Journal of Nonlinear and Convex Analysis
Volume 16, Number 7, 2015, 1305–1320

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

GENERALIZED INVEXITY AND GENERALIZED INVARIANT MONOTONE VECTOR FIELDS ON RIEMANNIAN MANIFOLDS WITH APPLICATIONS

SHENG-LAN CHEN AND NAN-JING HUANG*

Abstract. In this paper, we establish some connections between the generalized invexity for locally Lipschitz functions and the generalized invariant monotonicity for set-valued vector fields on Riemannian manifolds. As applications, we give some relationships between vector variatinal-like inequalites involving Clarke subdifferential and nonsmooth vector optimization on Riemannian manifolds by employing the pseudoinvexity and invariant pseudomonotonicity. The results presented in this paper extend the corresponding results given in the literature.

1. Introduction

The concept of convexity on linear spaces plays an important role in many aspects of optimization theory. This concept is often not enjoyed by the real problems ([11]). Therefore, various approaches have been proposed to weaken the convexity assumption. One of the useful generalizations is invexity introduced by Hanson [13]. His initial result inspired a great deal of subsequent work concerning further generalizations and applications in this category (see, for example, [2, 17, 26, 30, 32, 23] and the references therein).

A concept closely related to the convexity is the monotonicity of the mapping. It is well known that the convexity of a real-valued function is equivalent to the monotonicity of the corresponding gradient function ([16]). It is worth noting that monotonicity has played a central role in studying the existence and the sensitivity analysis of solutions for variational inequalities, variational inclusions, and complementary problems. The relation between generalized monotone operators and generalized convexity of functions has been studied by many authors in the last few years ([12, 28, 31]). Generalized invexity and its relation with generalized invariant monotonicity has been investigated in [31, 15]. At the same time, a large number of results have appeared in the literature reflecting the relationships between vector variational-like inequalities and vector optimization problems under the the assumptions of generalized invexity or invariant monotonicity ([3, 1, 24]).

On the other hand, Rapscátk [33] and Udriste [37] considered a generalization of convexity called geodesic convexity in Riemannian manifolds. In this setting

2010 *Mathematics Subject Classification.* 90C29, 58E35, 49J2.

Key words and phrases. Generalized invex functions, invariant monotone vector fields, generalized vector variational-like inequality, nonsmooth vector optimization problem, Riemannian manifold.

This work was supported by the National Natural Science Foundation of China (11171237,11471230).

*Corresponding author.

the linear space is replaced by a Riemannian manifold and the line segment by a geodesic. They also studied the monotonicity of the gradient of the geodesic convex functions. The concept of monotone vector field on Riemannian manifolds which was a generalization of monotone operator was introduced by Németh [27]. This notion has been extended by Da Cruz Neto et al. and Li et al. to the cases of set-valued mappings ([10, 20]). The concept of invex function on Riemannian manifolds was introduced by Pini [29], while Mititelu [25] investigated its generalization. Recently, Barani and Pouryayevali [6] introduced several notions of invexities for functions on Riemannian manifolds, and studied their relations with various concepts of invariant monotone vector fields defined on Riemannian manifolds. Besides, it is worth to mentioning that some concepts of nonsmooth analysis ([9]) have been extended from Euclidean spaces to Riemannian manifolds, in order to study optimization problems and related topics ([14, 19, 8, 5, 4, 21, 22, 34, 35, 36]). Very recently, Barani [5] proposed some notions of generalized convexity for locally Lipschitz functions and some concepts of generalized monotonicity for set-valued mappings on Hadamard manifolds, and studied the connections between of them.

In this paper, we introduce several kinds of generalized invexity for locally Lipschitz functions and generalized invariant monotonicity of set-valued vector field on Riemannian manifolds. By using the techniques of Barani [5], some necessary and sufficient conditions of being a locally Lipschitz function invex, or pseudoinvex are given in terms of invariant monotonicity, or pseudomonotonicity of its Clarke's subdifferential, respectively. As applications, we establish some relationships between a solution of generalized vector variational-like inequalities and an efficient or a weakly efficient solution to the nonsmooth vector optimization problem under the assumptions of pseudoinvexity or invariant pseudomonotonicity. The results presented in this paper extend some known results in [5, 6].

