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BILEVEL VECTOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS: DUALITY AND EXISTENCE

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Abstract. The aim of this paper is devoted to investigate the duality and existence of solutions for a class of bilevel vector pseudomonotone equilibrium problems without involving the information about the solution set of the lower-level equilibrium problem. Firstly, we propose the dual formulations of bilevel vector equilibrium problems (BVEP). Secondly, the primal-dual relationships are derived under cone-convexity and weak pseudo-monotonicity assumptions. Finally, the existence of solutions of BVEP are established without involving the information about the solution set of the lower-level problem.

1. Introduction

It is well-known that equilibrium problem is closely related to optimization and control problems, games theory, variational inequalities problems, complementarity problems and fixed point problems, as well as mechanics and physics (see [1, 6, 21, 23, 28, 29] and the references therein). Equilibrium problem which is also called Ky Fan inequality, was firstly introduced by Blum and Oettli. Thereafter, various types of equilibrium problems were intensively studied (see [2, 22, 25, 26] and the references therein). Dual optimization problems play an crucial role in the study of optimization and equilibrium theory and methods (see [1, 3, 20, 24, 27] and the references therein). In [25], Konnov and Yao investigated existence of solutions for generalized vector equilibrium problems by using dual method and Fan's lemma. Konnov and Schaible[24] proposed various duals for an abstract equilibrium problem, established the primal-dual relationships under some generalized convexity and monotonicity assumptions, and derived existence results by using the duality. Further, Ansari, Siddiqi and Wu[3] generalized the results of Konnov and Schaible [24] to generalized vector equilibrium problems in a real topological vector space. In [20], Farajzadeh and Lee considered the existence of solutions for dual vector equilibrium problems (DVEP) for a moving cone, and presented the relations between DVEP and its perturbations in a real topological vector space.

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In 2010, Moudafi [30] introduced a class of bilevel equilibrium problem (shortly, (BEP)) which is to find $x \in S_f$ such that

$$g(x, y) \geq 0, \ \forall \ y \in S_f,$$

where S_f is the solution set of the following equilibrium problem: find $u \in K$ such that

$$f(u, y) \geq 0, \ \forall \ y \in K,$$

where K is a nonempty, closed and convex subset of a Hilbert space and $f, g : K \times K \to R$ are two functions. He pointed out that this class is absorbing since it includes hierarchical optimization problems, optimization with equilibrium, variational inequalities, complementarity constraints as special cases. Also, by using the proximal method, an iterative algorithm to compute approximate solution of BEP and the weak convergence of the iterative sequence generated by the algorithm were suggested and derived, respectively. Since then, Ding[12, 13, 14, 15] and Ding, Liou and Yao[16] generalized the BEP to the bilevel generalized mixed equilibrium problems in reflexive Banach space, established the existence results of solutions for the mixed equilibrium problems and the bilevel mixed equilibrium problems by using minimax inequality. By using auxiliary principle technique, they also constructed some iterative algorithms for solving the mixed equilibrium problems and bilevel mixed equilibrium problems, and derived the strong convergence of the proposed algorithms under suitable assumptions. Chen et al.[8, 9] further explored the existence, well-posedness and algorithms for BEP by using fixed point method. Dinh and Muu [17] studied a class of bilevel pseudomonotone equilibrium problems by penalty function method, and proved that under the pseudo-∇-monotonicity, any stationary point of a regularized gap function is a solution of the penalized equilibrium problem. Chadli et al. [7] also discussed the existence and algorithmic aspects of a class of bilevel mixed equilibrium problems in Banach spaces, and then constructed an iterative algorithm by the auxiliary problem. They also proved that a sequence generated by the proposed algorithm is strongly convergent to a solution of the bilevel mixed equilibrium problem. Very recently, Anh, Kim and Muu[1] analyzed the convergence of an extragradient algorithm for a class of bilevel pseudomonotone variational inequality which is a special model of the BEP in [30]. In [2], Anh, Khanh and Van gave some sufficient conditions for the well-posedness and unique well-posedness to the bilevel equilibrium and optimization problems with equilibrium constraints under the assumptions of existence of solutions and the relaxed level closedness and pseudocontinuity. Very recently, Facchinei et al. [18, Math. Program. 145(2014):59–96] suggested some iterative algorithms for hemivariational inequalities with variational inequality constraints, which is also a special case of BEP in [30], by inexact Prox-Tikhonov method and distributed method, and applied to power control in ad-hoc networks. *It is worth noting that many authors studied the existence of solutions and iterative algorithms for bilevel equilibrium problems and bilevel variational inequalities involving the information about the solution set of the lower-level problem. Moreover, there are little results concerning the duality and existence of solution for bilevel vector equilibrium problems.*

