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APPROXIMATION METHOD FOR GENERAL MIXED EQUILIBRIUM MANN-TYPE VISCOSITY APPROXIMATION METHOD FOR GENERAL MIXED EQUILIBRIUM

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Abstract. In this paper, we introduce a Mann-type viscosity iterative algorithm for finding a common element of the set of solutions of a general mixed equilibrium problem, the set of solutions of a variational inequality for an inverse strongly monotone mapping, and the set of common fixed points of a strict pseudocontraction, one finite family of nonexpansive mappings and another infinite family of nonexpansive mappings in a real Hilbert space. The iterative algorithm is based on composite viscosity approximation method, Mann's iterative method, W-mapping approach to common fixed points of infinitely many nonexpansive mappings, and strongly positive bounded linear operator approach. We derive the strong convergence of the iterative algorithm to a common element of these sets, which also solves some hierarchical minimization. The result presented in this paper improves and extends some corresponding ones in the earlier and recent literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping V is called strongly positive on H if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Vx, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

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In particular, if $L = 1$ then A is called a nonexpansive mapping; if $L \in [0, 1)$ then A is called a contraction. A mapping $T : C \to C$ is called ξ-strictly pseudocontractive if there exists a constant $\xi \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \xi\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

In particular, if $\xi = 0$, then T is a nonexpansive mapping.

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $\overline{x} \in C$ such that

$$\langle A\overline{x}, y - \overline{x}\rangle \ge 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, A)$.

The VIP (1.1) was first discussed by Lions [35] and now is well known. Variational inequalities have extensively been investigated; see the monographs [3, 27–29, 32], and also the articles [6, 7, 9–12, 14, 19, 23, 24, 31, 43, 44, 49, 55, 57] (and the references therein). In 2003, for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ when $C \subset H$ is nonempty, closed and convex, $S : C \to C$ is nonexpansive and $A : C \to H$ is α-inverse strongly monotone, Takahashi and Toyoda [49] introduced the following Mann's type iterative algorithm:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, 2\alpha)$. It was shown in [49] that, if $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \ne \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$. Further, given a contractive mapping $f : C \to C$, an α-inverse strongly monotone mapping $A : C \to H$ and a nonexpansive mapping $T : C \to C$, Jung [31] introduced the following two-step iterative scheme by the composite viscosity approximation method

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n TP_C(y_n - \lambda_n Ay_n), \quad \forall n \ge 0, \end{cases} \tag{1.2}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1)$. It was proven in [31] that, if $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A) \ne \emptyset$, then the sequence $\{x_n\}$ converges strongly to $q = P_{\mathrm{Fix}(T)\cap \mathrm{VI}(C,A)} f(q)$.

Furthermore, if C is the fixed point set $\mathrm{Fix}(T)$ of a nonexpansive mapping T and S is another nonexpansive mapping (not necessarily with fixed points), the VIP (1.1) becomes the VIP of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - S)x^*, x - x^*\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.3}$$

This problem, introduced by Mainge and Moudafi [38, 39], is called hierarchical fixed point problem. It is clear that if S has fixed points, then they are solutions of VIP (1.3). In the literature, the recent research work shows that variational inequalities like (1.1) cover several topics, for example, monotone inclusions, convex optimization and quadratic minimization over fixed point sets; see [36, 41, 52, 53] for more details.

In [54], Yao, Liou and Marino [54] introduced two-step iterative algorithm that generates a sequence $\{x_n\}$ via the explicit scheme

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, \quad \forall n \geq 1. \end{cases} \tag{1.4}$$

Theorem 1.1 (**YLM** see [54])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S and T be two nonexpansive mappings of C into itself. Let $f : C \to C$ be a ρ-contraction and $\{\alpha_n\}$ and $\{\beta_n\}$ two real sequences in $(0,1)$. Assume that the sequence $\{x_n\}$ generated by scheme (1.4) is bounded and*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| = 0$, $\lim_{n\to\infty} \frac{1}{\beta_n}|1 - \frac{\alpha_{n-1}}{\alpha_n}| = 0$;

(iii) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, $\lim_{n\to\infty} \frac{\beta_n^2}{\alpha_n} = 0$;

(iv) $\mathrm{Fix}(T) \cap \mathrm{int}(C) \neq \emptyset$;

(v) *there exists a constant $k > 0$ such that $\|x - Tx\| \geq k \cdot \mathrm{dist}(x, \mathrm{Fix}(T))$ for each $x \in C$, where $\mathrm{dist}(x, \mathrm{Fix}(T)) = \inf_{y \in \mathrm{Fix}(T)} \|x - y\|$.*

Then the sequence $\{x_n\}$ strongly converges to $x^* = P_{\mathrm{Fix}(T)} f(x^*)$ which solves the VIP (1.3) with $S = f$.

In this paper, we consider the following general mixed equilibrium problem (GMEP) (see, also, [5, 34, 45]) of finding $x \in C$ such that

$$\Theta(x,y) + h(x,y) \geq 0, \quad \forall y \in C, \tag{1.5}$$

where $\Theta, h : C \times C \to \mathbf{R}$ are two bi-functions. We denote the set of solutions of GMEP (1.5) by GMEP(Θ, h). The GMEP (1.5) is very general, for examples, it includes the following equilibrium problems as special cases:

As an example, in [8,21,37,50] the authors considered and studied the generalized equilibrium problem (GEP) which is to find $x \in C$ such that

$$\Theta(x,y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C.$$

The set of solutions of GEP is denoted by GEP(Θ, A).

In [5,8,20,39], the authors considered and studied the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

The set of solutions of MEP is denoted by MEP(Θ, φ).

In [2,15,38,47], the authors considered and studied the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x,y) \geq 0, \quad \forall y \in C.$$

The set of solutions of EP is denoted by EP(Θ). It is worth to mention that the EP is an unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, etc.

Throughout this paper, it is assumed as in [25] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (θ1)-(θ3) and $h : C \times C \to \mathbf{R}$ is a bi-function with restrictions (h1)-(h3), where

(θ1) $\Theta(x,x) = 0$ for all $x \in C$;

(θ2) Θ is monotone (i.e., $\Theta(x,y) + \Theta(y,x) \le 0, \forall x, y \in C$) and upper hemicontinuous in the first variable, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} \Theta(tz + (1-t)x, y) \le \Theta(x,y);$$

(θ3) Θ is lower semicontinuous and convex in the second variable;

(h1) $h(x,x) = 0$ for all $x \in C$;

(h2) h is monotone and weakly upper semicontinuous in the first variable;

(h3) h is convex in the second variable.

For $r > 0$ and $x \in H$, let $T_r : H \to 2^C$ be a mapping defined by

$$T_r x = \{z \in C : \Theta(z,y) + h(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}$$

called the resolvent of Θ and h.

On the other hand, for a long time, many authors were interested in the construction of iterative algorithms that weakly or strongly converge to a common fixed point of a family of nonexpansive mappings; see e.g., [2, 4, 33].

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \ge 1$, define a mapping W_n on C as follows:

$$(1.6) \qquad \begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \cdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \cdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In 2013, Rattanaseeha [47] introduced an iterative algorithm:

$$(1.7) \qquad \begin{cases} x_1 \in H \text{ arbitrarily given}, \\ \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n V)W_n u_n], \quad \forall n \ge 1, \end{cases}$$

and proved the following strong convergence theorem.

Theorem 1.2 (see [47, Theorem 3.1])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (θ1)-(θ3). Let f be an α-contraction on H with $\alpha \in (0,1)$, and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{EP}(\Theta) \neq \emptyset$. Let $V : H \to H$ be a $\overline{\gamma}$-strongly positive bounded linear operator with $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of real numbers such that $0 < \lambda_n \le b < 1$, $n = 1, 2, \ldots$. Let W_n be the W-mapping of C into itself generated by (1.6). Let*

W be defined by $Wx = \lim_{n\to\infty} W_n x, \forall x \in C$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by (1.7), where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\}$ is a sequence in $(0,\infty)$ such that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, *(C2)* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and (C3)* $\lim_{n\to\infty} r_n = r > 0$.
Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (V - \gamma f))x^*$ is a unique solution of the VIP

$$\langle (V - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Vx, x\rangle - \Psi(x),$$

where Ψ is a potential function for γf.

In addition, Marino, Muglia and Yao [40] introduced a multi-step iterative scheme

$$(1.8) \qquad \begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, \quad i = 2, \ldots, N, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T y_{n,N}, \end{cases}$$

with $f : C \to C$ a ρ-contraction and $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, that generalizes the two-step iterative scheme (1.4) for two nonexpansive mappings to a finite family of nonexpansive mappings $T, S_i : C \to C$, $i = 1, \ldots, N$, and proved that the proposed scheme (1.8) converges strongly to a common fixed point of the mappings that is also an equilibrium point of the GMEP (1.5).

More recently, Marino, Muglia and Yao's multi-step iterative scheme (1.8) was extended to develop the following composite viscosity iterative algorithm by virtue of Jung's two-step iterative scheme (1.2).

Algorithm 1.3 (**CPY** (see (3.1) in [17])**.** Let $f : C \to C$ be a ρ-contraction and $A : C \to H$ be an α-inverse strongly monotone mapping. Let $S_i, T : C \to C$ be nonexpansive mappings for each $i = 1, \ldots, N$. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (θ1)-(θ3) and $h : C \times C \to \mathbf{R}$ be a bi-function with restrictions (h1)-(h3). Let $\{x_n\}$ be the sequence generated by

$$(1.9) \qquad \begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, \quad i = 2, \ldots, N, \\ y_n = \alpha_n f(y_{n,N}) + (1 - \alpha_n) T P_C(y_{n,N} - \lambda_n A y_{n,N}), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T P_C(y_n - \lambda_n A y_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ with

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1,$$

$\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ with

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

$\{\beta_{n,i}\}$ is a sequence in $(0,1)$ for each $i = 1,\dots,N$, and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$.

It was proven in [17] that the proposed scheme (1.9) converges strongly to a common fixed point of the mappings $T, S_i : C \to C, i = 1,\dots,N$, that is also an equilibrium point of the GMEP (1.5) and a solution of the VIP (1.1).

In this paper, we introduce a Mann-type viscosity iterative algorithm for finding a common element of the solution set GMEP(Θ, h) of GMEP (1.5), the solution set VI(C,A) of VIP (1.1) for an inverse-strongly monotone mapping $A : C \to H$, and the common fixed point set $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \cap_{i=1}^{N}\mathrm{Fix}(S_i) \cap \mathrm{Fix}(T)$ of a strict pseudocontraction $T : H \to H$, one finite family of nonexpansive mappings $S_i : C \to C,\ i = 1,\dots,N$ and another infinite family of nonexpansive mappings $T_n : C \to C,\ n = 1,2,\dots$, in the setting of the infinite-dimensional Hilbert space. The iterative algorithm is based on composite viscosity approximation method [31], Mann's iterative method, W-mapping approach to common fixed points of infinitely many nonexpansive mappings, and strongly positive bounded linear operator approach. Our aim is to prove the strong convergence of the iterative algorithm to an element of $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \cap_{i=1}^{N}\mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta,h) \cap \mathrm{VI}(C,A) \cap \mathrm{Fix}(T)$, which also solves some hierarchical minimization. The result presented in this paper improves and extends some corresponding ones in the earlier and recent literature. We observe that related results have been derived say in [1,2,10,13,16–18,22,26,38–40,49,51,54].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$ and $\omega_s(x_n)$ to denote the strong ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\},$$

and

$$\omega_s(x_n) := \{x \in H : x_{n_i} \to x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x,C).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1. *Given any $x \in H$ and $z \in C$. One has*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \le 0,\ \forall y \in C;$

(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C;$

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall y \in H$, *which hence implies that* P_C *is nonexpansive and monotone.*

Definition 2.2. A mapping $T : H \to H$ is said to be

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.3. A mapping $A : C \to H$ is said to be

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

On the other hand, it is obvious that if $A : C \to H$ is ζ-inverse-strongly monotone, then A is monotone and $\frac{1}{\zeta}$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I - \lambda A)u - (I - \lambda A)v\|^2 \\ &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle Au - Av, u - v\rangle + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\zeta)\|Au - Av\|^2. \end{aligned} \tag{2.1}$$

So, if $\lambda \leq 2\zeta$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

It is clear that, in a real Hilbert space H, $T : C \to C$ is ξ-strictly pseudocontractive if and only if the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \xi}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

This immediately implies that if T is a ξ-strictly pseudocontractive mapping, then $I - T$ is $\frac{1-\xi}{2}$-inverse strongly monotone; for further detail, we refer to [42] and the references therein. It is well known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings and that the class of pseudocontractions strictly includes the class of strict pseudocontractions.

