\\
&\lesssim\widetilde{\alpha}^{-q}\Big\{\sum_{l=-N}^{l_0}\sum_{\ell\in\mathbb{Z}_+}\Big[\int_{S_\ell(\vec{B})}\Big|S_L(f_l)(x)\Big|^2\,dx\Big]^{\frac12}\Big|S_\ell(\vec{B})\Big|^{\frac1q-\frac12}\Big\}^q\\
&\lesssim\widetilde{\alpha}^{-q}\Big[\sum_{l=-N}^{l_0}\sum_{\ell\in\mathbb{Z}_+}2^{\ell n(\frac1q-\frac12)}|\Omega|^{(\frac12-\frac1p)+(\frac1q-\frac12)}2^{-i\mathcal{N}}\Big\{\int_{S_\ell(\vec{B})}\Big[\int_0^r+\int_r^{2^{\ell-3}r}+\int_{2^{\ell-3}r}^\infty\Big]\\
&\quad\times\int_{\{y\in\mathbb{R}^n:\ |y-x|<t\}}\Big|t^2Le^{-t^2L}\mathcal{A}_r\Big(\chi_{S_i(\vec{B})}\varphi_l\Big)(y)\Big|^2\frac{dy\,dt\,dx}{t^{n+1}}\Big\}^{\frac12}\Big]^q\\
&=:\widetilde{\mathrm{Q}}_1+\widetilde{\mathrm{Q}}_2+\widetilde{\mathrm{Q}}_3.
\end{aligned}$$

We first estimate $\widetilde{\mathrm{Q}}_1$. For all $\ell\in\mathbb{Z}_+$, let E_ℓ be as in (4.18). It is easy to see that, for all $i\in\mathbb{N}$ and $\ell>i+1$,

$$\frac{r}{\mathrm{dist}(E_\ell,\,S_i(\vec{B}))}\lesssim2^{-\ell}.$$

Thus, by the fact $|\Omega|\sim2^{-l}$, Fubini's theorem, the Davies-Gaffney estimates, the quadratic estimates, $q\in(p,2)$ and the assumption $2^{l_0}\sim\widetilde{\alpha}^p$, we conclude that

$$\begin{aligned}
\widetilde{\mathrm{Q}}_1\lesssim&\widetilde{\alpha}^{-q}\Big[\sum_{l=-N}^{l_0}\sum_{\ell=0}^{i+1}2^{\ell n(\frac1q-\frac12)}2^{l(\frac1p-\frac1q)}2^{-i\mathcal{N}}\|\varphi_l\|_{L^2(S_i(\vec{B}))}\Big]^q\\
&+\widetilde{\alpha}^{-q}\Big[\sum_{l=-N}^{l_0}\sum_{\ell=i+2}^{\infty}2^{\ell n(\frac1q-\frac12)}2^{l(\frac1p-\frac1q)}2^{-i\mathcal{N}}\int_0^r\exp\Big\{-C\frac{[\mathrm{dist}(E_\ell,\,S_i(\vec{B}))]^2}{t^2}\Big\}\frac{dt}{t}\\
&\times\|\varphi_l\|_{L^2(S_i(\vec{B}))}\Big]^q
\end{aligned}$$

$$\lesssim \widetilde{\alpha}^{-q}\Big[\sum_{l=-\infty}^{l_0} 2^{l(\frac{1}{p}-\frac{1}{q})}\Big]^q + \widetilde{\alpha}^{-q}2^{l_0}\Big[\sum_{l=-\infty}^{l_0} 2^{l(\frac{1}{p}-\frac{1}{q})}\Big]^q \sim \widetilde{\alpha}^{-q}2^{l_0(\frac{q}{p}-1)} \sim \widetilde{\alpha}^{-p}.$$

Similar to the estimates of Q_1 and Q_2, we have $\widetilde{\mathrm{Q}}_2 + \widetilde{\mathrm{Q}}_3 \lesssim \widetilde{\alpha}^{-p}$. Thus,

$$\widetilde{\alpha}^p\Big|\Big\{x \in \mathbb{R}^n:\ S_L(f_1)(x) > \widetilde{\alpha}\Big\}\Big| \lesssim \widetilde{\alpha}^{p-q}2^{l_0(\frac{1}{p}-\frac{1}{q})q} \sim 1.$$

On the other hand, to estimate f_2, let $\widetilde{q} \in (0, p)$. Then

$$\begin{aligned}\mathrm{J} :=& \Big|\Big\{x \in \mathbb{R}^n:\ S_L\Big(\sum_{l=l_0+1}^{N} f_l\Big)(x) > \widetilde{\alpha}\Big\}\Big|\\ \lesssim& \,\widetilde{\alpha}^{-q}\Big[\sum_{l=-N}^{l_0}\sum_{\ell\in\mathbb{Z}_+}\Big\{\int_{S_\ell(\vec{B})}\Big[S_L(f_l)(x)\Big]^2\,dx\Big\}^{\frac{1}{2}}\Big|S_\ell(\vec{B})\Big|^{\frac{1}{q}-\frac{1}{2}}\Big]^q.\end{aligned}$$

Similar to the estimates of f_1, we also obtain

$$\mathrm{J} \lesssim \widetilde{\alpha}^{-q}2^{l(\frac{1}{p}-\frac{1}{q})\epsilon},$$

which implies that $\widetilde{\alpha}^p|\{x \in \mathbb{R}^n:\ S_L(f_2)(x) > \widetilde{\alpha}\}| \lesssim 1$. Combining the estimates of f_1 and f_2, we conclude that

$$\Big\|\sum_{l=-N}^{N} f_l\Big\|_{WH_L^p(\mathbb{R}^n)} \sim \widetilde{\alpha}^p\Big|\Big\{x \in \mathbb{R}^n:\ S_L\Big(\sum_{l=-N}^{N} f_l\Big)(x) > \widetilde{\alpha}\Big\}\Big| \lesssim 1,$$

where the implicit constants are independent of N. Thus, by letting $N \to \infty$, we obtain

$$\Big\|\sum_{l=-\infty}^{\infty} f_l\Big\|_{WH_L^p(\mathbb{R}^n)} \lesssim 1.$$

Thus, from (4.15), (4.16) and the assumption that $g \in (WH_L^p(\mathbb{R}^n))^*$, we deduce that

$$\|g\|_{W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}} \sim \int_0^\infty \frac{w_{\mathcal{N}}(\delta)}{\delta}\,d\delta \lesssim \lim_{N\to\infty}\Big|\Big\langle g,\ \sum_{l=-N}^{N} f_l\Big\rangle_{L^2(\mathbb{R}^n)}\Big| \lesssim \|g\|_{(WH_L^p(\mathbb{R}^n))^*},$$

which implies that $g \in W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}$ and

$$\|g\|_{W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}} \lesssim \|g\|_{(WH_L^p(\mathbb{R}^n))^*}.$$

This shows that $(WH_L^p(\mathbb{R}^n))^* \subset W\Lambda_{L,\mathcal{N}}^{n(\frac{1}{p}-1)}$ and hence finishes the proof of Theorem 4.6. $\square$

Remark 4.10. By Theorem 4.6, we see that $W\Lambda_{L,\mathcal{N}}^{\alpha}(\mathbb{R}^n)$ for $\alpha \in [0, \infty)$ is independent of the choice of $\mathcal{N} \in (n(\frac{1}{p}-\frac{1}{2}), \infty)$. Thus, we can write $W\Lambda_{L,\mathcal{N}}^{\alpha}(\mathbb{R}^n)$ simply by $W\Lambda_L^{\alpha}(\mathbb{R}^n)$.

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