Motivated and inspired by the ongoing research in this direction, the aim of this paper is devoted to investigate the duality and existence of solution for a class of bilevel vector pseudomonotone equilibrium problems without involving the information about the solution set of the lower-level equilibrium problem. Firstly, we propose the dual formulations of bilevel vector equilibrium problems (BVEP). Secondly, the primal-dual relationships are derived under cone-convexity and weak pseudo-monotonicity assumptions. Finally, existence of solutions of BVEP are established without involving the information about the solution set of the lower-level problem.

Throughout this paper, let E, H and Z be finite dimensional Euclidean spaces, K be a nonempty, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings, and let $C \subseteq H$ and $Q \subseteq Z$ be closed, convex and pointed cones with nonempty interior $\mathrm{int}C \neq \emptyset$ and $\mathrm{int}Q \neq \emptyset$. Recall that a subset B of H is said to be a convex and pointed cone if $B + B = B, B \cap (-B) = \{0\}$ and $\mu b \in B$ for all $\mu > 0$ and $b \in B$. The dual cone of B is denoted by

$$B^* = \{u \in H : x^T u \geq 0, \forall\ x \in B\}.$$

Consider the following *bilevel vector equilibrium problem* (shortly, (BVEP)):
Find $x^* \in S_\Psi$ such that

$$\Phi(x^*, y) \notin -\mathrm{int}\, C,\ \forall\ y \in S_\Psi, \tag{1.1}$$

where S_Ψ is the solution set of the lower-level equilibrium problem:
Find $y^* \in K$ such that

$$\Psi(y^*, z) \notin -\mathrm{int}\, Q,\ \forall\ z \in K. \tag{1.2}$$

Denote the solution set of the BVEP (1.1) with (1.2) by S.

Special cases:

(I) If $\Phi(x, y) = f(y) - f(x)$, where $f : K \to H$ is vector-valued, then the BVEP (1.1) with (1.2) reduces to the following multiobjective programming with equilibrium constraints (MPEC):

$$\text{“}C - \min\text{”}\ f(y)\ \text{ subject to }\ y \in S_\Psi. \tag{1.3}$$

where S_Ψ is the solution set of the lower-level equilibrium problem (1.2).

The MPEC (1.3) cover various types of optimization with equilibrium, variational inequality, complementarity and inclusions as constraints (see [2, 11, 31, 27, 29] and the references therein).

(II) If $H = (-\infty, +\infty)$ and $C = Q = [0, +\infty)$, then the BVEP (1.1) with (1.2) reduces to the following bilevel equilibrium problem:
Find $x^* \in S_\Psi$ such that

$$\Phi(x^*, y) \geq 0,\ \ \forall\ y \in S_\Psi, \tag{1.4}$$

where $S_\Psi = \{y \in K : \Psi(y, z) \geq 0,\ \ \forall\ z \in K\}$.

For a suitable choice of Φ and Ψ, this class, which was firstly introduced by Moudafi [30], includes many types of bilevel equilibrium problems such as bilevel generalized mixed (quasi) equilibrium problem, bilevel generalized mixed quasi-variational-like inequality problem, bilevel mixed equilibrium problem, bilevel pseudomonotone equilibrium problem and variational inequality with variational inequality constraints (see [9, 12, 13, 14, 15, 16, 17] and the references therein), and has been greatly applied to economics and management sciences, decision-making disciplines, engineering, power control systems and so on (see [4, 18] and the references therein).

2. Notions and facts

The following notions and results, which are mostly well known, are recalled here for the reader's convenience.

Definition 2.1. Let $\psi : K \times K \to Z$ be a vector-valued mapping. ψ is called:

(1) *Q-convex* with respect to the second argument if, for any given $x \in K$,

$$t\psi(x, y) + (1 - t)\psi(x, w) - \psi(x, ty + (1 - t)w) \in Q, \ \forall\, y, w \in K, t \in (0, 1);$$

(2) *affine* with respect to the second argument if, for any given $x \in K$,

$$\psi(x, ty + (1 - t)w) = t\psi(x, y) + (1 - t)\psi(x, w), \ \forall\, y, w \in K, t \in (-\infty, +\infty);$$

(3) *hemicontinuous* with respect to the first argument if, for any given $x \in K$,

$$\lim_{t \searrow 0} \psi(ty + (1 - t)w, x) = \psi(w, x), \ \forall\, y, w \in K.$$

It is easy to see that if $\psi : K \times K \to Z$ is affine with respect to the second argument, then it is Q-convex with respect to the second argument.