Proposition 2.4 (see [42, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) *If T is a ξ-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \leq \frac{1+\xi}{1-\xi}\|x - y\|, \quad \forall x, y \in C.$$

(ii) *If T is a ξ-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - T)x_n \to 0$, then $(I - T)\tilde{x} = 0$.*

(iii) *If T is ξ-(quasi-)strict pseudocontraction, then the fixed-point set $\mathrm{Fix}(T)$ of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well defined.*

Proposition 2.5 (see [55]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a ξ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)\xi \leq \gamma$. Then*

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.6. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.7. *Let H be a real Hilbert space. Then the following hold:*

(a) *$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$;*

(b) *$\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$;*

(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

Lemma 2.8 (see [48]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists where $U_{n,k}$ is defined as in (1.6).*

Remark 2.9 (see [56, Remark 3.1]). It can be known from Lemma 2.8 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \epsilon.$$

Remark 2.10 (see [56, Remark 3.2])**.** Utilizing Lemma 2.8, we define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C.$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 1\}$. Hence, it is clear from Remark 2.9 that for an arbitrary $\epsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Lemma 2.11 (see [48])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, $\mathrm{Fix}(W) = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.*

Lemma 2.12 (see [30, Demiclosedness principle])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with $\mathrm{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.13. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2.1 (i)) implies*

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \forall \lambda > 0.$$

Lemma 2.14 (see [41])**.** *Let V be a $\overline{\gamma}$-strongly positive bounded linear operator on H and assume $0 < \rho \leq \|V\|^{-1}$. Then $\|I - \rho V\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.15 (see [53])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - s_n)a_n + s_n b_n + t_n, \quad \forall n \geq 1,$$

where $\{s_n\}, \{t_n\}$ and $\{b_n\}$ satisfy the following conditions:

(i) $\{s_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} s_n = \infty$;
(ii) *either* $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=1}^{\infty} |s_n b_n| < \infty$;
(iii) $t_n \geq 0$ *for all* $n \geq 1$, *and* $\sum_{n=1}^{\infty} t_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

In the sequel, we will indicate with $\mathrm{GMEP}(\Theta, h)$ the solution set of GMEP (1.2).

Lemma 2.16 (see [25])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (θ1)-(θ3) and $h : C \times C \to \mathbf{R}$ is a bi-function with restrictions (h1)-(h3). Moreover, let us suppose that*

(H) for fixed $r > 0$ and $x \in C$, there exist a bounded $K \subset C$ and $\hat{x} \in K$ such that for all $z \in C \setminus K$, $-\Theta(\hat{x}, z) + h(z, \hat{x}) + \frac{1}{r}\langle \hat{x} - z, z - x\rangle < 0$.

For $r > 0$ and $x \in H$, the mapping $T_r : H \to 2^C$ (i.e., the resolvent of Θ and h) has the following properties:

(i) $T_r x \neq \emptyset$;
(ii) *$T_r x$ is a singleton;*
(iii) *T_r is firmly nonexpansive;*
(iv) $\mathrm{GMEP}(\Theta, h) = \mathrm{Fix}(T_r)$ *and it is closed and convex.*

Lemma 2.17 (see [25])**.** *Let us suppose that (θ1)-(θ3), (h1)-(h3) and (H) hold. Let $x, y \in H$, $r_1, r_2 > 0$. Then*

$$\|T_{r_2}y - T_{r_1}x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}y - y\|.$$

Lemma 2.18 (see [40])**.** *Suppose that the hypotheses of Lemma 2.16 are satisfied. Let $\{r_n\}$ be a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$. Suppose that $\{x_n\}$ is a bounded sequence. Then the following statements are equivalent and true:*

(a) *if $\|x_n - T_{r_n}x_n\| \to 0$ as $n \to \infty$, each weak cluster point of $\{x_n\}$ satisfies the problem*

$$\Theta(x, y) + h(x, y) \geq 0, \quad \forall y \in C,$$

i.e., $\omega_w(x_n) \subseteq \mathrm{GMEP}(\Theta, h)$.

(b) *the demiclosedness principle holds in the sense that, if $x_n \rightharpoonup x^*$ and $\|x_n - T_{r_n}x_n\| \to 0$ as $n \to \infty$, then $(I - T_{r_k})x^* = 0$ for all $k \geq 1$.*

Finally, recall that a set-valued mapping $\widetilde{T} : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in \widetilde{T}x$ and $g \in \widetilde{T}y$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $\widetilde{T} : H \to 2^H$ is maximal if its graph $G(\widetilde{T})$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $\widetilde{T}$ is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for all $(y, g) \in G(\widetilde{T})$ implies $f \in \widetilde{T}x$. Let $A : C \to H$ be a monotone, L-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$. Define

$$\widetilde{T}v = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

It is known in [46] that in this case $\widetilde{T}$ is maximal monotone, and

$$0 \in \widetilde{T}v \ \Leftrightarrow \ v \in \mathrm{VI}(C, A). \tag{2.2}$$

3. Main results

We now propose the following Mann-type viscosity iterative scheme:

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1}S_1u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_iu_n + (1 - \beta_{n,i})y_{n,i-1}, \quad i = 2, \ldots, N, \\ y_n = \alpha_n\gamma f(y_{n,N}) + (I - \alpha_n\mu V)W_nP_C(y_{n,N} - \nu_nAy_{n,N}), \\ x_{n+1} = \beta_ny_n + \gamma_nP_C(y_n - \nu_nAy_n) + \delta_nTP_C(y_n - \nu_nAy_n), \end{cases} \tag{3.1}$$

for all $n \geq 1$, where

$A : C \to H$ is an α-inverse-strongly monotone mapping;

V is a $\overline{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ is an l-Lipschitz continuous mapping with $0 \leq \gamma l < \mu\overline{\gamma}$;

$T : H \to H$ is a ξ-strict pseudocontraction and $S_i : C \to C$ is a nonexpansive mapping for each $i = 1, \ldots, N$;

$\Theta, h : C \times C \to \mathbf{R}$ are two bi-functions satisfying the hypotheses of Lemma 2.16;

$\{\nu_n\}$ is a sequence in $(0, 2\alpha)$ with $0 < \liminf_{n\to\infty} \nu_n \leq \limsup_{n\to\infty} \nu_n < 2\alpha$;

$\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

$\{\gamma_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ with $\beta_n + \gamma_n + \delta_n = 1, \forall n \geq 1$;

$\{\beta_{n,i}\}_{i=1}^N$ are sequences in $(0, 1)$ and $(\gamma_n + \delta_n)\xi \leq \gamma_n, \forall n \geq 1$;

$\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\liminf_{n\to\infty} \delta_n > 0$.

We start our main result from the following series of propositions.

Proposition 3.1. *Let us suppose that* $\Omega = \cap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T) \neq \emptyset$. *Then the sequences* $\{x_n\}, \{y_n\}, \{y_{n,i}\}$ *for all* i, $\{u_n\}$ *are bounded.*

Proof. Since $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [c, d] \subset (0, 1)$ and $0 < \alpha_n\mu \leq \|V\|^{-1}$ for all $n \geq 1$. Since V is a $\overline{\gamma}$-strongly positive bounded linear operator on H, by Lemma 2.14 we know that

$$\|I - \alpha_n\mu V\| \leq 1 - \alpha_n\mu\overline{\gamma}, \quad \forall n \geq 1.$$

Let us observe that, if $p \in \Omega$, then

$$\|y_{n,1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

For all from $i = 2$ to $i = N$, by induction, one proves that

$$\|y_{n,i} - p\| \leq \beta_{n,i}\|u_n - p\| + (1 - \beta_{n,i})\|y_{n,i-1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

Thus we obtain that for every $i = 1, \ldots, N$,

$$\|y_{n,i} - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.2}$$

Let $\tilde{y}_{n,N} = P_C(y_{n,N} - \nu_n Ay_{n,N})$ and $\tilde{y}_n = P_C(y_n - \nu_n Ay_n)$ for every $n \geq 1$. Since $I - \nu_n A$ is nonexpansive and $p = P_C(p - \nu_n Ap)$ (due to Lemma 2.13), we have

$$\begin{aligned} \|\tilde{y}_{n,N} - p\| &= \|P_C(y_{n,N} - \nu_n Ay_{n,N}) - P_C(p - \nu_n Ap)\| \\ &\leq \|(y_{n,N} - \nu_n Ay_{n,N}) - (p - \nu_n Ap)\| \\ &\leq \|y_{n,N} - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \end{aligned} \tag{3.3}$$

Moreover, from $p = W_n p$ we get

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n\gamma(f(y_{n,N}) - f(p)) + (I - \alpha_n\mu V)(W_n\tilde{y}_{n,N} - p) + \alpha_n(\gamma f - \mu V)p\| \\ &\leq \alpha_n\gamma\|f(y_{n,N}) - f(p)\| + \|I - \alpha_n\mu V\|\|W_n\tilde{y}_{n,N} - p\| + \alpha_n\|(\gamma f - \mu V)p\| \\ &\leq \alpha_n\gamma l\|y_{n,N} - p\| + (1 - \alpha_n\mu\overline{\gamma})\|\tilde{y}_{n,N} - p\| + \alpha_n\|(\gamma f - \mu V)p\| \\ &\leq \alpha_n\gamma l\|y_{n,N} - p\| + (1 - \alpha_n\mu\overline{\gamma})\|y_{n,N} - p\| + \alpha_n\|(\gamma f - \mu V)p\| \\ &= (1 - \alpha_n(\mu\overline{\gamma} - \gamma l))\|y_{n,N} - p\| + \alpha_n\|(\gamma f - \mu V)p\| \\ &= (1 - \alpha_n(\mu\overline{\gamma} - \gamma l))\|y_{n,N} - p\| + \alpha_n(\mu\overline{\gamma} - \gamma l)\frac{\|(\gamma f - \mu V)p\|}{\mu\overline{\gamma} - \gamma l} \end{aligned}$$

$$\leq \max\Big\{\|y_{n,N} - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\overline{\gamma} - \gamma l}\Big\}$$
$$\leq \max\Big\{\|x_n - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\overline{\gamma} - \gamma l}\Big\}.$$

Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 1$, utilizing Proposition 2.5 we obtain from the last inequality

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n(y_n - p) + \gamma_n(P_C(y_n - \nu_n Ay_n) - p) \\
&\quad +\delta_n(TP_C(y_n - \nu_n Ay_n) - p)\| \\
&\leq \beta_n\|y_n - p\| + \|\gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)\| \\
&\leq \beta_n\|y_n - p\| + (\gamma_n + \delta_n)\|\tilde{y}_n - p\| \\
&\leq \beta_n\|y_n - p\| + (\gamma_n + \delta_n)\|y_n - p\| \\
&= \|y_n - p\| \\
&\leq \max\Big\{\|x_n - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\overline{\gamma} - \gamma l}\Big\}.
\end{aligned}$$

By induction, we get

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\overline{\gamma} - \gamma l}\}, \quad \forall n \geq 1.$$

This implies that $\{x_n\}$ is bounded and so are $\{Ay_{n,N}\}, \{Ay_n\}, \{\tilde{y}_{n,N}\}, \{\tilde{y}_n\}$, $\{u_n\}$, $\{y_n\}, \{y_{n,i}\}$ for each $i = 1, \ldots, N$. Since $\|W_n\tilde{y}_{n,N} - p\| \leq \|y_{n,N} - p\|$ and $\|T\tilde{y}_n - p\| \leq \frac{1+\xi}{1-\xi}\|y_n - p\|$, $\{W_n\tilde{y}_{n,N}\}$ and $\{T\tilde{y}_n\}$ are also bounded. □

Proposition 3.2. *Let us suppose that $\Omega \neq \emptyset$. Moreover, let us suppose that the following hold:*

(H0) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(H1) $\sum_{n=2}^{\infty} |\nu_n - \nu_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\nu_n - \nu_{n-1}|}{\alpha_n} = 0$;
(H2) $\sum_{n=2}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$;
(H3) $\sum_{n=2}^{\infty} |\beta_{n,i} - \beta_{n-1,i}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\alpha_n} = 0$ *for each* $i = 1, \ldots, N$;
(H4) $\sum_{n=2}^{\infty} |r_n - r_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|r_n - r_{n-1}|}{\alpha_n} = 0$;
(H5) $\sum_{n=2}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0$;
(H6) $\sum_{n=2}^{\infty} |\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}| < \infty$ *or* $\lim_{n\to\infty} \frac{1}{\alpha_n}|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}| = 0$.