2. Preliminaries

In this section, we recall some definitions and known results about Riemannian manifolds which will be used throughout the paper. It can be found in many introductory books on Riemannian geometry, such as in [7, 18, 19, 33, 37].

Let M be a C^∞ smooth manifold modelled on a Hilbert space H, either finite dimensional or infinite dimensional, endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_pM \cong H$. The corresponding norm is denoted by $\| \cdot \|_p$ and the length of a piecewise C^1 curve $\gamma : [a, b] \to M$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt.$$

For any two points p, $q \in M$, we define

$$d(p, q) = \inf\{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve joining } p \text{ to } q\}.$$

Then d is a distance which induces the original topology on M. On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M. We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ, that is, γ satisfies the equation $\nabla_{d\gamma(t)/dt} d\gamma(t)/dt = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p,q)$ is a geodesic and is called a minimal geodesic. The existence theorem for ordinary differential equation implies that for every $v \in TM$, there exists an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma(v) : J(v) \to M$ with $d\gamma_v(0)/dt = v$. This implies that there is an open neighborhood $\tilde{T}M$ of the submanifold M of TM such that for every $v \in \tilde{T}M$, the geodesic $\gamma_v(t)$ is defined for $|t| < 2$. The exponential mapping $exp : \tilde{T}M \to M$ is then defined as $\exp(v) = J_v(1)$ and the restriction of exp to a fiber T_pM in $\tilde{T}M$ is denoted by $\exp_p$ for every $p \in M$. We use parallel transport of vectors along geodesic. Recall that for a given curve $\gamma : I \to M$, a number $t_0 \in I$ and a vector $v_0 \in T_{\gamma(t_0)}M$, there exists exactly one parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = v_0$. Moreover, the mapping defined by $v_0 \to V(t)$ is a linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t)}M$ for each $t \in I$. We denote this mapping by $P^t_{t_0,\gamma}$ and we call it the parallel translation from $T_{\gamma(t_0)}M$ to $T_{\gamma(t)}M$ along the curve γ.

We recall that a finite dimensional Riemannian manifold is complete if its geodesic are defined for any values of t. The Hopf-Rinow's theorem asserts that if the Riemannian manifold M is complete, then any pair of points in M can be joined by a minimal geodesic segment.

Recall that a real-valued function f defined on a Riemannian manifold M is said to satisfy a Lipschitz condition of rank K on a given subset S of M if $|f(x) - f(y)| \leq Kd(x,y)$ for every $x, y \in S$, where d is the Riemannian distance on M. A function f is said to be Lipschitz near $x \in M$ if it satisfies the Lipschitz condition of some rank on an open neighborhood of x. A function f is said to be locally Lipschitz on M if f is Lipschitz near x for every $x \in M$.

Throughout this article, unless stated otherwise, we always suppose that M is a Riemannian manifold and $f : M \to R$ is a given function.

Definition 2.1 ([36])**.** Let f be a locally Lipschitz on M. The generalized directional derivative $f^\circ(y;v)$ of f at $x \in M$ in the direction $v \in T_xM$, denoted by $f^\circ(x;v)$, is defined as

$$f^\circ(x;v) = \limsup_{y\to x, t\downarrow 0} \frac{f\circ\varphi^{-1}(\varphi(y) + td\varphi(x)(v)) - f\circ\varphi^{-1}(\varphi(y))}{t},$$

where (φ, U) is a chart at x.

Definition 2.2 ([36])**.** Let f be a locally Lipschitz on M. The generalized gradient (or Clarke subdifferential) of f at $y \in M$ is the subset $\partial_c f(y)$ of T_yM^* defined by

$$\partial_c f(y) = \{\zeta \in T_yM^* | f^\circ(y;v) \geq \langle \zeta, v\rangle,\ \forall v \in T_yM\},$$

It is worth mentioning that the Clarke subdifferential set $\partial_c f(y)$ is a nonempty subset of T_yM^* (see [14]).