Definition 2.2 ([22]). Let $\psi : K \times K \to Z$ be a vector-valued mapping. ψ is called:

(1) *weakly Q-pseudomonotone* if, for any $x, y \in K$,

$$\psi(x, y) \notin -\operatorname{int} Q \ \Rightarrow \ \psi(y, x) \notin \operatorname{int} Q;$$

(2) *Q-pseudomonotone* if, for any $x, y \in K$,

$$\psi(x, y) \notin -\operatorname{int} Q \ \Rightarrow \ \psi(y, x) \in -Q;$$

(3) *strictly Q-pseudomonotone* if, for any $x, y \in K, x \neq y$,

$$\psi(x, y) \notin -\operatorname{int} Q \ \Rightarrow \ \psi(y, x) \in -\operatorname{int} Q.$$

It is easy to see that

the strict C-pseudomonotonicity $\Rightarrow$ C-pseudomonotonicity $\Rightarrow$ the weak C-pseudomonotonicity.

Fact 2.3 ([10]). Let Δ be a convex cone of Z with $\operatorname{int} \Delta \neq \emptyset$ and its dual cone Δ^*. The following hold:

(1) If $u \in \operatorname{int} \Delta$, then $x^T u > 0$ for all $x \in \Delta^* \setminus \{0\}$, where the superscript T denotes the transpose;

(2) If $x \in \operatorname{int} \Delta^*$, then $x^T u > 0$ for all $u \in \Delta \setminus \{0\}$.

Fact 2.4 ([19]). Let D be a nonempty, convex subset of a finite dimensional Euclidean space E, $F : D \to 2^E$ be a KKM mapping, i.e., for every finite subset $\{x_1, x_2, \dots, x_m\}$ of $D, \operatorname{co}\{x_1, x_2, \dots, x_m\}$ is contained in $\bigcup_{i=1}^m F(x_i)$ where co denotes the convex hull, such that for any $x \in D, F(x)$ is closed and $F(x^*)$ is bounded for some $x^* \in D$. Then there exists $y^* \in D$ such that $y^* \in F(x)$ for all $x \in D$, i.e., $\bigcap_{x\in D} F(x) \neq \emptyset$.

3. Duality for (BVEP)

In this section, we propose the dual of bilevel vector equilibrium problem (shortly, (DBVEP)), and establish the equivalence between DBVEP and BVEP under some suitable conditions.

Motivated by Konnov and Schaible [24] and Ansari, Siddiqi and Wu [3], we propose the following the dual formulation of BVEP:

Find $x^* \in S^d_\Psi$ such that

$$\Phi(y, x^*) \notin \operatorname{int} C, \ \ \forall\, y \in S^d_\Psi, \tag{3.1}$$

where S^d_Ψ is the solution set of the lower-level equilibrium problem:

Find $y^* \in K$ such that

$$\Psi(z, y^*) \notin \operatorname{int} Q, \ \ \forall\, z \in K. \tag{3.2}$$

Denote the solution set of the DBVEP (3.1) with (3.2) by S^d.

We now establish the equivalence between DBVEP and BVEP.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings. Assume that the following conditions hold:*

1. *$\Psi(x, x) \in Q$ and $\Phi(x, x) \in C$ for all $x \in K$;*
2. *Ψ and Φ are hemicontinuous with respect to the first argument;*
3. *Ψ and Φ are Q-convex and C-convex with respect to the second argument, respectively;*
4. *Ψ and Φ are weakly Q-pseudomonotone and weakly C-pseudomonotone, respectively.*

If S_Ψ is nonempty closed and convex, then BVEP and DBVEP are equivalent, i.e., $S = S^d$.

Proof. Let $x^* \in S$. Then $\Phi(x^*, y) \notin -\operatorname{int} C$ for all $y \in S_\Psi$ and $\Psi(x^*, z) \notin -\operatorname{int} Q$ for all $z \in K$. This together with condition (4) yields that

$$\Phi(y, x^*) \notin \operatorname{int} C, \ \forall\, y \in S_\Psi,$$

and

$$\Psi(z, x^*) \notin \operatorname{int} Q, \ \forall\, z \in K.$$

If $S_\Psi = S^d_\Psi$, then $x^* \in S^d$.