Then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, *i.e.,* $\{x_n\}$ *is asymptotically regular.*

Proof. First, it is known that $\{\beta_n\} \subset [c, d] \subset (0, 1)$ as in the proof of Proposition 3.1. Taking into account $\liminf_{n\to\infty} r_n > 0$, we may assume, without loss of generality, that $\{r_n\} \subset [\epsilon, \infty)$ for some $\epsilon > 0$. First, we write $x_n = \beta_{n-1}y_{n-1} + (1 - \beta_{n-1})w_{n-1}$, $\quad \forall n \geq 2$, where $w_{n-1} = \frac{x_n - \beta_{n-1}y_{n-1}}{1-\beta_{n-1}}$. It follows that for all $n \geq 2$

$$\begin{aligned}
w_n - w_{n-1} &= \frac{x_{n+1} - \beta_n y_n}{1 - \beta_n} - \frac{x_n - \beta_{n-1}y_{n-1}}{1 - \beta_{n-1}} \\
&= \frac{\gamma_n\tilde{y}_n + \delta_n T\tilde{y}_n}{1 - \beta_n} - \frac{\gamma_{n-1}\tilde{y}_{n-1} + \delta_{n-1}T\tilde{y}_{n-1}}{1 - \beta_{n-1}}
\end{aligned}$$

$$= \frac{\gamma_n(\tilde{y}_n - \tilde{y}_{n-1}) + \delta_n(T\tilde{y}_n - T\tilde{y}_{n-1})}{1-\beta_n} + \Big(\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big)\tilde{y}_{n-1} + \Big(\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\Big)T\tilde{y}_{n-1}. \tag{3.4}$$

Since $(\gamma_n + \delta_n)\xi \le \gamma_n$ for all $n \ge 1$, utilizing Proposition 2.5 we have

$$\|\gamma_n(\tilde{y}_n - \tilde{y}_{n-1}) + \delta_n(T\tilde{y}_n - T\tilde{y}_{n-1})\| \le (\gamma_n + \delta_n)\|\tilde{y}_n - \tilde{y}_{n-1}\|.$$

This together with (3.4), implies that

$$\begin{aligned} \|w_n - w_{n-1}\| &\le \frac{\|\gamma_n(\tilde{y}_n - \tilde{y}_{n-1}) + \delta_n(T\tilde{y}_n - T\tilde{y}_{n-1})\|}{1-\beta_n} \\ &\quad + \Big|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|\|\tilde{y}_{n-1}\| \\ &\quad + \Big|\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\Big|\|T\tilde{y}_{n-1}\| \\ &\le \frac{(\gamma_n + \delta_n)\|\tilde{y}_n - \tilde{y}_{n-1}\|}{1-\beta_n} \\ &\quad + \Big|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|(\|\tilde{y}_{n-1}\| + \|T\tilde{y}_{n-1}\|) \\ &= \|\tilde{y}_n - \tilde{y}_{n-1}\| + \Big|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|(\|\tilde{y}_{n-1}\| + \|T\tilde{y}_{n-1}\|). \end{aligned} \tag{3.5}$$

Next, we estimate $\|y_n - y_{n-1}\|$. From (3.1), we have

$$\begin{cases} y_n = \alpha_n\gamma f(y_{n,N}) + (I - \alpha_n\mu V)W_n\tilde{y}_{n,N}, \\ y_{n-1} = \alpha_{n-1}\gamma f(y_{n-1,N}) + (I - \alpha_{n-1}\mu V)W_{n-1}\tilde{y}_{n-1,N}, \quad \forall n \ge 2. \end{cases}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= (I - \alpha_n\mu V)(W_n\tilde{y}_{n,N} - W_{n-1}\tilde{y}_{n-1,N}) \\ &\quad + (\alpha_n - \alpha_{n-1})(\gamma f(y_{n-1,N}) - \mu V W_{n-1}\tilde{y}_{n-1,N}) \\ &\quad + \alpha_n\gamma(f(y_{n,N}) - f(y_{n-1,N})). \end{aligned} \tag{3.6}$$

Utilizing the nonexpansivity of W_n, T_n and $U_{n,i}$, we have from (1.6)

$$\begin{aligned} \|W_n\tilde{y}_{n-1,N} - W_{n-1}\tilde{y}_{n-1,N}\| &= \|\lambda_1 T_1 U_{n,2}\tilde{y}_{n-1,N} - \lambda_1 T_1 U_{n-1,2}\tilde{y}_{n-1,N}\| \\ &\le \lambda_1\|U_{n,2}\tilde{y}_{n-1,N} - U_{n-1,2}\tilde{y}_{n-1,N}\| \\ &= \lambda_1\|\lambda_2 T_2 U_{n,3}\tilde{y}_{n-1,N} - \lambda_2 T_2 U_{n-1,3}\tilde{y}_{n-1,N}\| \\ &\le \lambda_1\lambda_2\|U_{n,3}\tilde{y}_{n-1,N} - U_{n-1,3}\tilde{y}_{n-1,N}\| \\ &\le \cdots \\ &\le \lambda_1\lambda_2\cdots\lambda_{n-1}\|U_{n,n}\tilde{y}_{n-1,N} - U_{n-1,n}\tilde{y}_{n-1,N}\| \\ &\le \widehat{M}\prod_{i=1}^{n-1}\lambda_i, \end{aligned} \tag{3.7}$$

where $\sup_{n\ge1}\{\|U_{n+1,n+1}\tilde{y}_{n,N}\| + \|U_{n,n+1}\tilde{y}_{n,N}\|\} \le \widehat{M}$ for some $\widehat{M} > 0$. Moreover, it is easy to see that

$$\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| \le \|(y_{n,N} - \nu_n A y_{n,N}) - (y_{n-1,N} - \nu_{n-1} A y_{n-1,N})\|$$

$$\begin{aligned} &\le \|(y_{n,N} - \nu_n A y_{n,N}) - (y_{n-1,N} - \nu_n A y_{n-1,N})\| \\ &\quad + |\nu_{n-1} - \nu_n| \|A y_{n-1,N}\| \\ &\le \|y_{n,N} - y_{n-1,N}\| + |\nu_{n-1} - \nu_n| \|A y_{n-1,N}\|, \end{aligned} \tag{3.8}$$

and similarly,

$$\|\tilde{y}_n - \tilde{y}_{n-1}\| \le \|y_n - y_{n-1}\| + |\nu_{n-1} - \nu_n| \|A y_{n-1}\|. \tag{3.9}$$

Combining (3.6)-(3.8), we get from $\{\lambda_n\} \subset (0, b] \subset (0, 1)$

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \|I - \alpha_n \mu V\| \|W_n \tilde{y}_{n,N} - W_{n-1} \tilde{y}_{n-1,N}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(y_{n-1,N}) - \mu V W_{n-1} \tilde{y}_{n-1,N}\| \\ &\quad + \alpha_n \gamma \|f(y_{n,N}) - f(y_{n-1,N})\| \\ &\le (1 - \alpha_n \mu \overline{\gamma})(\|W_n \tilde{y}_{n,N} - W_n \tilde{y}_{n-1,N}\| + \|W_n \tilde{y}_{n-1,N} - W_{n-1} \tilde{y}_{n-1,N}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(y_{n-1,N}) - \mu V W_{n-1} \tilde{y}_{n-1,N}\| + \alpha_n \gamma l \|y_{n,N} - y_{n-1,N}\| \\ &\le (1 - \alpha_n \mu \overline{\gamma}) \Big(\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| + \widehat{M} \prod_{i=1}^{n-1} \lambda_i \Big) \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(y_{n-1,N}) - \mu V W_{n-1} \tilde{y}_{n-1,N}\| + \alpha_n \gamma l \|y_{n,N} - y_{n-1,N}\| \\ &\le (1 - \alpha_n \mu \overline{\gamma}) \Big(\|y_{n,N} - y_{n-1,N}\| + |\nu_{n-1} - \nu_n| \|A y_{n-1,N}\| + \widehat{M} \prod_{i=1}^{n-1} \lambda_i \Big) \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(y_{n-1,N}) - \mu V W_{n-1} \tilde{y}_{n-1,N}\| + \alpha_n \gamma l \|y_{n,N} - y_{n-1,N}\| \\ &\le (1 - \alpha_n(\mu \overline{\gamma} - \gamma l)) \|y_{n,N} - y_{n-1,N}\| + |\nu_{n-1} - \nu_n| \|A y_{n-1,N}\| + \widehat{M} \prod_{i=1}^{n-1} \lambda_i \\ &\quad + |\alpha_n \alpha_{n-1}| \|\gamma f(y_{n-1,N}) - \mu V W_{n-1} \tilde{y}_{n-1,N}\| \\ &\le (1 - \alpha_n(\mu \overline{\gamma} - \gamma l)) \|y_{n,N} - y_{n-1,N}\| \\ &\quad + \widetilde{M}(|\nu_{n-1} - \nu_n| + |\alpha_n - \alpha_{n-1}| + b^{n-1}), \end{aligned} \tag{3.10}$$

where $\sup_{n \ge 1} \{\|\gamma f(y_{n,N}) - \mu V W_n \tilde{y}_{n,N}\| + \|A y_{n,N}\| + \widehat{M}\} \le \widetilde{M}$ for some $\widetilde{M} > 0$. Furthermore, observe that $x_{n+1} = \beta_n y_n + (1 - \beta_n) w_n$ and $x_n = \beta_{n-1} y_{n-1} + (1 - \beta_{n-1}) w_{n-1}$. Simple calculations show that

$$x_{n+1} - x_n = (1 - \beta_n)(w_n - w_{n-1}) + \beta_n (y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(y_{n-1} - w_{n-1}). \tag{3.11}$$

Combining (3.5) and (3.9)-(3.11), we get from $\{\lambda_n\} \subset (0, b] \subset (0, 1)$

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (1 - \beta_n) \|w_n - w_{n-1}\| + \beta_n \|y_n - y_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|y_{n-1} - w_{n-1}\| \\ &\le (1 - \beta_n) [\|\tilde{y}_n - \tilde{y}_{n-1}\| + \Big| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \Big| (\|\tilde{y}_{n-1}\| \\ &\quad + \|T \tilde{y}_{n-1}\|)] + \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|y_{n-1} - w_{n-1}\| \\ &\le (1 - \beta_n) \Big[\|y_n - y_{n-1}\| + |\nu_{n-1} - \nu_n| \|A y_{n-1}\| \\ &\quad + \Big| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \Big| (\|\tilde{y}_{n-1}\| + \|T \tilde{y}_{n-1}\|) \Big] \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&+\beta_n\|y_n-y_{n-1}\|+|\beta_n-\beta_{n-1}|\|y_{n-1}-w_{n-1}\| \\
&\le \|y_n-y_{n-1}\|+|\nu_{n-1}-\nu_n|\|Ay_{n-1}\| \\
&+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|(\|\tilde{y}_{n-1}\|+\|T\tilde{y}_{n-1}\|) \\
&+|\beta_n-\beta_{n-1}|\|y_{n-1}-w_{n-1}\| \\
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\|y_{n,N}-y_{n-1,N}\| \\
&+\widetilde{M}(|\nu_{n-1}-\nu_n|+|\alpha_n-\alpha_{n-1}|+b^{n-1})+|\nu_{n-1}-\nu_n|\|Ay_{n-1}\| \\
&+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|(\|\tilde{y}_{n-1}\|+\|T\tilde{y}_{n-1}\|) \\
&+|\beta_n-\beta_{n-1}|\|y_{n-1}-w_{n-1}\| \\
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\|y_{n,N}-y_{n-1,N}\| \\
&+\widetilde{M_0}[|\nu_{n-1}-\nu_n|+|\alpha_n-\alpha_{n-1}| \\
&+b^{n-1}+|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|+|\beta_n-\beta_{n-1}|],
\end{aligned}
$$

where $\sup_{n\ge1}\{\widetilde{M}+\|Ay_n\|+\|\tilde{y}_n\|+\|T\tilde{y}_n\|+\|y_n-w_n\|\}\le\widetilde{M_0}$ for some $\widetilde{M_0}>0$.

In the meantime, by the definition of $y_{n,i}$ one obtains that, for all $i=N,\dots,2$

$$
\begin{aligned}
\|y_{n,i}-y_{n-1,i}\| &\le \beta_{n,i}\|u_n-u_{n-1}\|+\|S_iu_{n-1}-y_{n-1,i-1}\||\beta_{n,i}-\beta_{n-1,i}| \\
&\quad +(1-\beta_{n,i})\|y_{n,i-1}-y_{n-1,i-1}\|.
\end{aligned}
\tag{3.13}
$$

In the case $i=1$, we have

$$
\begin{aligned}
\|y_{n,1}-y_{n-1,1}\| &\le \beta_{n,1}\|u_n-u_{n-1}\|+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}| \\
&\quad +(1-\beta_{n,1})\|u_n-u_{n-1}\| \\
&= \|u_n-u_{n-1}\|+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}|.
\end{aligned}
\tag{3.14}
$$

Substituting (3.14) in all (3.13)-type one obtains for $i=2,\dots,N$

$$
\begin{aligned}
\|y_{n,i}-y_{n-1,i}\| &\le \|u_n-u_{n-1}\|+\sum_{k=2}^{i}\|S_ku_{n-1}-y_{n-1,k-1}\||\beta_{n,k}-\beta_{n-1,k}| \\
&\quad +\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}|.
\end{aligned}
$$

This together with (3.12) implies that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\|y_{n,N}-y_{n-1,N}\|+\widetilde{M_0}\Big[|\nu_n-\nu_{n-1}| \\
&\quad +|\alpha_n-\alpha_{n-1}|+b^{n-1}+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|+|\beta_n-\beta_{n-1}|\Big] \\
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\Big[\|u_n-u_{n-1}\|+\sum_{k=2}^{N}(\|S_ku_{n-1}-y_{n-1,k-1}\| \\
&\quad \times|\beta_{n,k}-\beta_{n-1,k}|)+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}|\Big] \\
&\quad +\widetilde{M_0}\Big[|\nu_n-\nu_{n-1}|+|\alpha_n-\alpha_{n-1}|+b^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|+|\beta_n-\beta_{n-1}|\Big] \\
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\|u_n-u_{n-1}\|+\sum_{k=2}^{N}(\|S_k u_{n-1}-y_{n-1,k-1} \\
&\times|\beta_{n,k}-\beta_{n-1,k}|)+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}| \\
&+\widetilde{M}_0\Big[|\nu_n-\nu_{n-1}|+|\alpha_n-\alpha_{n-1}| \\
&+|\beta_n-\beta_{n-1}|+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|+b^{n-1}\Big].
\end{aligned}
\tag{3.15}
$$