Lemma 2.3 ([14, Lebourg's Mean Value Theorem])**.** *Let M be a finite dimensional Riemannian manifold, $x, y \in M$ and $\gamma : [0,1] \to M$ be a smooth path joining x and y. Let f be a Lipschitz function around $\gamma[0,1]$. Then there exist $0 < t_0 < 1$ and $\xi \in \partial f(\gamma(t_0))$ such that*

$$f(y) - f(x) = \langle \xi, \gamma'(t_0)\rangle.$$

Definition 2.4 ([29])**.** Let M be a Riemannian manifold and $\gamma : [0,1] \to M$ be a smooth curve on M such that $\gamma(0) = y$ and $\gamma(1) = x$. Then γ is said to possess the property (P) with respect to $x, y \in M$ if

$$\gamma'(s)(t-s) = \eta(\gamma(t), \gamma(s)), \quad \forall s, t \in [0,1],$$

where $\eta : M \times M \to TM$ is a function satisfying $\eta(x,y) \in T_yM$ for every $x, y \in M$.

Definition 2.5. Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a mapping such that $\eta(x,y) \in T_yM$ for every $x, y \in M$. Then η is said to be integrable if, for any $x, y \in M$, there exists a geodesic γ possessing the property (P) with respect to $x, y \in M$.

Remark 2.6 ([6])**.** Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be integrable. Then,

$$\eta(x,y) = \eta(\gamma(1), \gamma(0)) = \gamma'(0).$$

Moreover, one has

$$\eta(y, \gamma(s)) = -s\gamma'(s) = -sP^s_{0,\gamma}[\gamma'(0)] = -sP^s_{0,\gamma}[\eta(x,y)]$$

and

$$\eta(\gamma(1), \gamma(s)) = (1-s)\gamma'(s) = (1-s)P^s_{0,\gamma}[\eta(x,y)].$$

Now we present the following definitions.

Definition 2.7. Let f be locally Lipschitz on M and $\eta : M \times M \to TM$ be a mapping such that $\eta(x,y) \in T_yM$ for any $x, y \in M$. Then f is said to be

(i) invex (IX) with respect to η on M if, for any $x, y \in K$ and $\xi \in \partial_c f(y)$,

$$f(x) - f(y) \geq \langle \xi, \eta(x,y) \rangle_y. \tag{2.1}$$

(ii) strictly invex (SIX) on M w.r.t. η if inequality (2.1) is strict for all $x, y \in M$ with $x \neq y$;

(iii) strongly invex (SGIX) w.r.t. η on M if there exists a constant $\alpha > 0$ such that, for any $x, y \in M$ and $\xi \in \partial_c f(y)$,

$$f(x) - f(y) \geq \langle \xi, \eta(x,y) \rangle_y + \alpha \|\eta(x,y)\|^2_y.$$

(iv) pseudoinvex (PIX) w.r.t. η on M if, for any $x, y \in M$ and $\xi \in \partial_c f(y)$,

$$\langle \xi, \eta(x,y) \rangle_y \geq 0 \Rightarrow f(x) \geq f(y).$$

(v) strictly pseudoinvex (SPIX) w.r.t. η on M if, for all $x, y \in M$ with $x \neq y$ and $\xi \in \partial_c f(y)$,

$$\langle \xi, \eta(x,y) \rangle_y \geq 0 \Rightarrow f(x) > f(y).$$

(vi) strongly pseudoinvex (SGPIX) w.r.t. η on M if there exists a constant $\alpha > 0$ such that, for any $x, y \in M$ and $\xi \in \partial_c f(y)$,

$$\langle \xi, \eta(x,y) \rangle_y \geq 0 \Rightarrow f(x) \geq f(y) + \alpha \|\eta(x,y)\|^2_y.$$

Remark 2.8. (i) The notions of various types of invexity and pseudoinvexity for differentiable functions on Riemannian manifolds were introduced by Barani et al. [6].

(ii) The concepts given in Definition 2.7 are natural extension of those from Euclidean spaces to Riemannian manifolds.