To this end, we prove that $S_\Psi = S^d_\Psi$. Then $S_\Psi \subseteq S^d_\Psi$ follows from the condition (4). On the other hand, let $\bar{y} \in S^d_\Psi$. For any $z \in K$, set $z_\lambda = \lambda z + (1-\lambda)\bar{y}$ for all $\lambda \in (0, 1)$. Then $z_\lambda \in K$ for all $\lambda \in (0, 1)$ and, $\Psi(z_\lambda, z_\lambda) \in Q$. Since Ψ is Q-convex with respect to the second argument, one has

$$\lambda\Psi(z_\lambda, z) + (1-\lambda)\Psi(z_\lambda, \bar{y}) - \Psi(z_\lambda, z_\lambda) \in Q.$$

Taking into account $\Psi(z_\lambda, z_\lambda) \in Q$, we obtain

$$\lambda\Psi(z_\lambda, z) + (1-\lambda)\Psi(z_\lambda, \bar{y}) \in Q + \Psi(z_\lambda, z_\lambda) \subseteq Q. \tag{3.3}$$

Claim that $\Psi(z_\lambda, z) \notin -\operatorname{int} Q$. Suppose that $\Psi(z_\lambda, z) \in -\operatorname{int} Q$. Then

$$-\lambda\Psi(z_\lambda, z) \in \operatorname{int} Q. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$(1-\lambda)\Psi(z_\lambda, \bar{y}) \in Q - \lambda\Psi(z_\lambda, z) \subseteq Q + \operatorname{int} Q \subseteq \operatorname{int} Q.$$

This combine with $1 - \lambda > 0$ that $\Psi(z_\lambda, \bar{y}) \in \operatorname{int} Q$, which contradicts $\bar{y} \in S_\Psi^d$. Therefore

$$\Psi(z_\lambda, z) \in Z \setminus (-\operatorname{int} Q). \tag{3.5}$$

Since Ψ is hemicontinuous with respect to the first argument, and from (3.5), one has

$$\lim_{\lambda \searrow 0} \Psi(z_\lambda, z) = \Psi(\bar{y}, z) \in Z \setminus (-\operatorname{int} Q), \ \forall\, z \in K.$$

This shows that $\Psi(\bar{y}, z) \notin -\operatorname{int} Q$ for all $z \in K$. Then $\bar{y} \in S_\Psi$ and so, $S_\Psi^d \subseteq S_\Psi$. Therefore, $S_\Psi^d = S_\Psi$.

Conversely, let $x^* \in S^d$. Then $x^* \in S_\Psi^d$ and

$$\Phi(y, x^*) \notin \operatorname{int} C, \quad \forall\, y \in S_\Psi^d.$$

According to $S_\Psi = S_\Psi^d$, $x^* \in S_\Psi$. Hence

$$\Psi(x^*, z) \notin -\operatorname{int} Q, \ \forall\, z \in K. \tag{3.6}$$

Let $\bar{K} = S_\Psi^d$. By the same argument, $\Phi(x^*, y) \notin -\operatorname{int} C$ for all $y \in \bar{K}$. Therefore $x^* \in S_\Psi$ such that $\Phi(x^*, y) \notin -\operatorname{int} C$ for all $y \in S_\Psi$. This combine with (3.6) implies that $x^* \in S$. □

Remark 3.2. The dual formulation of BVEP is distinct from that of Konnov and Schaible [24], Ansari, Siddiqi and Wu [3] and Huang, Li and Thompson[22]. Generally, the dual of DBVEP is not the primal BVEP unless their lower-level equilibrium problems are equivalent.

Corollary 3.3. *Let K be a nonempty, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings. Assume that the conditions (1),(2),(4) of Theorem 3.1, and the following hold:*

(3)′ *Ψ and Φ and affine with respect to the second argument.*

If S_Ψ is nonempty closed and convex, then BVEP and DBVEP are equivalent, i.e., $S = S^d$.

Proof. Follows readily from Theorem 3.1. □

Corollary 3.4. *Let K be a nonempty, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings. Assume that the following conditions hold:*

1. *$\Psi(x, x) \in Q$ for all $x \in K$;*
2. *Ψ is hemicontinuous with respect to the first argument;*
3. *Ψ is Q-convex with respect to the second argument;*
4. *Ψ is weakly Q-pseudomonotone.*

Then $S_\Psi = S_\Psi^d$.