By Lemma 2.17, we know that

$$
\|u_n-u_{n-1}\|\le\|x_n-x_{n-1}\|+L\Big|1-\frac{r_{n-1}}{r_n}\Big|,
\tag{3.16}
$$

where $L=\sup_{n\ge1}\|u_n-x_n\|$. So, substituting (3.16) in (3.15) we obtain

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\le \Big(1-\alpha_n(\mu\overline{\gamma}-\gamma l))(\|x_n-x_{n-1}\|+L\Big|1-\frac{r_{n-1}}{r_n}|\Big) \\
&\quad+\sum_{k=2}^{N}\|S_ku_{n-1}-y_{n-1,k-1}\||\beta_{n,k}-\beta_{n-1,k}| \\
&\quad+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}|+\widetilde{M}_0\Big[|\nu_n-\nu_{n-1}| \\
&\quad+|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|+b^{n-1}\Big] \\
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\|x_n-x_{n-1}\|+L\Big|1-\frac{r_{n-1}}{r_n}\Big| \\
&\quad+\sum_{k=2}^{N}\|S_ku_{n-1}-y_{n-1,k-1}\||\beta_{n,k}-\beta_{n-1,k}| \\
&\quad+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}\|+\widetilde{M}_0\Big[|\nu_n-\nu_{n-1}| \\
&\quad+|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|\Big]+\widetilde{M}_0b^{n-1} \qquad (3.17)\\
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\|x_n-x_{n-1}\|+\widetilde{M}_1\Big[\frac{|r_n-r_{n-1}|}{r_n} \\
&\quad+\sum_{k=2}^{N}|\beta_{n,k}-\beta_{n-1,k}|+|\beta_{n,1}-\beta_{n-1,1}|+|\nu_n-\nu_{n-1}| \\
&\quad+|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}\Big|+\Big|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\Big|\Big]+\widetilde{M}_1b^{n-1} \\
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\|x_n-x_{n-1}\|+\widetilde{M}_1\Big[\frac{|r_n-r_{n-1}|}{\gamma} \\
&\quad+\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|+|\nu_n-\nu_{n-1}|+|\alpha_n-\alpha_{n-1}|
\end{aligned}
$$

$$+ |\beta_n - \beta_{n-1}| + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|\Big] + \widetilde{M}_1 b^{n-1},$$

where $\gamma > 0$ is a minorant for $\{r_n\}$ and $\sup_{n\geq 1}\{L + \widetilde{M}_0 + \sum_{k=2}^N \|S_k u_n - y_{n,k-1}\| + \|S_1 u_n - u_n\|\} \leq \widetilde{M}_1$ for some $\widetilde{M}_1 > 0$. By hypotheses (H0)-(H6) and Lemma 2.15, we obtain the claim. □

Proposition 3.3. *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that $\{x_n\}$ is asymptotically regular. Then $\|x_n - u_n\| = \|x_n - T_{r_n} x_n\| \to 0$ as $n \to \infty$.*

Proof. Take a fixed $p \in \Omega$ arbitrarily. We recall that, by the firm nonexpansivity of T_{r_n}, a standard calculation (see [26]) shows that for $p \in \mathrm{GMEP}(\Theta, h)$

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.18}$$

Utilizing Lemmas 2.6 and 2.7 (b), we obtain from $0 \leq \gamma l < \mu\overline{\gamma}$, (3.1), (3.2) and (3.18) that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n\gamma(f(y_{n,N}) - f(p)) + (I - \alpha_n\mu V)(W_n\tilde{y}_{n,N} - p) \\
&\quad + \alpha_n(\gamma f - \mu V)p\|^2 \\
&\leq \|\alpha_n\gamma(f(y_{n,N}) - f(p)) + (I - \alpha_n\mu V)(W_n\tilde{y}_{n,N} - p)\|^2 \\
&\quad + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\
&\leq [\alpha_n\gamma\|f(y_{n,N}) - f(p)\| + \|I - \alpha_n\mu V\|\|W_n\tilde{y}_{n,N} - p\|]^2 \\
&\quad + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\
&\leq [\alpha_n\gamma l\|y_{n,N} - p\| + (1 - \alpha_n\mu\overline{\gamma})\|\tilde{y}_{n,N} - p\|]^2 \\
&\quad + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\
&= [\alpha_n\mu\overline{\gamma}\frac{\gamma l}{\mu\overline{\gamma}}\|y_{n,N} - p\| + (1 - \alpha_n\mu\overline{\gamma})\|\tilde{y}_{n,N} - p\|]^2 \\
&\quad + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \qquad (3.19)\\
&\leq \alpha_n\mu\overline{\gamma}\frac{(\gamma l)^2}{(\mu\overline{\gamma})^2}\|y_{n,N} - p\|^2 + (1 - \alpha_n\mu\overline{\gamma})\|\tilde{y}_{n,N} - p\|^2 \\
&\quad + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\
&\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + \|\tilde{y}_{n,N} - p\|^2 + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\
&\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + \|y_{n,N} - p\|^2 + \nu_n(\nu_n - 2\alpha)\|Ay_{n,N} - Ap\|^2 \\
&\quad + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\
&\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + \|u_n - p\|^2 + \nu_n(\nu_n - 2\alpha)\|Ay_{n,N} - Ap\|^2 \\
&\quad + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\
&\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad + \nu_n(\nu_n - 2\alpha)\|Ay_{n,N} - Ap\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\|.
\end{aligned}$$

Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 1$, utilizing Proposition 2.5 we have from (3.1) and (3.19) that

$$\|x_{n+1} - p\|^2 = \|\beta_n(y_n - p) + \gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)\|^2$$

$$
\begin{aligned}
&= \left\| \beta_n(y_n - p) + (\gamma_n + \delta_n) \frac{1}{\gamma_n + \delta_n} [\gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)] \right\|^2 \\
&\le \beta_n \|y_n - p\|^2 + (\gamma_n + \delta_n) \left\| \frac{1}{\gamma_n + \delta_n} [\gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)] \right\|^2 \\
&\le \beta_n \|y_n - p\|^2 + (\gamma_n + \delta_n) \|\tilde{y}_n - p\|^2 \\
&= \beta_n \|y_n - p\|^2 + (1 - \beta_n) \|\tilde{y}_n - p\|^2 \\
&\le \beta_n \|y_n - p\|^2 + (1 - \beta_n)[\|y_n - p\|^2 + \nu_n(\nu_n - 2\alpha)\|Ay_n - Ap\|^2] \qquad (3.20)\\
&= \|y_n - p\|^2 + (1 - \beta_n)\nu_n(\nu_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&\le \alpha_n \mu \overline{\gamma} \|y_{n,N} - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad + \nu_n(\nu_n - 2\alpha)\|Ay_{n,N} - Ap\|^2 + 2\alpha_n \|(\gamma f - \mu V)p\| \|y_n - p\| \\
&\quad + (1 - \beta_n)\nu_n(\nu_n - 2\alpha)\|Ay_n - Ap\|^2.
\end{aligned}
$$

So, we deduce that

$$
\begin{aligned}
&\|x_n - u_n\|^2 + \nu_n(2\alpha - \nu_n)\|Ay_{n,N} - Ap\|^2 \\
&\quad + (1 - \beta_n)\nu_n(2\alpha - \nu_n)\|Ay_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \mu \overline{\gamma} \|y_{n,N} - p\|^2 \\
&\quad + 2\alpha_n \|(\gamma f - \mu V)p\| \|y_n - p\| \\
&\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad + \alpha_n \mu \overline{\gamma} \|y_{n,N} - p\|^2 + 2\alpha_n \|(\gamma f - \mu V)p\| \|y_n - p\|.
\end{aligned}
$$

By Propositions 3.1 and 3.2 we know that the sequences $\{x_n\}, \{y_n\}$ and $\{y_{n,N}\}$ are bounded, and that $\{x_n\}$ is asymptotically regular. Therefore, from $\alpha_n \to 0$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$ and $0 < \liminf_{n\to\infty} \nu_n \le \limsup_{n\to\infty} \nu_n < 2\alpha$, we obtain that

$$
\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|Ay_{n,N} - Ap\| = \lim_{n\to\infty} \|Ay_n - Ap\| = 0. \qquad (3.21)
$$

□

Remark 3.4. By the last proposition we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$, i.e., the sets of strong/weak cluster points of $\{x_n\}$ and $\{u_n\}$ coincide.

Of course, if $\beta_{n,i} \to \beta_i \neq 0$ as $n \to \infty$, for all indices i, the assumptions of Proposition 3.2 are enough to assure that

$$
\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,i}} = 0, \quad \forall i \in \{1, \dots, N\}.
$$

In the next proposition, we examine the case in which at least one sequence $\{\beta_{n,k_0}\}$ is a null sequence.

Proposition 3.5. *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that (H0) holds. Moreover, for an index $k_0 \in \{1, \dots, N\}$, $\lim_{n\to\infty} \beta_{n,k_0} = 0$ and the following hold:*

(H7) for each $i \in \{1, \dots, N\}$,

$$
\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|r_n - r_{n-1}|}{\alpha_n \beta_{n,k_0}}
$$

$$= \lim_{n\to\infty} \frac{1}{\alpha_n \beta_{n,k_0}} \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| = \lim_{n\to\infty} \frac{b^n}{\alpha_n \beta_{n,k_0}}$$
$$= \lim_{n\to\infty} \frac{|\nu_n - \nu_{n-1}|}{\alpha_n \beta_{n,k_0}} = 0;$$

(H8) there exists a constant $\tau > 0$ such that $\frac{1}{\alpha_n}\left|\frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}}\right| < \tau$ for all $n \geq 1$. Then

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} = 0.$$

Proof. We start by (3.17). Dividing both the terms by β_{n,k_0} we have

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} &\leq (1 - \alpha_n(\mu\overline{\gamma} - \gamma l)) \frac{\|x_n - x_{n-1}\|}{\beta_{n,k_0}} \\ &+ \widetilde{M}_1 \Bigg[\frac{|r_n - r_{n-1}|}{\gamma \beta_{n,k_0}} + \frac{\sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}|}{\beta_{n,k_0}} \\ &+ \frac{|\nu_n - \nu_{n-1}|}{\beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}} \\ &+ \frac{\left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|}{\beta_{n,k_0}} + \frac{b^{n-1}}{\beta_{n,k_0}} \Bigg]. \end{aligned} \tag{3.22}$$

So, by (H8) we have

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} &\leq (1 - \alpha_n(\mu\overline{\gamma} - \gamma l)) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + (1 - \alpha_n(\mu\overline{\gamma} - \gamma l)) \|x_n - x_{n-1}\| \times \\ &\times \left| \frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}} \right| + \widetilde{M}_1 \Bigg[\frac{|r_n - r_{n-1}|}{\gamma \beta_{n,k_0}} + \frac{\sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}|}{\beta_{n,k_0}} \\ &+ \frac{|\nu_n - \nu_{n-1}|}{\beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}} \\ &+ \frac{\left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|}{\beta_{n,k_0}} + \frac{b^{n-1}}{\beta_{n,k_0}} \Bigg] \\ &\leq (1 - \alpha_n(\mu\overline{\gamma} - \gamma l)) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + \|x_n - x_{n-1}\| \left| \frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}} \right| \\ &+ \widetilde{M}_1 \Bigg[\frac{|r_n - r_{n-1}|}{\gamma \beta_{n,k_0}} + \frac{\sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\nu_n - \nu_{n-1}|}{\beta_{n,k_0}} \\ &+ \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}} + \frac{\left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|}{\beta_{n,k_0}} + \frac{b^{n-1}}{\beta_{n,k_0}} \Bigg] \end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n(\mu\overline{\gamma}-\gamma l))\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}}+\alpha_n\tau\|x_n-x_{n-1}\| \\
&\quad+\widetilde{M}_1\Bigg[\frac{|r_n-r_{n-1}|}{\gamma\beta_{n,k_0}}+\frac{\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}}+\frac{|\nu_n-\nu_{n-1}|}{\beta_{n,k_0}} \\
&\quad+\frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}}+\frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}+\frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}}+\frac{b^{n-1}}{\beta_{n,k_0}}\Bigg] \\
&=(1-\alpha_n(\mu\overline{\gamma}-\gamma l))\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}}+\alpha_n(\mu\overline{\gamma}-\gamma l)\times \\
&\quad\times\frac{1}{\mu\overline{\gamma}-\gamma l}\Bigg\{\tau\|x_n-x_{n-1}\|+\widetilde{M}_1\Bigg[\frac{|r_n-r_{n-1}|}{\gamma\alpha_n\beta_{n,k_0}} \\
&\quad+\frac{\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|}{\alpha_n\beta_{n,k_0}}+\frac{|\nu_n-\nu_{n-1}|}{\alpha_n\beta_{n,k_0}}+\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n\beta_{n,k_0}} \\
&\quad+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n\beta_{n,k_0}}+\frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\alpha_n\beta_{n,k_0}}+\frac{b^{n-1}}{\alpha_n\beta_{n,k_0}}\Bigg]\Bigg\}.
\end{aligned}
$$