Remark 2.9. By Definition 2.7, it is clear that SIX $\Rightarrow$ SPIX, and

$$\begin{array}{ccc} (SGIX) & \Longrightarrow & (IX) \\ \Downarrow & & \Downarrow \\ (SGPIX) & \Longrightarrow & (PIX) \end{array}$$

Definition 2.10. Let $A : M \to 2^{TM}$ be a set-valued vector field such that $A(x) \subseteq T_xM$ for all $x \in M$. Then A is said to be

(i) invariant monotone (IM) on M w.r.t. η if

$$\langle u, \eta(x, y)\rangle_y + \langle v, \eta(y, x)\rangle_x \leq 0, \quad \forall u \in A(y), \forall v \in A(x). \tag{2.2}$$

(ii) strictly invariant monotone (SIM) w.r.t. η on M if inequality (2.2) is strict for all $x, y \in M$ with $x \neq y$.

(iii) strongly invariant monotone (SGIM) w.r.t η on M if there exists a constant $\alpha > 0$ such that

$$\langle u, \eta(x, y)\rangle_y + \langle v, \eta(y, x)\rangle_x \leq -\alpha(\|\eta(x, y)\|_y^2 + \|\eta(y, x)\|_x^2), \quad \forall x, y \in M.$$

(iv) invariant pseudomonotone (IPM) on M w.r.t. η if, for any $x, y \in M$ and $v \in A(x)$,

$$\langle v, \eta(y, x)\rangle_x \geq 0 \Rightarrow \langle u, \eta(x, y)\rangle_y \leq 0, \quad \forall u \in A(y).$$

(v) strictly invariant pseudomonotone on (SIPM) M w.r.t. η if, for any $x, y \in M$ with $x \neq y$ and any $v \in A(x)$,

$$\langle v, \eta(y, x)\rangle_x \geq 0 \Rightarrow \langle u, \eta(x, y)\rangle_y < 0, \quad \forall u \in A(y).$$

(vi) strongly invariant pseudomonotone (SGIPM) on M w.r.t. η if, for any $x, y \in M$ and $v \in A(x)$,

$$\langle v, \eta(y, x)\rangle_x \geq 0 \Rightarrow \langle u, \eta(x, y)\rangle_y \leq -\alpha\|\eta(x, y)\|_y^2, \quad \forall u \in A(y).$$

Remark 2.11. (i) Barani et al. [6] presented the notions of several sorts of invariant monotonicity and invariant pseudomonotonicity for all univalued vector fields on Riemannian manifolds.

(ii) The concepts given in Definition 2.10 also extend the corresponding ones in Euclidean spaces.

Remark 2.12. By Definition 2.10, we have the following implications:

$$\begin{array}{ccc} (SGIM) & \Longrightarrow & (IM) \\ \Downarrow & & \Downarrow \\ (SGPIM) & \Longrightarrow & (IPM) \end{array}$$

3. Main Results

In this section, we establish some connections between the generalized invexity for locally Lipschitz functions and the generalized invariant monotonicity for set-valued vector fields on Riemannian manifolds.

Theorem 3.1. *Let M be a Riemannian manifold and f locally Lipschitz on M. If f is (strongly, strictly) invex w.r.t. η on M, then $\partial_c f$ is (strongly, strictly) invariant monotone w.r.t. η on M.*

Proof. We prove only the assertion strongly and with $\alpha = 0$ and by replacing $\leq$ and $\geq$ by $<$ and $>$, the other cases can be proved similarly. Suppose that f is strongly invex w.r.t. η on M with a constant $\alpha > 0$. For any given $x, y \in M$, it follows from the strong invexity of f that

$$f(x) - f(y) \geq \langle \xi, \eta(x,y)\rangle_y + \alpha \|\eta(x,y)\|_y^2, \quad \forall \xi \in \partial_c f(y)$$

and

$$f(y) - f(x) \geq \langle \gamma, \eta(y,x)\rangle_x + \alpha \|\eta(y,x)\|_x^2, \quad \forall \gamma \in \partial_c f(x).$$

By adding the above two inequalities, we have

$$\langle \xi, \eta(x,y)\rangle_y + \langle \gamma, \eta(y,x)\rangle_x \leq -\alpha(\|\eta(y,x)\|_x^2 + \|\eta(x,y)\|_y^2).$$

Thus the conclusion follows. □

Theorem 3.2. *Let M be a finite dimensional Riemannian manifold and f locally Lipschitz on M. Suppose that $\eta : M \times M \to TM$ is integrable. If $\partial_c f$ is (strongly, strictly) invariant monotone w.r.t. η on M, then f is (strongly, strictly) invex w.r.t. η on M.*