Proof. Inspect the proof of Theorem 3.1. □

4. The existence results for BVEP

In this section, we firstly study the nonemptiness and convexity of the set of solutions for the lower-level equilibrium problem (1.2) of BVEP under some suitable conditions, and then provide the existence of solutions for BVEP.

Lemma 4.1. *Let K be a nonempty, closed and convex subset of E and $\Psi : K\times K \to Z$ be continuous and affine with respect to the first argument such that $\Psi(y, z) \in Q \cup (-intQ)$ for all $y, z \in K$. Then S_Ψ is closed and conex.*

Proof. Let $\{y_n\} \subseteq S_\Psi$ such that $y_n \to \bar{y} \in K$. Then $\Psi(y_n, z) \notin -\text{int}\, Q$ for all $z \in K$. It follows from the continuity of Ψ that $\Psi(\bar{y}, z) \in Z \setminus (-\text{int}\, Q)$ for all $z \in K$. This shows that $\bar{y} \in S_\Psi$.

Let $y_1, y_2 \in S_\Psi$ and set $y_\iota = \iota y_1 + (1-\iota)y_2$ for $\iota \in (0, 1)$. Then

$$\Psi(y_i, z) \notin -\text{int}Q,\ \forall\ z \in K, i = 1, 2. \tag{4.1}$$

Since $\Psi : K \times K \to Z$ is affine with respect to the first argument, we have

$$\Psi(y_\iota, z) = \iota\Psi(y_1, z) + (1-\iota)\Psi(y_2, z),\ \forall\ z \in K. \tag{4.2}$$

Note that $\Psi(y, z) \in Q \cup (-\text{int}\, Q)$ for all $y, z \in K$. So, from (4.1), (4.2), we have $\Psi(y_\iota, z) \in Q$. Consequently, $\Psi(y_\iota, z) \notin -\text{int}\, Q$ for all $z \in K$. Therefore $y_\iota \in S_\Psi$ for $\iota \in (0, 1)$ and so, S_Ψ is closed and convex. □

We next prove the solvability, convexity and boundedness of the solution set S_Ψ of the lower-level equilibrium problem (1.2).

Lemma 4.2. *Let K be a nonempty, bounded, closed and convex subset of E. Assume that the conditions (1)-(4) of Corollary 3.4 and the following hold:*

(5) *Ψ is continuous with respect to the second argument such that $\Psi(y, z) \in (-Q) \cup (intQ)$ for all $y, z \in K$.*

Then S_Ψ is nonempty, bounded, closed and convex.

Proof. Let us first show that S_Ψ is closed and convex. Take $\{y_n\} \subseteq S_\Psi$ such that $y_n \to \bar{y} \in K$. Then $\{y_n\} \subseteq S^d_\Psi$ by Corollary 3.4. Hence $\Psi(z, y_n) \notin \text{int}Q$ for all $z \in K$. Moreover, one has

$$\Psi(z, y_n) \in Z \setminus \text{int}\, Q,\ \forall\ z \in K.$$

By the continuity of Ψ with respect to the second argument, we have

$$\Psi(z, \bar{y}) \in Z \setminus \text{int}\, Q,\ \forall\ z \in K.$$

So, $\Psi(z, \bar{y}) \notin \text{int}\, Q$ for all $z \in K$. This yields that $\bar{y} \in S^d_\Psi$ and so, $\bar{y} \in S_\Psi \subseteq K$. Therefore S_Ψ is closed and bounded.

Suppose that S_Ψ is not convex. Then there exist $\hat{y}, \tilde{y} \in S_\Psi$ and $\lambda \in (0, 1)$ such that $\lambda\hat{y} + (1-\lambda)\tilde{y} \notin S_\Psi$. According to Corollary 3.4, there exists $\hat{z} \in K$ such that $\Psi(\hat{z}, \lambda\hat{y} + (1-\hat{y})\tilde{y}) \in \text{int}\, Q$. Again, from $\hat{y}, \tilde{y} \in S_\Psi$ and the condition (5), $\Psi(\hat{z}, \hat{y}) \in -Q$ and $\Psi(\hat{z}, \tilde{y}) \in -Q$. By Fact 2.3, for any $u \in Q^* \setminus \{0\}$,

$$\Psi(\hat{z}, \lambda\hat{y} + (1-\hat{y})\tilde{y})^T u > 0. \tag{4.3}$$

and

(4.4) $(\lambda\Psi(\hat{z},\hat{y})+(1-\lambda)\Psi(\hat{z},\tilde{y}))^T u=\lambda\Psi(\hat{z},\hat{y})^T u+(1-\lambda)\Psi(\hat{z},\tilde{y})^T u\leq 0.$