Therefore, utilizing Lemma 2.15, from (H0), (H7) and the asymptotical regularity of $\{x_n\}$ (due to Proposition 3.2), we deduce that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}}=0.$$

□

Proposition 3.6. *Let us suppose that $\Omega\neq\emptyset$. Let us suppose that (H0)-(H6) hold. Then, $\|\tilde{y}_{n,N}-y_{n,N}\|\to 0$ and $\|\tilde{y}_n-y_n\|\to 0$ as $n\to\infty$.*

Proof. Let $p\in\Omega$. Taking into account the firm nonexpansivity of P_C, we have

$$
\begin{aligned}
\|\tilde{y}_{n,N}-p\|^2 &= \|P_C(y_{n,N}-\nu_nAy_{n,N})-P_C(p-\nu_nAp)\|^2 \\
&\le \langle (y_{n,N}-\nu_nAy_{n,N})-(p-\nu_nAp),\tilde{y}_{n,N}-p\rangle \\
&= \frac{1}{2}[\|y_{n,N}-p-\nu_n(Ay_{n,N}-Ap)\|^2+\|\tilde{y}_{n,N}-p\|^2 \\
&\quad-\|y_{n,N}-p-\nu_n(Ay_{n,N}-Ap)-(\tilde{y}_{n,N}-p)\|^2] \\
&\le \frac{1}{2}[\|y_{n,N}-p\|^2+\|\tilde{y}_{n,N}-p\|^2-\|y_{n,N}-\tilde{y}_{n,N}-\nu_n(Ay_{n,N}-Ap)\|^2] \\
&= \frac{1}{2}[\|y_{n,N}-p\|^2+\|\tilde{y}_{n,N}-p\|^2-\|y_{n,N}-\tilde{y}_{n,N}\|^2 \\
&\quad+2\nu_n\langle y_{n,N}-\tilde{y}_{n,N},Ay_{n,N}-Ap\rangle-\nu_n^2\|Ay_{n,N}-Ap\|^2] \\
&\le \frac{1}{2}[\|y_{n,N}-p\|^2+\|\tilde{y}_{n,N}-p\|^2-\|y_{n,N}-\tilde{y}_{n,N}\|^2 \\
&\quad+2\nu_n\langle y_{n,N}-\tilde{y}_{n,N},Ay_{n,N}-Ap\rangle],
\end{aligned}
$$

which hence leads to

$$\begin{aligned} \|\tilde{y}_{n,N}-p\|^2 \quad \leq \quad & \|y_{n,N}-p\|^2-\|y_{n,N}-\tilde{y}_{n,N}\|^2 \\ & +2\nu_n\|y_{n,N}-\tilde{y}_{n,N}\|\|Ay_{n,N}-Ap\|. \end{aligned} \tag{3.23}$$

Similarly, we get

$$\|\tilde{y}_n-p\|^2 \leq \|y_n-p\|^2-\|y_n-\tilde{y}_n\|^2+2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\|. \tag{3.24}$$

So, it follows from (3.2), (3.19)-(3.20) and (3.23)-(3.24) that

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \beta_n\|y_n-p\|^2+(1-\beta_n)\|\tilde{y}_n-p\|^2 \\ &\leq \beta_n\|y_n-p\|^2+(1-\beta_n)[\|y_n-p\|^2-\|y_n-\tilde{y}_n\|^2 \\ &\quad +2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\|] \\ &\leq \|y_n-p\|^2-(1-\beta_n)\|y_n-\tilde{y}_n\|^2+2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\| \\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N}-p\|^2+\|\tilde{y}_{n,N}-p\|^2+2\alpha_n\langle(\gamma f-\mu V)p, y_n-p\rangle \\ &\quad -(1-\beta_n)\|y_n-\tilde{y}_n\|^2+2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\| \\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N}-p\|^2+\|y_{n,N}-p\|^2-\|y_{n,N}-\tilde{y}_{n,N}\|^2 \\ &\quad +2\nu_n\|y_{n,N}-\tilde{y}_{n,N}\|\|Ay_{n,N}-Ap\|+2\alpha_n\langle(\gamma f-\mu V)p, y_n-p\rangle \\ &\quad -(1-\beta_n)\|y_n-\tilde{y}_n\|^2+2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\| \\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N}-p\|^2+\|x_n-p\|^2-\|y_{n,N}-\tilde{y}_{n,N}\|^2 \\ &\quad +2\nu_n\|y_{n,N}-\tilde{y}_{n,N}\|\|Ay_{n,N}-Ap\|+2\alpha_n\|(\gamma f-\mu V)p\|\|y_n-p\| \\ &\quad -(1-\beta_n)\|y_n-\tilde{y}_n\|^2+2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\|, \end{aligned}$$

which together with $\{\beta_n\} \subset [c,d] \subset (0,1)$, implies that

$$\begin{aligned} \|y_{n,N}-\tilde{y}_{n,N}\|^2+(1-d)\|y_n-\tilde{y}_n\|^2 &\leq \|y_{n,N}-\tilde{y}_{n,N}\|^2+(1-\beta_n)\|y_n-\tilde{y}_n\|^2 \\ &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\mu\overline{\gamma}\|y_{n,N}-p\|^2 \\ &\quad +2\alpha_n\|(\gamma f-\mu V)p\|\|y_n-p\| \\ &\quad +2\nu_n\|y_{n,N}-\tilde{y}_{n,N}\|\|Ay_{n,N}-Ap\| \\ &\quad +2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\| \\ &\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\ &\quad +\alpha_n\mu\overline{\gamma}\|y_{n,N}-p\|^2 \\ &\quad +2\alpha_n\|(\gamma f-\mu V)p\|\|y_n-p\| \\ &\quad +2\nu_n\|y_{n,N}-\tilde{y}_{n,N}\|\|Ay_{n,N}-Ap\| \\ &\quad +2\nu_n\|y_n-\tilde{y}_n\|\|Ay_n-Ap\|. \end{aligned}$$

As $\alpha_n \to 0$, $\|x_{n+1}-x_n\| \to 0$ (due to Proposition 3.2), and $\{x_n\}, \{y_n\}, \{y_{n,N}\}$, $\{\tilde{y}_{n,N}\}, \{\tilde{y}_n\}$ are bounded, we conclude from (3.21) that

$$\lim_{n\to\infty}\|y_{n,N}-\tilde{y}_{n,N}\| = \lim_{n\to\infty}\|y_n-\tilde{y}_n\| = 0. \tag{3.25}$$

□

Proposition 3.7. *Let us suppose that* $\Omega \neq \emptyset$. *Let us suppose that* $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ *for each* $i = 1, \ldots, N$. *Moreover, suppose that (H0)-(H6) are satisfied. Then,* $\lim_{n\to\infty} \|S_i u_n - u_n\| = 0$ *for each* $i = 1, \ldots, N$ *provided* $\|Ty_n - y_n\| \to$ *as* $n \to \infty$.

Proof. First of all, observe that

$$\begin{aligned} x_{n+1} - x_n &= \beta_n(y_n - x_n) + \gamma_n(\tilde{y}_n - x_n) + \delta_n(T\tilde{y}_n - x_n) \\ &= \beta_n(y_n - x_n) + \gamma_n(\tilde{y}_n - y_n) + \gamma_n(y_n - x_n) + \delta_n(T\tilde{y}_n - Ty_n) \\ &\quad + \delta_n(Ty_n - y_n) + \delta_n(y_n - x_n) \\ &= y_n - x_n + \gamma_n(\tilde{y}_n - y_n) + \delta_n(T\tilde{y}_n - Ty_n) + \delta_n(Ty_n - y_n). \end{aligned}$$

In terms of (3.25) and Proposition 3.2 we know that $\|y_n - \tilde{y}_n\| \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 1$, by Propositions 2.5 we have

$$\begin{aligned} \|y_n - x_n\| &= \|x_{n+1} - x_n - \gamma_n(\tilde{y}_n - y_n) - \delta_n(T\tilde{y}_n - Ty_n) - \delta_n(Ty_n - y_n)\| \\ &\leq \|x_{n+1} - x_n\| + \|\gamma_n(\tilde{y}_n - y_n) + \delta_n(T\tilde{y}_n - Ty_n)\| + \|\delta_n(Ty_n - y_n)\| \\ &\leq \|x_{n+1} - x_n\| + (\gamma_n + \delta_n)\|\tilde{y}_n - y_n\| + \delta_n\|Ty_n - y_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\tilde{y}_n - y_n\| + \|Ty_n - y_n\|, \end{aligned}$$

which together with $\|Ty_n - y_n\| \to 0$, implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.26}$$

Let us show that for each $i \in \{1, \ldots, N\}$, one has $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$. Let $p \in \Omega$. When $i = N$, by Lemma 2.7 (b) we have from (3.2), (3.3) and (3.19)

$$\begin{aligned} \|y_n - p\|^2 &\leq \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + \|\tilde{y}_{n,N} - p\|^2 + 2\alpha_n\langle(\gamma f - \mu V)p, y_n - p\rangle \\ &\leq \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \|y_{n,N} - p\|^2 \\ &= \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \beta_{n,N}\|S_N u_n - p\|^2 \\ &\quad + (1 - \beta_{n,N})\|y_{n,N-1} - p\|^2 - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\ &\leq \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \beta_{n,N}\|u_n - p\|^2 \\ &\quad + (1 - \beta_{n,N})\|u_n - p\|^2 - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\ &= \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \|u_n - p\|^2 \\ &\quad - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\ &\leq \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \|x_n - p\|^2 \\ &\quad - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2. \end{aligned}$$

So, we have

$$\begin{aligned} \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 &\leq \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\ &\quad + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\ &\quad + \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|). \end{aligned}$$

As $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_{n,N} \leq \limsup_{n\to\infty} \beta_{n,N} < 1$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (due to (3.26)), it is known that $\{\|S_N u_n - y_{n,N-1}\|\}$ is a null sequence.

Let $i \in \{1, \dots, N-1\}$. Then one has

$$\begin{aligned}\|y_n - p\|^2 &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \|y_{n,N} - p\|^2\\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \beta_{n,N}\|S_N u_n - p\|^2\\ &\quad + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2\\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \beta_{n,N}\|x_n - p\|^2\\ &\quad + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2\\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \beta_{n,N}\|x_n - p\|^2\\ &\quad + (1-\beta_{n,N})[\beta_{n,N-1}\|S_{N-1}u_n - p\|^2 + (1-\beta_{n,N-1})\|y_{n,N-2} - p\|^2]\\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\|\\ &\quad + (\beta_{n,N} + (1-\beta_{n,N})\beta_{n,N-1})\|x_n - p\|^2\\ &\quad + \prod_{k=N-1}^{N}(1-\beta_{n,k})\|y_{n,N-2} - p\|^2,\end{aligned}$$

and so, after $(N-i+1)$-iterations,

$$\begin{aligned}\|y_n - p\|^2 &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\|\\ &\quad + (\beta_{n,N} + \sum_{j=i+2}^{N}(\prod_{l=j}^{N}(1-\beta_{n,l}))\beta_{n,j-1})\|x_n - p\|^2\\ &\quad + \prod_{k=i+1}^{N}(1-\beta_{n,k})\|y_{n,i} - p\|^2\\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\|\\ &\quad + (\beta_{n,N} + \sum_{j=i+2}^{N}(\prod_{l=j}^{N}(1-\beta_{n,l}))\beta_{n,j-1})\|x_n - p\|^2\\ &\quad + \prod_{k=i+1}^{N}(1-\beta_{n,k})[\beta_{n,i}\|S_i u_n - p\|^2 + (1-\beta_{n,i})\|y_{n,i-1} - p\|^2\\ &\quad - \beta_{n,i}(1-\beta_{n,i})\|S_i u_n - y_{n,i-1}\|^2]\\ &\leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| + \|x_n - p\|^2\\ &\quad - \beta_{n,i}\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2.\end{aligned} \tag{3.27}$$

Again we obtain that

$$\beta_{n,i}\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2 \leq \alpha_n\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\|$$

$$
\begin{aligned}
&+ \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\le \alpha_n \mu \overline{\gamma} \|y_{n,N} - p\|^2 + 2\alpha_n \|(\gamma f - \mu V)p\| \|y_n - p\| \\
&+ \|x_n - y_n\| (\|x_n - p\| + \|y_n - p\|).
\end{aligned}
$$

As $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$ for each $i = 1, \dots, N-1$, and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (due to (3.26)), it is known that

$$\lim_{n\to\infty} \|S_i u_n - y_{n,i-1}\| = 0.$$

Obviously for $i = 1$, we have $\|S_1 u_n - u_n\| \to 0$.