Proof. We prove only the assertion strongly and with $\alpha = 0$ and by replacing $\geq$ and $\leq$ by $>$ and $<$, the other cases can be proved similarly. Let $\partial_c f$ be strongly invariant monotone w.r.t. η on M with constant $\alpha > 0$. For any given $x, y \in M$, since η is integrable, there exists a geodesic $\gamma : [0,1] \to M$ possessing property (P) such that $\gamma(0) = y$ and $\gamma(1) = x$. Now define a geodesic $\beta : [0,1] \to M$ as

$$\beta(s) = \gamma(s + (1-s)t), \quad \forall s \in [0,1].$$

It follows from Lemma 2.3 that there exist $l \in (t,1)$ and $\xi \in \partial_c f(\beta(l))$ such that

$$f(x) - f(\gamma(t)) = \langle \xi, \beta'(l)\rangle_{\beta(l)} = (1-t)\langle \xi, \gamma'(a)\rangle_{z_1}, \tag{3.1}$$

where $a = l + (1-l)t > t$ and $z_1 = \beta(l) = \gamma(a)$.

Similarly, if we consider the geodesic $\theta : [0,1] \to M$ defined by

$$\theta(s) = \gamma(st), \quad \forall s \in [0,1],$$

then Lemma 2.3 implies that there exist $h \in (0,t)$ and $\zeta \in \partial_c f(\theta(h))$ such that

$$f(\gamma(t)) - f(y) = \langle \zeta, \theta'(h)\rangle_{\theta(h)} = t\langle \zeta, \gamma'(b)\rangle_{z_2}, \tag{3.2}$$

where $b = ht < t$ and $z_2 = \gamma(b) = \theta(h)$.

Since $\partial_c f$ is strongly invariant monotone, we know that, for $\xi \in \partial_c f(z_1)$ and any $\vartheta \in \partial_c f(y)$,

$$\langle \xi, \eta(y,z_1)\rangle_{z_1} + \langle \vartheta, \eta(z_1,y)\rangle_y \leq -\alpha(\|\eta(y,z_1)\|_{z_1}^2 + \|\eta(z_1,y)\|_y^2). \tag{3.3}$$

By property (P), Remark 2.6 and the parallel translation, one has

$$\eta(y,z_1) = \eta(\gamma(0),\gamma(a)) = -a\gamma'(a) = -aP_{0,\gamma}^{a}[\eta(x,y)] \tag{3.4}$$

and

$$\eta(z_1,y) = a\eta(x,y). \tag{3.5}$$

It follows from (3.3)-(3.5) that

$$\langle \xi, -P^a_{0,\gamma}[\eta(x,y)]\rangle_{z_1} + \langle \vartheta, \eta(x,y)\rangle_y \leq -\alpha a(\|P^a_{0,\gamma}[\eta(x,y)]\|^2_{z_1} + \|\eta(x,y)\|^2_y)$$

and so

$$-\langle \xi, P^a_{0,\gamma}[\eta(x,y)]\rangle_{z_1} + \langle \vartheta, \eta(x,y)\rangle_y \leq -2a\alpha\|\eta(x,y)\|^2_y.$$

Since $P^a_{0,\gamma}[\eta(x,y)] = \gamma'(a)$, we get

$$\langle \xi, \gamma'(a)\rangle_{z_1} \geq \langle \vartheta, \eta(x,y)\rangle_y + 2a\alpha\|\eta(x,y)\|^2_y. \tag{3.6}$$

Similarly, we can show that

$$\langle \zeta, \gamma'(b)\rangle_{z_2} \geq \langle \vartheta, \eta(x,y)\rangle_y + 2b\alpha\|\eta(x,y)\|^2_y. \tag{3.7}$$

It follows from (3.1), (3.2), (3.6) and (3.7) that

$$f(x) - f(\gamma(t)) \geq (1-t)\langle \vartheta, \eta(x,y)\rangle_y + 2(1-t)a\alpha\|\eta(x,y)\|^2_y$$

and

$$f(\gamma(t)) - f(y) \geq t\langle \vartheta, \eta(x,y)\rangle_y + 2tb\alpha\|\eta(x,y)\|^2_y.$$