Since Ψ is Q-convex with respect to the second argument,

$$\lambda\Psi(\hat{z},\hat{y})+(1-\lambda)\Psi(\hat{z},\tilde{y})-\Psi(\hat{z},\lambda\hat{y}+(1-\hat{y})\tilde{y})\in Q.$$

This shows that

$$[\lambda\Psi(\hat{z},\hat{y})+(1-\lambda)\Psi(\hat{z},\tilde{y})-\Psi(\hat{z},\lambda\hat{y}+(1-\hat{y})\tilde{y})]^T u\geq 0,\ \forall\ u\in Q^*\setminus\{0\}.$$

Further, we have

$$(\lambda\Psi(\hat{z},\hat{y})+(1-\lambda)\Psi(\hat{z},\tilde{y}))^T u-\Psi(\hat{z},\lambda\hat{y}+(1-\hat{y})\tilde{y})^T u\geq 0,\ \forall\ u\in Q^*\setminus\{0\}.$$

It follows from (4.4) that

$$\Psi(\hat{z},\lambda\hat{y}+(1-\hat{y})\tilde{y})^T u\leq 0,\ \forall\ u\in Q^*\setminus\{0\},$$

which contradicts (4.3). Therefore S_Ψ is convex.

Finally, we show that $S_\Psi\neq\emptyset$. Define the mappings $\Gamma,\Upsilon:K\to 2^K$ by

$$\Gamma(z)=\{y\in K:\Psi(y,z)\notin -\operatorname{int}Q\},\ \forall\ z\in K$$

and

$$\Upsilon(z)=\{y\in K:\Psi(z,y)\notin \operatorname{int}Q\},\ \forall\ z\in K.$$

Clearly, $S_\Psi=\bigcap_{z\in K}\Gamma(z)$ and $\bigcap_{z\in K}\Upsilon(z)=S_\Psi^d$. It follows from the conditions (1) and (4) that

$$z\in\Gamma(z)\subseteq\Upsilon(z),\ \ \forall\ z\in K. \tag{4.5}$$

Therefore $\Psi(z,z)\in Q\cap(-Q)=\{0\}$ by the condition (5) and so, $\Psi(z,z)=0$ for all $z\in K$.

We claim that Γ is a KKM mapping. In fact, if there exists a finite subset $\{z_1,z_2,\ldots,z_m\}$ of K, $\operatorname{co}\{z_1,z_2,\ldots,z_m\}\not\subseteq\bigcup_{i=1}^m\Gamma(z_i)$. That is, there exists $\tilde{z}=\sum_{i=1}^m\iota_i z_i$, where $\sum_{i=1}^m\iota_i=1$ and $\iota_i\geq 0, i=1,2,\ldots,m$, such that $\tilde{z}\notin\Gamma(z_i)$ for $i=1,2,\ldots,m$. This implies that $\Psi(\tilde{z},z_i)\in -\operatorname{int}Q$ for $i=1,2,\ldots,m$ and so,

$$\sum_{i=1}^m\iota_i\Psi(\tilde{z},z_i)\in -\operatorname{int}Q. \tag{4.6}$$

By the Q-convexity of Ψ with respect to the second argument, we have

$$\sum_{i=1}^m\iota_i\Psi(\tilde{z},z_i)-\Psi(\tilde{z},\tilde{z})\in Q.$$

This, together with $\Psi(\tilde{z},\tilde{z})=0$, yields that $\sum_{i=1}^m\iota_i\Psi(\tilde{z},z_i)\in Q$, which contradicts (4.6). Therefore Γ and Υ are KKM mappings by (4.5). Similar to the proof of the closedness of S_Ψ, for each $z\in K$, $\Upsilon(z)$ is closed. From the boundedness of K, $\Upsilon(z)$ is bounded for each $z\in K$. By Fact 2.4, $S_\Psi^d=\bigcap_{z\in K}\Upsilon(z)\neq\emptyset$. Hence $S_\Psi\neq\emptyset$. □

Lemma 4.3. *Let K be a nonempty, closed and convex subset of E. Assume that the conditions (1)-(5) of Lemma 4.2 and the following hold:*

(6) *there exists a nonempty, bounded, closed and convex subset Ω of E such that for any $\tilde{y}\in K\setminus\Omega$, there is $\tilde{z}\in\Omega$ satisfying $\Psi(\tilde{y},\tilde{z})\in -\operatorname{int}Q$.*

Then S_Ψ is nonempty, bounded, closed and convex.