To conclude, we have that

$$\|S_2 u_n - u_n\| \le \|S_2 u_n - y_{n,1}\| + \|y_{n,1} - u_n\| = \|S_2 u_n - y_{n,1}\| + \beta_{n,1} \|S_1 u_n - u_n\|$$

from which $\|S_2 u_n - u_n\| \to 0$. Thus by induction $\|S_i u_n - u_n\| \to 0$ for all $i = 2, \dots, N$ since it is enough to observe that

$$
\begin{aligned}
\|S_i u_n - u_n\| &\le \|S_i u_n - y_{n,i-1}\| + \|y_{n,i-1} - S_{i-1} u_n\| + \|S_{i-1} u_n - u_n\| \\
&\le \|S_i u_n - y_{n,i-1}\| + (1 - \beta_{n,i-1}) \|S_{i-1} u_n - y_{n,i-2}\| \\
&\quad + \|S_{i-1} u_n - u_n\|.
\end{aligned}
$$

□

Remark 3.8. As an example, we consider $N = 2$ and the sequences:
(a) $\nu_n = \alpha - \frac{1}{n}$, $\forall n > \frac{1}{\alpha}$;
(b) $\alpha_n = \frac{1}{\sqrt{n}}$, $r_n = 2 - \frac{1}{n}$, $\forall n > 1$;
(c) $\beta_{n,1} = \frac{1}{2} - \frac{1}{n}$, $\beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}$, $\forall n > 2$;
(d) $\beta_n = \frac{1}{2} + \frac{2}{n}$, $\gamma_n = \delta_n = \frac{1}{4} - \frac{1}{n}$, $\forall n > 4$.
Then they satisfy the hypotheses on the parameter sequences in Proposition 3.7.

Proposition 3.9. *Let us suppose that $\Omega \neq \emptyset$ and $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose there exists $k \in \{1, \dots, N\}$ such that $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1, \dots, N\}$ the largest index such that $\beta_{n,k_0} \to 0$ as $n \to \infty$. Suppose that*
(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;
(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;
(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0, 1)$.
Moreover, suppose that (H0), (H7) and (H8) hold. Then, $\lim_{n\to\infty} \|S_i u_n - u_n\| = 0$ *for each* $i = 1, \dots, N$ *provided* $\|T y_n - y_n\| \to 0$ *as* $n \to \infty$.

Proof. First of all we note that if (H7) holds then also (H1)-(H6) are satisfied. So $\{x_n\}$ is asymptotically regular.

Let k_0 be as in the hypotheses. As in Proposition 3.7, for every index $i \in \{1, ..., N\}$ such that $\beta_{n,i} \to \beta_i \neq 0$ (which leads to $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$), one has $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$.

For all the other indices $i \le k_0$, we can prove that $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$ in a similar manner. By the relation (due to (3.20) and (3.27))

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n \|y_n - p\|^2 + (1 - \beta_n) \|\tilde{y}_n - p\|^2 \\
&\le \beta_n \|y_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2
\end{aligned}
$$

$$= \|y_n - p\|^2$$
$$\leq \alpha_n \mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\|$$
$$+ \|x_n - p\|^2 - \beta_{n,i}\prod_{k=i}^{N}(1 - \beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2,$$

we immediately obtain that

$$\prod_{k=i}^{N}(1 - \beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2 \leq \frac{\alpha_n}{\beta_{n,i}}[\mu\overline{\gamma}\|y_{n,N} - p\|^2 + 2\|(\gamma f - \mu V)p\|\|y_n - p\|]$$
$$+ \frac{\|x_n - x_{n+1}\|}{\beta_{n,i}}(\|x_n - p\| + \|x_{n+1} - p\|).$$

By Proposition 3.5 or by hypothesis (ii) on the sequences, we have

$$\frac{\|x_n - x_{n+1}\|}{\beta_{n,i}} = \frac{\|x_n - x_{n+1}\|}{\beta_{n,k_0}} \cdot \frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0.$$

So, the conclusion follows. □

Remark 3.10. Let us consider $N = 3$ and the following sequences:
(a) $\nu_n = \alpha - \frac{1}{n^2}, \quad \forall n > \frac{1}{\alpha^{1/2}}$;
(b) $\alpha_n = \frac{1}{n^{1/2}}, \quad r_n = 2 - \frac{1}{n^2}, \quad \forall n > 1$;
(c) $\beta_n = \frac{1}{2} + \frac{2}{n^2}, \quad \gamma_n = \delta_n = \frac{1}{4} - \frac{1}{n^2}, \quad \forall n > 2$;
(d) $\beta_{n,1} = \frac{1}{n^{1/4}}, \quad \beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}, \quad \beta_{n,3} = \frac{1}{n^{1/3}}, \quad \forall n > 1$.
It is easy to see that all hypotheses (i)-(iii), (H0), (H7) and (H8) of Proposition 3.9 are satisfied.

Remark 3.11. Under the hypotheses of Proposition 3.9, analogously to Proposition 3.7, one can see that

$$\lim_{n\to\infty} \|S_i u_n - y_{n,i-1}\| = 0, \quad \forall i \in \{2, \ldots, N\}.$$

Corollary 3.12. *Let us suppose that the hypotheses of either Proposition 3.7 or Proposition 3.9 are satisfied. Then* $\omega_w(x_n) = \omega_w(u_n) = \omega_w(y_{n,1})$, $\omega_s(x_n) = \omega_s(u_n) = \omega_s(y_{n,1})$ *and* $\omega_w(x_n) \subset \Omega$.

Proof. By Remark 3.4, we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$. Note that by Remark 3.11,

$$\lim_{n\to\infty} \|S_N u_n - y_{n,N-1}\| = 0.$$

In the meantime, it is known that

$$\lim_{n\to\infty} \|S_N u_n - u_n\| = \lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Hence we have

$$\lim\nolimits_{n\to\infty} \|S_N u_n - y_n\| = 0. \tag{3.28}$$

Furthermore, it follows from (3.1) that

$$\lim_{n\to\infty} \|y_{n,N} - y_{n,N-1}\| = \lim_{n\to\infty} \beta_{n,N}\|S_N u_n - y_{n,N-1}\| = 0,$$

which together with $\lim_{n\to\infty}\|S_N u_n - y_{n,N-1}\| = 0$, yields

$$\lim_{n\to\infty}\|S_N u_n - y_{n,N}\| = 0. \tag{3.29}$$

Combining (3.28) and (3.29), we conclude that

$$\lim_{n\to\infty}\|y_n - y_{n,N}\| = 0, \tag{3.30}$$

which together with $\lim_{n\to\infty}\|x_n - y_n\| = 0$, leads to

$$\lim_{n\to\infty}\|x_n - y_{n,N}\| = 0. \tag{3.31}$$

Now we observe that

$$\|x_n - y_{n,1}\| \le \|x_n - u_n\| + \|y_{n,1} - u_n\| = \|x_n - u_n\| + \beta_{n,1}\|S_1 u_n - u_n\|.$$

By Propositions 3.3 and 3.7, $\|x_n - u_n\| \to 0$ and $\|S_1 u_n - u_n\| \to 0$ as $n \to \infty$, and hence

$$\lim_{n\to\infty}\|x_n - y_{n,1}\| = 0.$$

So we get $\omega_w(x_n) = \omega_w(y_{n,1})$ and $\omega_s(x_n) = \omega_s(y_{n,1})$.

In addition, it is easy to see from (3.1) and $\alpha_n \to 0$ that

$$\lim_{n\to\infty}\|y_n - W_n\tilde{y}_{n,N}\| = \lim_{n\to\infty}\alpha_n\|\gamma f(y_{n,N}) - \mu V W_n\tilde{y}_{n,N}\| = 0. \tag{3.32}$$

Since

$$\begin{aligned}\|y_{n,N} - W_n y_{n,N}\| &\le \|y_{n,N} - y_n\| + \|y_n - W_n\tilde{y}_{n,N}\| + \|W_n\tilde{y}_{n,N} - W_n y_{n,N}\| \\ &\le \|y_{n,N} - y_n\| + \|y_n - W_n\tilde{y}_{n,N}\| + \|\tilde{y}_{n,N} - y_{n,N}\|,\end{aligned}$$

from (3.25), (3.30) and (3.32), it follows that

$$\lim_{n\to\infty}\|y_{n,N} - W_n y_{n,N}\| = 0. \tag{3.33}$$

Taking into account that $\|y_{n,N} - W y_{n,N}\| \le \|y_{n,N} - W_n y_{n,N}\| + \|W_n y_{n,N} - W y_{n,N}\|$, from Remark 2.10 and the boundedness of $\{y_{n,N}\}$ we immediately get

$$\lim_{n\to\infty}\|y_{n,N} - W y_{n,N}\| = 0. \tag{3.34}$$

Next, let us show that $\omega_w(x_n) \subset \Omega$. Indeed, let $p \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Since $p \in \omega_w(u_n)$, by Proposition 3.7 and Lemma 2.12 (demiclosedness principle), we have $p \in \mathrm{Fix}(S_i)$ for each $i = 1, \ldots, N$, i.e., $p \in \cap_{i=1}^N \mathrm{Fix}(S_i)$. Taking into account $p \in\in \omega_w(y_{n,N})$ (due to (3.31)) and $\|y_{n,N} - W y_{n,N}\| \to 0$ (due to (3.34)), by Lemma 2.12 (demiclosedness principle) we know that $p \in \mathrm{Fix}(W) = \cap_{n=1}^\infty \mathrm{Fix}(T_n)$ (due to Lemma 2.11). Also, since $p \in \omega_w(y_n)$ (due to $\|x_n - y_n\| \to 0$), in terms of $\|Ty_n - y_n\| \to 0$ and Proposition 2.4, we get $p \in \mathrm{Fix}(T)$. Moreover, by Lemma 2.18 and Proposition 3.3 we know that $p \in \mathrm{GMEP}(\Theta, h)$. Furthermore, we prove that $p \in \mathrm{VI}(C, A)$. As a matter of fact, since $p \in \omega_w(y_{n,N})$ (due to (3.31)), there exists a subsequence $\{y_{n_i,N}\}$ of $\{y_{n,N}\}$ such that $y_{n_i,N} \rightharpoonup p$. So, from (3.25) we know that $\tilde{y}_{n_i,N} \rightharpoonup p$. Let

$$\widetilde{T}v = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Let $(v, u) \in G(\widetilde{T})$. Since $u - Av \in N_C v$ and $\tilde{y}_{n,N} \in C$, we have

$$\langle v - \tilde{y}_{n,N}, u - Av\rangle \ge 0.$$

On the other hand, from $\tilde{y}_{n,N} = P_C(y_{n,N} - \nu_n A y_{n,N})$ and $v \in C$, we have

$$\langle v - \tilde{y}_{n,N}, \tilde{y}_{n,N} - (y_{n,N} - \nu_n A y_{n,N})\rangle \geq 0,$$

and hence

$$\langle v - \tilde{y}_{n,N}, \frac{\tilde{y}_{n,N} - y_{n,N}}{\nu_n} + A y_{n,N}\rangle \geq 0.$$

Therefore we have

$$\begin{aligned}
\langle v - \tilde{y}_{n_i,N}, u\rangle &\geq \langle v - \tilde{y}_{n_i,N}, Av\rangle \\
&\geq \langle v - \tilde{y}_{n_i,N}, Av\rangle - \Big\langle v - \tilde{y}_{n_i,N}, \frac{\tilde{y}_{n_i,N} - y_{n_i,N}}{\nu_{n_i}} + A y_{n_i,N}\Big\rangle \\
&= \langle v - \tilde{y}_{n_i,N}, Av - A\tilde{y}_{n_i,N}\rangle + \langle v - \tilde{y}_{n_i,N}, A\tilde{y}_{n_i,N} - A y_{n_i,N}\rangle \\
&\quad - \Big\langle v - y_{n_i,N}, \frac{\tilde{y}_{n_i,N} - y_{n_i,N}}{\nu_{n_i}}\Big\rangle \\
&\geq \langle v - \tilde{y}_{n_i,N}, A\tilde{y}_{n_i,N} - A y_{n_i,N}\rangle - \Big\langle v - \tilde{y}_{n_i,N}, \frac{\tilde{y}_{n_i,N} - y_{n_i,N}}{\nu_{n_i}}\Big\rangle.
\end{aligned}$$

From (3.25) and since A is Lipschitz continuous, we get $\lim_{n\to\infty} \|A\tilde{y}_{n.N} - Ay_{n,N}\| = 0$. From $\tilde{y}_{n_i,N} \rightharpoonup p,\ 0 < \liminf_{n\to\infty} \nu_n \leq \limsup_{n\to\infty} \nu_n < 2\alpha$ and (3.25), we have

$$\langle v - p, u\rangle \geq 0.$$

Since $\widetilde{T}$ is maximal monotone, we have $p \in \widetilde{T}^{-1}0$ and hence $p \in \mathrm{VI}(C, A)$. Consequently, it is known that $p \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \cap_{i=1}^{N}\mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T) =: \Omega$.