By adding the above two inequalities, for any $\vartheta \in \partial_c f(y)$,

$$\begin{aligned} f(x) - f(y) &\geq \langle \vartheta, \eta(x,y)\rangle_y + 2\alpha[(1-t)a + tb]\|\eta(x,y)\|^2_y \\ &\geq \langle \vartheta, \eta(x,y)\rangle_y + 2\alpha b\|\eta(x,y)\|^2_y \\ &= \langle \vartheta, \eta(x,y)\rangle_y + 2\alpha ht\|\eta(x,y)\|^2_y \\ &> \langle \vartheta, \eta(x,y)\rangle_y + 2\alpha h^2\|\eta(x,y)\|^2_y. \end{aligned}$$

Thus, f is strongly invex. This completes the proof. □

Theorem 3.3. *Let M be a finite dimensional Riemannian manifold and f locally Lipschitz w.r.t. η on M. Suppose that η is integrable. Then $\partial_c f$ is strictly invariant pseudomonotone w.r.t. η on M if and only if f is strictly pseudoinvex w.r.t. η.*

Proof. Suppose that f is strictly pseudoinvex w.r.t. η on M. Let $x, y \in M$ with $x \neq y$ and $\xi \in \partial_c f(y)$ such that

$$\langle \xi, \eta(x,y)\rangle_y \geq 0.$$

Then the strict pseudoinvexity of f implies that

$$f(x) > f(y). \tag{3.8}$$

We need to show that $\langle \zeta, \eta(y,x)\rangle_x < 0$ for all $\zeta \in \partial_c f(x)$. Assume that $\langle \zeta_0, \eta(y,x)\rangle_x \geq 0$ for some $\zeta_0 \in \partial_c f(x)$. Then the strict pseudoinvexity of f yields that $f(y) > f(x)$, which contradicts (3.8).

Conversely, suppose that $\partial_c f$ is strictly invariant pseudomonotone w.r.t. η on M. Let $x, y \in M$ with $x \neq y$ and $\xi \in \partial_c f(y)$ such that

$$\langle \xi, \eta(x,y)\rangle_y \geq 0. \tag{3.9}$$

Since η is integrable, there exists a geodesic $\gamma : [0,1] \to M$ possessing property (P) such that $\gamma(0) = y$ and $\gamma(1) = x$. We need to show that $f(x) > f(y)$. Assume that $f(x) \leq f(y)$. Then it follows from Lemma 2.3 that there exists $t_0 \in (0,1)$ and $\zeta \in \partial_c f(\gamma(t_0))$ such that

$$f(x) - f(y) = \langle \zeta, \gamma'(t_0)\rangle_{\gamma(t_0)} \leq 0.$$

This together with property (P) implies that

$$\langle \zeta, \gamma'(t_0) \rangle_{\gamma(t_0)} = -\frac{1}{t_0} \langle \zeta, \eta(y, \gamma(t_0)\rangle_{\gamma(t_0)} \leq 0$$

and so

$$\langle \zeta, \eta(y, \gamma(t_0)) \rangle_{\gamma(t_0)} \geq 0, \quad \zeta \in \partial_c f(\gamma(t_0)). \tag{3.10}$$

Since $\partial_c f$ is strictly invariant pseudomonotone, from (3.10), one has

$$\langle \xi, \eta(\gamma(t_0), y) \rangle_y < 0, \quad \forall\, \xi \in \partial_c f(y).$$

By property (P), we conclude that

$$\langle \xi, t_0 \gamma'(0) \rangle_y = t_0 \langle \xi, \eta(x, y) \rangle_y < 0, \quad \forall\, \xi \in \partial_c f(y),$$

which contradicts (3.9). This completes the proof. □

Theorem 3.4. *Let M be a finite dimensional Riemannian manifold and f locally Lipschitz on M. Suppose that η is integrable. If $\partial_c f$ is strongly invariant pseudomonotone w.r.t. η on M, then f is strongly pseudoinvex w.r.t. η on M.*