Proof. Set $G(z) = \{y \in \Omega : \Psi(z, y) \notin \operatorname{int} Q\}$ for all $z \in K$. Similar to the proof of Lemma 4.2, we have that S_Ψ is closed and convex, and that $G(z)$ is closed and bounded for each $z \in K$. The boundedness of S_Ψ results from the condition (6). $S_\Psi \neq \emptyset$ can be proved as the one of [22, Theorem 4.5] and so it is omitted here. □

Remark 4.4. (i) If Ψ is strictly Q-pseudomonotone in Lemma 4.2 and Lemma 4.3, then S_Ψ is a singleton. Indeed, if there exist $y_1, y_2 \in S_\Psi$ and $y_1 \neq y_2$, $\Psi(y_1, y_2) \notin -\operatorname{int} Q$ and $\Psi(y_2, y_1) \notin -\operatorname{int} Q$. By the strictly Q-pseudomonotonicity of Ψ, $\Psi(y_2, y_1) \in -\operatorname{int} Q$ which is a contradiction.

(ii) In the setting of finite dimensional Euclidean spaces, compared with Theorem 3.1 of [5], the condition (6) is weaker than the coercivity condition (C) in [5]. Lemma 4.3 does not require the condition: "For all $c \notin Q$ and for all $x \in K$, the set $\{y \in K : c - \Psi(x, y) \in \operatorname{int} Q\}$ is convex". Moreover, under the assumptions of Theorem 3.1 and Theorem 3.2 of [5], the solution set S_Ψ of the lower-level equilibrium problem (1.2) is not convex in general (see [5, Remark 3.4]). So, Lemma 4.3 is different from Theorem 3.1 of [5].

We now show the characterizations of solution of BVEP.

Theorem 4.5. *Let K be a nonempty, bounded, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings. Assume that the conditions (1)-(4) of Theorem 3.1 and the following hold:*

(5) *Ψ and Φ are continuous with respect to the second argument such that $\Psi(x, y) \in (-Q) \cup (intQ)$ and $\Phi(x, y) \in (-C) \cup (intC)$ for all $x, y \in K$.*

Then the solution set S of BVEP is nonempty, bounded, closed and convex.

Proof. Follows readily from Lemma 4.2. □

Theorem 4.6. *Let K be a nonempty, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings. Assume that the conditions (1)-(4) of Theorem 3.1 and the following hold:*

(5) *Ψ and Φ are continuous with respect to the second argument such that $\Psi(x, y) \in (-Q) \cup (intQ)$ and $\Phi(x, y) \in (-C) \cup (intC)$ for all $x, y \in K$;*
(6) *there exists a nonempty, bounded, closed and convex subset Ω of E such that for any $\tilde{y} \in K \setminus \Omega$, there is $\tilde{z} \in \Omega$ satisfying $\Psi(\tilde{y}, \tilde{z}) \in -intQ$.*

Then the solution set S of BVEP is nonempty, bounded, closed and convex.

Proof. Combine Lemma 4.2 with Lemma 4.3. □

The following results deduce directly from Remark 4.4, Theorem 4.5 and Theorem 4.6.

Corollary 4.7. *Let K be a nonempty, bounded, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings. Assume that Φ is strictly C-pseudomonotone, and all conditions of Theorem 4.5 hold. Then BVEP admits a unique solution.*

Corollary 4.8. *Let K be a nonempty, closed and convex subset of E, $\Phi : K \times K \to H$ and $\Psi : K \times K \to Z$ be vector-valued mappings. Assume that Φ is strictly C-pseudomonotone, and all conditions of Theorem 4.6 hold. Then BVEP admits a unique solution.*

Remark 4.9. By using dual results of BVEP, the existence of solution of DBVEP can be derived from Theorems 3.1, 4.5, 4.6 and Corollaries 3.3, 4.7 and 4.8.

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