Theorem 3.13. *Let us suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \ldots, N$, be sequences in $(0, 1)$ such that $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ for each index i. Moreover, let us suppose that (H0)-(H6) hold. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$, defined by scheme (3.1), all converge strongly to $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ if and only if $\|y_n - Ty_n\| \to 0$ as $n \to \infty$, where $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ is the unique solution of the VIP*

$$\langle(\gamma f - \mu V)x^*, x - x^*\rangle \leq 0, \quad \forall x \in \Omega, \tag{3.35}$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle - \Psi(x), \tag{3.36}$$

where Ψ is a potential function for γf.

Proof. First of all, we note that V is a $\overline{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ is an l-Lipschitz continuous mapping with $0 \leq \gamma l < \mu\overline{\gamma}$. It is clear that

$$\langle(\mu V - \gamma f)x - (\mu V - \gamma f)y, x - y\rangle \geq (\mu\overline{\gamma} - \gamma l)\|x - y\|^2, \quad \forall x, y \in H.$$

Hence we deduce that $\mu V - \gamma f$ is $(\mu\overline{\gamma} - \gamma l)$-strongly monotone. In the meantime, it is easy to see that $\mu V - \gamma f$ is $(\mu\|V\| + \gamma l)$-Lipschitz continuous with constant $\mu\|V\| + \gamma l > 0$. Thus, there exists a unique solution x^* in Ω to the VIP (3.35). Equivalently, x^* is the unique solution of the minimization problem (3.36).

Now, observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f-\mu V)x^*, x_n-x^*\rangle=\lim_{i\to\infty}\langle(\gamma f-\mu V)x^*, x_{n_i}-x^*\rangle. \tag{3.37}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some $p\in H$. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup p$. Then by Corollary 3.12, we get $p\in\omega_w(x_n)\subset\Omega$. Hence, from (3.35) and (3.37), we have

$$\limsup_{n\to\infty}\langle(\gamma f-\mu V)x^*, x_n-x^*\rangle=\langle(\gamma f-\mu V)x^*, p-x^*\rangle\le 0. \tag{3.38}$$

Since (H1)-(H6) hold, the sequence $\{x_n\}$ is asymptotically regular (according to Proposition 3.2). In terms of (3.21) and (3.26), $\|x_n-u_n\|\to 0$ and $\|x_n-y_n\|\to 0$ as $n\to\infty$.

Let us show that $\|x_n-x^*\|\to 0$ as $n\to\infty$. Indeed, putting $p=x^*$, we deduce from (3.3), (3.19) and (3.20) that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&\le\beta_n\|y_n-x^*\|^2+(1-\beta_n)\|\tilde{y}_n-x^*\|^2\\
&\le\beta_n\|y_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2=\|y_n-x^*\|^2\\
&\le\alpha_n\mu\overline{\gamma}\frac{(\gamma l)^2}{(\mu\overline{\gamma})^2}\|y_{n,N}-x^*\|^2+(1-\alpha_n\mu\overline{\gamma})\|\tilde{y}_{n,N}-x^*\|^2\\
&\quad+2\alpha_n\langle(\gamma f-\mu V)x^*, y_n-x^*\rangle\\
&\le\alpha_n\frac{(\gamma l)^2}{\mu\overline{\gamma}}\|x_n-x^*\|^2+(1-\alpha_n\mu\overline{\gamma})\|x_n-x^*\|^2\\
&\quad+2\alpha_n\langle(\gamma f-\mu V)x^*, y_n-x^*\rangle\\
&=\Big(1-\alpha_n\frac{(\mu\overline{\gamma})^2-(\gamma l)^2}{\mu\overline{\gamma}}\Big)\|x_n-x^*\|^2\\
&\quad+\alpha_n\frac{(\mu\overline{\gamma})^2-(\gamma l)^2}{\mu\overline{\gamma}}\frac{2\mu\overline{\gamma}}{(\mu\overline{\gamma})^2-(\gamma l)^2}\langle(\gamma f-\mu V)x^*, y_n-x^*\rangle.
\end{aligned} \tag{3.39}$$

Since $\sum_{n=1}^\infty\alpha_n=\infty$ and $\|x_n-y_n\|\to 0$, we obtain that $\sum_{n=1}^\infty\alpha_n\frac{(\mu\overline{\gamma})^2-(\gamma l)^2}{\mu\overline{\gamma}}=\infty$ and

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{2\mu\overline{\gamma}}{(\mu\overline{\gamma})^2-(\gamma l)^2}\langle(\gamma f-\mu V)x^*, y_n-x^*\rangle\\
&=\limsup_{n\to\infty}\frac{2\mu\overline{\gamma}}{(\mu\overline{\gamma})^2-(\gamma l)^2}(\langle(\gamma f-\mu V)x^*, x_n-x^*\rangle+\langle(\gamma f-\mu V)x^*, y_n-x_n\rangle)\\
&=\limsup_{n\to\infty}\frac{2\mu\overline{\gamma}}{(\mu\overline{\gamma})^2-(\gamma l)^2}\langle(\gamma f-\mu V)x^*, x_n-x^*\rangle\le 0.
\end{aligned}$$

(due to (3.38)). Applying Lemma 2.15 to (3.39), we infer that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

In a similar way, we can conclude another theorem as follows.

Theorem 3.14. *Let us suppose that $\Omega\neq\emptyset$. Let $\{\alpha_n\},\{\beta_{n,i}\}, i=1,\dots,N$, be sequences in $(0,1)$ such that $\beta_{n,i}\to\beta_i$ for each index i as $n\to\infty$. Suppose that there exists $k\in\{1,\dots,N\}$ for which $\beta_{n,k}\to 0$ as $n\to\infty$. Let $k_0\in\{1,\dots,N\}$ the* *largest index for which $\beta_{n,k_0} \to 0$. Moreover, let us suppose that (H0), (H7) and (H8) hold and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$*;*

(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to \beta_i$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$*;*

(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0,1)$.

Then the sequences $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *defined by scheme (3.1) all converge strongly to* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *if and only if* $\|y_n - Ty_n\| \to 0$ *as* $n \to \infty$*, where* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *is the unique solution of the VIP*

$$\langle (\gamma f - \mu V)x^*, x - x^* \rangle \le 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \Omega} \frac{\mu}{2} \langle Vx, x \rangle - \Psi(x),$$

where Ψ *is a potential function for* γf*.*

Remark 3.15. According to the above argument process for Theorems 3.13 and 3.14, we can readily see that if in scheme (3.1), the iterative step $y_n = \alpha_n \gamma f(y_{n,N}) + (I - \alpha_n \mu V) W_n P_C(y_{n,N} - \nu_n A y_{n,N})$ is replaced by the iterative one $y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu V) W_n P_C(y_{n,N} - \nu_n A y_{n,N})$, then Theorems 3.13 and 3.14 remain valid.

Remark 3.16. Theorems 3.13 and 3.14 improve, extend, supplement and develop [17, Theorems 3.1 and 3.2] and [40, Theorems 3.12 and 3.13] in the following aspects.

(i) The multi-step iterative scheme (3.1) in [17] is extended to develop our Mann-type viscosity iterative scheme (3.1) by virtue of W-mapping approach to common fixed points of infinitely many nonexpansive mappings, and strongly positive bounded linear operator approach. The iterative scheme (3.1) is based on composite viscosity approximation method [31], Mann's iterative method, W-mapping approach to common fixed points of infinitely many nonexpansive mappings, and strongly positive bounded linear operator approach.

(ii) The argument techniques in our Theorems 3.13 and 3.14 are very different from those techniques in [17, Theorems 3.1 and 3.2] and [40, Theorems 3.12 and 3.13] because we make use of the properties of strict pseudocontractions (see Propositions 2.4 and 2.5), the ones of W-mappings (see Remarks 2.9 and 2.10 and Lemmas 2.8 and 2.11), the ones of the resolvent operator associated with Θ and h (see Lemmas 2.16-2.18), the inclusion problem $0 \in \widetilde{T}x^*$ ($\Leftrightarrow \quad x^* \in \mathrm{VI}(C, A)$) (see (2.2)), the ones of strongly positive boundedness linear operators (see Lemma 2.14), and the convergence criteria for nonnegative real sequences (see Lemma 2.15).

(iii) The problem of finding an element of $\cap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T)$ (where T is a strict pseudocontraction) in our Theorems 3.13 and 3.14 is more general and more subtle than the one of finding an element of $\mathrm{Fix}(T) \cap \cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h)$ in [40, Theorems 3.12 and 3.13] (where T is a nonexpansive mapping) and the one of finding an element of $\mathrm{Fix}(T) \cap \cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \mathrm{VI}(C, A)$ in [17, Theorems 3.1 and 3.2] (where T is a nonexpansive mapping).

(iv) Our Theorems 3.13 and 3.14 generalizes [17, Theorems 3.1 and 3.2] and [40, Theorems 3.12 and 3.13] from the nonexpansive mapping T to the strict pseudocontraction T and from the nonexpansive mapping T to infinitely many nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$. In the meantime, these theorems extend not only [40, Theorems 3.12 and 3.13] to the setting of VIP (1.1), hierarchical minimization (3.36) and infinitely many nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$, but also [17, Theorems 3.1 and 3.2] to the setting of hierarchical minimization (3.36) and infinitely many nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$.

4. Applications

For a given nonlinear mapping $A : C \to H$, we consider the variational inequality problem (VIP) of finding $\overline{x} \in C$ such that

$$\langle A\overline{x}, y - \overline{x}\rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

We will indicate with $\mathrm{VI}(C, A)$ the set of solutions of the VIP (4.1).

Recall that if u is a point in C, then the following relation holds:

$$u \in \mathrm{VI}(C, A) \;\Leftrightarrow u = P_C(I - \lambda A)u, \quad \forall \lambda > 0. \tag{4.2}$$

An operator $A : C \to H$ is said to be an α-inverse strongly monotone operator if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

As an example, we recall that the α-inverse strongly monotone operators are firmly nonexpansive mappings if $\alpha \geq 1$ and that every α-inverse strongly monotone operator is also $\frac{1}{\alpha}$-Lipschitz continuous (see [49]).

Let us observe also that, if A is α-inverse strongly monotone, the mappings $P_C(I - \lambda A)$ are nonexpansive for all $\lambda \in (0, 2\alpha]$ since they are compositions of nonexpansive mappings (see p. 419 in [49]).

Let us consider $\tilde{S}_1, \ldots, \tilde{S}_M$ a finite number of nonexpansive self-mappings on C and $A_1, \ldots, A_N$ be a finite number of α-inverse strongly monotone operators. Let $T : H \to H$ be a ξ-strict pseudocontraction on H with fixed points. Let us consider the following mixed problem of finding $x^* \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\varTheta, h) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T)$ such that

$$\begin{cases} \langle (I - \tilde{S}_1)x^*, y - x^*\rangle \geq 0, \\ \quad \forall y \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\varTheta, h) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T), \\ \langle (I - \tilde{S}_2)x^*, y - x^*\rangle \geq 0, \\ \quad \forall y \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\varTheta, h) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T), \\ \cdots \\ \langle (I - \tilde{S}_M)x^*, y - x^*\rangle \geq 0, \\ \quad \forall y \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\varTheta, h) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T), \\ \langle A_1x^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \\ \langle A_2x^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \\ \cdots \\ \langle A_Nx^*, y - x^*\rangle \geq 0, \quad \forall y \in C. \end{cases} \tag{4.3}$$

Let us call (SVI) the set of solutions of the $(M + N)$-system. This problem is equivalent to finding a common fixed point of T, $\{P_C(I - \lambda A_i)\}_{i=1}^{N}$ and

$\{P_{\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)\cap\mathrm{GMEP}(\Theta,h)\cap\mathrm{VI}(C,A)\cap\mathrm{Fix}(T)}\tilde{S}_i\}_{i=1}^{M}$. So we claim that the following holds.

Theorem 4.1. *Let us suppose that $\Omega = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap (\mathrm{SVI}) \cap \mathrm{GMEP}(\Theta, h) \cap \mathrm{VI}(C, A)\cap\mathrm{Fix}(T) \neq \emptyset$. Fix $\lambda > 0$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \ldots, (M+N)$, be sequences in $(0, 1)$ such that $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ for all indices i. Moreover, let us suppose that (H0)-(H6) hold. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ explicitly defined by scheme*

$$(4.4)\quad \begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1}P_{\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)\cap\mathrm{GMEP}(\Theta,h)\cap\mathrm{VI}(C,A)\cap\mathrm{Fix}(T)}\tilde{S}_1 u_n \\ \quad +(1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}P_{\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)\cap\mathrm{GMEP}(\Theta,h)\cap\mathrm{VI}(C,A)\cap\mathrm{Fix}(T)}\tilde{S}_i u_n \\ \quad +(1 - \beta_{n,i})y_{n,i-1}, (i = 2, \ldots, M) \\ y_{n,M+j} = \beta_{n,M+j}P_C(I - \lambda A_j)u_n + (1 - \beta_{n,M+j})y_{n,M+j-1}, \\ \quad (j = 1, \ldots, N) \\ y_n = \alpha_n\gamma f(y_{n,M+N}) + (I - \alpha_n\mu V)W_nP_C(y_{n,M+N} - \nu_nAy_{n,M+N}), \\ x_{n+1} = \beta_ny_n + \gamma_nP_C(y_n - \nu_nAy_n) + \delta_nTP_C(y_n - \nu_nAy_n), \end{cases}$$

all converge strongly to $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ if and only if $\|y_n - Ty_n\| \to 0$ as $n \to \infty$, where $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ is the unique solution of the VIP

$$\langle(\gamma f - \mu V)x^*, x - x^*\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle - \Psi(x),$$

where Ψ is a potential function for γf.