Proof. Let $x, y \in M$ and $\xi \in \partial_c f(y)$ such that

$$\langle \xi, \eta(x, y) \rangle_y \geq 0. \tag{3.11}$$

Then exists a geodesic $\gamma : [0, 1] \to M$ possessing property (P) such that $\gamma(0) = y$, $\gamma(1) = x$. Let $\beta : [0, 1] \to M$ be defined as

$$\beta(s) = \gamma(st + (1 - s)t), \quad \forall s \in [0, 1].$$

Then Lemma 2.3 shows that there exist $l \in (t, 1)$ and $\zeta_1 \in \partial_c f(\beta(l))$ such that

$$f(x) - f(\gamma(t)) = \langle \zeta_1, \beta'(l) \rangle_{\beta(l)} = (1 - t) \langle \zeta_1, \gamma'(a) \rangle_{z_1}, \tag{3.12}$$

where $a = l + (1 - l)t > t$ and $z_1 = \beta(l) = \gamma(a)$. Moreover, let $\theta : [0, 1] \to M$ be defined as

$$\theta(s) = \gamma(st), \quad \forall s \in [0, 1].$$

Then it follows from Lemma 2.3 that there exist $h \in (0, t)$ and $\zeta_2 \in \partial_c f(\theta(h))$ such that

$$f(\gamma(t)) - f(y) = \langle \zeta_2, \theta'(h) \rangle_{\theta(h)} = t \langle \zeta_2, \gamma'(b) \rangle_{z_2}, \tag{3.13}$$

where $b = ht < t$ and $z_2 = \gamma(ht) = \theta(h)$.

Now from (3.12), (3.13) and the property (P), one has

$$f(x) - f(\gamma(t)) = -\frac{1 - t}{a} \langle \zeta_1, \eta(y, z_1) \rangle_{z_1}, \quad \zeta_1 \in \partial_c f(z_1) \tag{3.14}$$

and

$$f(\gamma(t)) - f(y) = -\frac{t}{b} \langle \zeta_2, \eta(y, z_2) \rangle_{z_2}, \quad \zeta_2 \in \partial_c f(z_2). \tag{3.15}$$

Again, the property (P) together with (3.11) yields that

$$0 \leq \langle \xi, \eta(x, y) \rangle_y = \frac{1}{a} \langle \xi, \eta(z_1, y) \rangle_y = \frac{1}{b} \langle \xi, \eta(z_2, y) \rangle_y. \tag{3.16}$$

Since $\partial_c f$ is strongly invariant monotone on M w.r.t. η, it follows from (3.16) and property (P) that

$$\begin{aligned} \langle \zeta_1, \eta(y, z_1) \rangle_{z_1} &\leq -\alpha \|\eta(y, z_1)\|_{z_1}^2 \\ &= -\alpha \left\| -a P_{0,\gamma}^a [\eta(x, y)] \right\|_{z_1}^2 \\ &= -\alpha a^2 \|\eta(x, y)\|_y^2 \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \langle \zeta_2, \eta(y, z_2) \rangle_{z_2} &\leq -\alpha \|\eta(y, z_2)\|_{z_2}^2 \\ &= -\alpha \left\| -b P_{0,\gamma}^b [\eta(x, y)] \right\|_{z_2}^2 \\ &= -\alpha b^2 \|\eta(x, y)\|_y^2. \end{aligned} \tag{3.18}$$

By (3.14), (3.15), (3.17) and (3.18), one has

$$\begin{aligned} f(x) - f(y) &\geq [(1-t)\alpha a + t\alpha b] \|\eta(x, y)\|_y^2 \\ &> \alpha b \|\eta(x, y)\|_y^2 \\ &= \alpha h t \|\eta(x, y)\|_y^2 \\ &> \alpha h^2 \|\eta(x, y)\|_y^2. \end{aligned}$$

This completes proof. □

Theorem 3.5. *Let M be a finite dimensional Riemannian manifold and f locally Lipschitz on M. Suppose that η is integrable. If $\partial_c f$ is invariant pseudomonotone, then for any $x, y \in M$,*

$$f(x) \leq f(y) \Longrightarrow f(\gamma(t)) \leq f(y), \quad \forall t \in [0, 1] \tag{3.19}$$

and

$$f(x) < f(y) \Longrightarrow f(\gamma(t)) < f(y), \quad \forall t \in [0, 1], \tag{3.20}$$

where $\gamma : [0, 1] \to M$ is a smooth path joining y and x.