Theorem 4.2. *Let us suppose that $\Omega \neq \emptyset$. Fix $\lambda > 0$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \ldots, (M + N)$, be sequences in $(0, 1)$ and $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose that there exists $k \in \{1, \ldots, M + N\}$ such that $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1, \ldots, M + N\}$ be the largest index for which $\beta_{n,k_0} \to 0$. Moreover, let us suppose that (H0), (H7) and (H8) hold and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;

(ii) *if* $i \leq k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;

(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ explicitly defined by scheme (4.4) all converge strongly to $x^* = P_\Omega(I-(\mu V -\gamma f))x^*$ if and only if $\|y_n-Ty_n\| \to 0$ as $n \to \infty$, where $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ is the unique solution of the VIP

$$\langle(\gamma f - \mu V)x^*, x - x^*\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle - \Psi(x),$$

where Ψ is a potential function for γf.

Remark 4.3. If in system (4.3), $A = A_1 = \cdots = A_N = 0$, $T_n \equiv I, \forall n \geq 1$, and T is a nonexpansive mapping, we obtain a system of hierarchical fixed point problems introduced by Mainge and Moudafi [38, 39].

On the other hand, if $S : C \to C$ is a κ-strictly pseudocontractive mapping, that is, there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C,$$

then $A = I - S$ is $\frac{1-\kappa}{2}$-inverse strongly monotone; see [42].

Utilizing Theorems 3.13 and 3.14, we also give two strong convergence theorems for finding a common element of the solution set GMEP(Θ, h) of GMEP (1.5) and the common fixed point set $\cap_{n=1}^{\infty}\text{Fix}(T_n) \cap \cap_{i=1}^{N}\text{Fix}(S_i) \cap \text{Fix}(S)$ of a κ-strict pseudocontraction $S : C \to C$, one finite family of nonexpansive mappings $S_i : C \to C, i = 1, \dots, N$ and another infinite family of nonexpansive mappings $T_n : C \to C, n = 1, 2, \dots$.

Theorem 4.4. *Let $\alpha = \frac{1-\kappa}{2}$. Let us suppose that $\Omega = \cap_{n=1}^{\infty}\text{Fix}(T_n) \cap \cap_{i=1}^{N}\text{Fix}(S_i) \cap \text{Fix}(S) \cap \text{GMEP}(\Theta, h) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, N$, be sequences in $(0, 1)$ such that $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ for all indices i. Moreover, let us suppose that there hold (H0)-(H6) with $\gamma_n = 0, \forall n \geq 1$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ generated explicitly by*

$$(4.5) \quad \begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1}S_1u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_iu_n + (1 - \beta_{n,i})y_{n,i-1}, \quad i = 2, \dots, N, \\ y_n = \alpha_n\gamma f(y_{n,N}) + (I - \alpha_n\mu V)W_n((1 - \nu_n)y_{n,N} + \nu_nSy_{n,N}), \\ x_{n+1} = \beta_ny_n + (1 - \beta_n)((1 - \nu_n)y_n + \nu_nSy_n), \quad \forall n \geq 1, \end{cases}$$

all converge strongly to $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$, which is the unique solution of the VIP

$$\langle(\gamma f - \mu V)x^*, x - x^*\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle - \Psi(x),$$

where Ψ is a potential function for γf.

Proof. In Theorem 3.13, put $A = I - S$ and $T \equiv I$. Then A is $\frac{1-\kappa}{2}$-inverse strongly monotone. Hence we deduce that $\text{Fix}(S) = \text{VI}(C, A)$, and

$$\begin{cases} P_C(y_{n,N} - \nu_nAy_{n,N}) = (1 - \nu_n)y_{n,N} + \nu_nSy_{n,N}, \\ P_C(y_n - \nu_nAy_n) = (1 - \nu_n)y_n + \nu_nSy_n. \end{cases}$$

Thus, in terms of Theorem 3.13, we obtain the desired result. □

Theorem 4.5. *Let $\alpha = \frac{1-\kappa}{2}$. Let us suppose that $\Omega = \cap_{n=1}^{\infty}\text{Fix}(T_n) \cap \cap_{i=1}^{N}\text{Fix}(S_i) \cap \text{Fix}(S) \cap \text{GMEP}(\Theta, h) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, N$, be sequences in $(0, 1)$ such that $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose that there exists $k \in \{1, \dots, N\}$ for which $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1, \dots, N\}$ be the largest index for which $\beta_{n,k_0} \to 0$. Moreover, let us suppose that there hold (H0), (H7) and (H8) with $\gamma_n = 0, \forall n \geq 1$ and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;

(ii) *if* $i \leq k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;

(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$, *and* $\{u_n\}$ *generated explicitly by (4.5), all converge strongly to* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$, *which is the unique solution of the VIP*

$$\langle (\gamma f - \mu V)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \Omega} \frac{\mu}{2} \langle Vx, x \rangle - \Psi(x),$$

where Ψ *is a potential function for* γf.

References

[1] A. E. Al-Mazrooei, B. A. Bin Dehaish, A. Latif and J. C. Yao, *Hybrid viscosity approximation methods for general systems of variational inequalities in Banach spaces*, Fixed Point Theory Appl. (2013), 2013:**258**, 22 pp.

[2] S. Atsushiba and W. Takahashi, *Strong convergence theorems for a finite family of nonexpansive mappings and applications*, Indian J. Math. **41** (1999), 435–453.

[3] C. Baiocchi and A. Capelo, *Variational and Quasi-Variational Inequalities*, Wiley, New York, 1984.

[4] R. E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.

[5] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[6] L. C. Ceng, Q. H. Ansari, N. C. Wong and J. C. Yao, *An extragradient-like approximation method for variational inequalities and fixed point problems*, Fixed Point Theory Appl. (2011), 2011:**22**, 18 pp.

[7] L. C. Ceng, Q. H. Ansari, M. M. Wong and J. C. Yao, *Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems*, Fixed Point Theory **13** (2012), 403–422.

[8] L. C. Ceng, Q. H. Ansari and S. Schaible, *Hybrid extragradient-like methods for generalized mixed equilibrium problems, system of generalized equilibrium problems and optimization problems*, J. Glob. Optim. **53** (2012), 69–96.

[9] L. C. Ceng, Q. H. Ansari and J. C. Yao, *Relaxed extragradient iterative methods for variational inequalities*, Appl. Math. Comput. **218** (2011), 1112–1123.

[10] L. C. Ceng, S. M. Guu and J. C. Yao, *Finding common solutions of a variational inequality, a general system of variational inequalities, and a fixed-point problem via a hybrid extragradient method*, Fixed Point Theory Appl. (2011) Art. ID 626159, 22 pp.

[11] L. C. Ceng, N. Hadjisavvas and N. C. Wong, *Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems*, J. Glob. Optim. **46** (2010), 635–646.

[12] L. C. Ceng, Z. R. Kong and C. F. Wen, *On general systems of variational inequalities*, Comput. Math. Appl. **66** (2013), 1514–1532.

[13] L. C. Ceng, A. Latif, Q. H. Ansari and J. C. Yao,*Hybrid extragradient method for hierarchical variational inequalities*, Fixed Point Theory Appl. (2014), 2014:**222**, 35 pp.

[14] L. C. Ceng and J. C. Yao, *An extragradient-like approximation method for variational inequality problems and fixed point problems*, Appl. Math. Comput. **190** (2007), 205–215.

[15] L. C. Ceng, A. Petrusel and J. C. Yao, *Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, J. Optim. Theory Appl. **143** (2009), 37–58.

[16] L. C. Ceng, A. Petrusel and J. C. Yao, *Composite viscosity approximation methods for equilibrium problem, variational inequality and common fixed points*, J. Nonlinear Convex Anal. **15** (2014), 219–240.

[17] L. C. Ceng, A. Petrusel and J. C. Yao, *Composite viscosity approximation methods for equilibrium problem, variational inequality and common fixed points*, J. Nonlinear Convex Anal. **15** (2014), 219–240.

[18] L. C. Ceng, D. R. Sahu and J. C. Yao, *A unified extragradient method for systems of hierarchical variational inequalities in a Hilbert space*, J. Inequal. Appl. (2014), 2014:**460**, 32 pp.

[19] L. C. Ceng, M. Teboulle and J. C. Yao, *Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed point problems*, J. Optim. Theory Appl. **146** (2010), 19–31.

[20] L. C. Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), 186–201.

[21] L. C. Ceng and J. C. Yao, *A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem*, Nonlinear Anal. **72** (2010), 1922–1937.

[22] L. C. Ceng and J. C. Yao, *On the triple hierarchical variational inequalities with constraints of mixed equilibria, variational inclusions and systems of generalized equilibria*, Tamkang J. Math. **45** (2014), 297–334.

[23] L. C. Ceng, C. Y. Wang and J. C. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res. **67** (2008), 375–390.

[24] L. C. Ceng, H. K. Xu and J. C. Yao, *A hybrid steepest-descent method for variational inequalities in Hilbert spaces*, Appl. Anal. **87** (2008), 575–589.

[25] F. Cianciaruso, G. Marino, L. Muglia and Y. Yao, *A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem*, Fixed Point Theory Appl. (2010) Art. ID 383740, 19 pp.

[26] V. Colao, G. Marino and H. K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl. **344** (2008), 340–352.

[27] R. W. Cottle, F. Giannessi and J. L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Applications*, Wiley, New York, 1980.

[28] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vols. I and II, Springer, Berlin, 2003.

[29] F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum, New York, 1995.

[30] K. Goebel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University Press, Cambridge, England, 1990.

[31] J. S. Jung, *A new iteration method for nonexpansive mappings and monotone mappings in Hilbert spaces*, J. Inequal. Appl. (2010) Art. ID 251761, 16 pp.

[32] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[33] P. K. F. Kuhfittig, *Common fixed points of nonexpansive mappings by iteration*, Pacific. J. Math. **97** (1981), 137–139.

[34] A. Lahmdani, O. Chadli and J. C. Yao, *Existence of solutions for noncoercive hemivariational inequalities by an equilibrium approach under pseudomonotone perturbation*, J. Optim. Theory Appl. **160** (2014), 49–66.

[35] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[36] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (1) (2000), 46–55.

[37] A. Moudafi, *Weak convergence theorems for nonexpansive mappings and equilibrium problems*, J. Nonlinear Convex Anal. **9** (2008), 37–43.

[38] A. Moudafi and P.-E. Mainge, *Towards viscosity approximations of hierarchical fixed point problems*, Fixed Point Theory Appl. (2006) Art. ID 95453, 10 pp.

[39] A. Moudafi and P.-E. Mainge, *Strong convergence of an iterative method for hierarchical fixed point problems*, Pac. J. Optim. **3** (2007), 529–538.

[40] G. Marino, L. Muglia and Y. Yao, *Viscosity methods for common solutions of equilibrium and variational inequality problems via multi-step iterative algorithms and common fixed points*, Nonlinear Anal. **75** (2012), 1787–1798.

[41] G. Marino and H. K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **318** (2006), 43–52.

[42] G. Marino and H. K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Math. Appl. **329** (2007), 336–346.

[43] N. Nadezhkina and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **128** (2006), 191–201.

[44] N. Nadezhkina and W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM J. Optim. **16** (2006), 1230–1241.

[45] W. Oettli, *A remark on vector-valued equilibria and generalized monotonicity*, Acta Math. Vietnam. **22** (1997), 215–221.

[46] R. T. Rockafellar, *Monotone operators and the proximal point algorithms*, SIAM J. Control Optim. **14** (1976), 877–898.

[47] K. Rattanaseeha, *The general iterative methods for equilibrium problems and fixed point problems of countable family of nonexpansive mappings in Hilbert spaces*, J. Inequal. Appl. (2013) 2013:**153**, 20 pp.

[48] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwan. J. Math. **5** (2001), 387–404.

[49] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[50] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), 1025–1033.

[51] W. Takahashi, N. C. Wong and J. C. Yao, Weak and strong convergence theorems for commutative families of positively homogeneous nonexpansive mappings in Banach spaces, J. Nonlinear Convex Anal. **15** (2014), 557–572.

[52] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

[53] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc. **66** (2002), 240–256.

[54] Y. Yao, Y. C. Liou and G. Marino, *Two-step iterative algorithms for hierarchical fixed point problems and variational inequality problems*, J. Appl. Math. Comput. **31** (2009), 433–445.

[55] Y. Yao, Y. C. Liou and S. M. Kang, *Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method*, Comput. Math. Appl. **59** (2010), 3472–3480.

[56] Y. Yao, Y. C. Liou and J. C. Yao, *Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings*, Fixed Point Theory Appl. (2007) Art. ID 064363, 12 pp.

[57] L. C. Zeng, J.C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwan. J. Math. **10** (2006), 1293–1303.

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