Proof. For any given $x, y \in M$, since η is integrable, there exists a geodesic $\gamma : [0, 1] \to M$ possessing property (P) such that $\gamma(0) = y$ and $\gamma(1) = x$. If (3.19) does not hold, then there exists $\bar{t} \in (0, 1)$ such that

$$f(\gamma(\bar{t})) > f(y) \geq f(x). \tag{3.21}$$

Define $\beta : [0, 1] \to M$ as

$$\beta(s) = \gamma(s\bar{t}), \quad \forall s \in [0, 1].$$

Then by Lemma 2.3, there exist $l \in (0, \bar{t})$ and $\xi \in \partial_c f(\beta(l))$ such that

$$f(\gamma(\bar{t})) - f(y) = \bar{t} \langle \xi, \gamma'(a) \rangle_{z_1}, \tag{3.22}$$

where $a = l\bar{t} < \bar{t}$ and $z_1 = \gamma(a) = \beta(l)$. Similarly, define $\theta : [0, 1] \to M$ as

$$\theta(s) = \gamma((1-s)\bar{t} + s), \quad \forall s \in [0, 1].$$

Then it follows from Lemma 2.3 that there exist $h \in (\bar{t}, 1)$ and $\zeta \in \partial_c f(\theta(h))$ such that

$$f(x) - f(\gamma(\bar{t}) = (1 - \bar{t}) \langle \zeta, \gamma'(b) \rangle_{z_2}, \tag{3.23}$$

where $b = (1-h)\bar{t} + h > \bar{t}$ and $z_2 = \gamma(b) = \theta(h)$.

On the other hand, property (P) implies that

$$\langle \xi, \eta(\gamma(b), \gamma(a)) \rangle_{\gamma(a)} = (b-a)\langle \xi, \gamma'(a) \rangle_{\gamma(a)}$$

and

$$\langle \zeta, \eta(\gamma(a), \gamma(b)) \rangle_{\gamma(b)} = (a-b)\langle \zeta, \gamma'(b) \rangle_{\gamma(b)}.$$

This together with (3.21)-(3.23) yields

$$\langle \xi, \eta(\gamma(b), \gamma(a)) \rangle_{\gamma(a)} > 0, \quad \xi \in \partial_c f(\gamma(a))$$

and

$$\langle \zeta, \eta(\gamma(a), \gamma(b)) \rangle_{\gamma(b)} > 0, \quad \zeta \in \partial_c f(\gamma(b)),$$

which contradicts the invariant pseudomonotonicity of $\partial_c f$ w.r.t. η on M, and so (3.19) holds. Similarly, we can show that (3.20) holds. This completes the proof. □

Remark 3.6. (i) Theorems 3.1-3.4 generalize and improve Theorems 4.1, 4.2, 5.1 and 5.2 of Barani [6] from smooth cases to nonsmooth ones.
(ii) Theorems 3.1 and 3.2 also generalize Theorem 4.3 in [5], in which the convexity of f was replaced by invexity of f.

4. Applications to the vector variational-like inequality and vector optimization

In this section, we give some relationships between vector variational-like inequalites involving Clarke subdifferential and nonsmooth vector optimization on Riemannian manifolds.

Let M be a Riemannian manifold, $f : M \to R^p$ be a vector-valued function, and $\eta : M \times M \to TM$ be a given mapping. We consider the following vector optimization problem:

$$\text{(VOP)} \quad \min \quad f(x) = (f_1(x), \cdots, f_p(x)) \\ \text{s.t. } x \in M.$$

Definition 4.1. [2] A point $\bar{x} \in M$ is said to be an efficient (or Pareto) solution (respectively, weak efficient solution) of (VOP) if

$$f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \ldots, f_p(y) - f_p(\bar{x})) \notin -R^p_+ \backslash \{0\}, \quad \forall y \in M$$

(respectively, $f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \ldots, f_p(y) - f_p(\bar{x})) \notin -intR^p_+$, $\forall y \in M$), where R^p_+ is the nonnegative orthant of R^p.

It is clear that every efficient solution is a weakly efficient solution. For further details on vector optimization theory, we refer to [2, 17, 23, 32] and the references therein.

We now consider the following two types of generalized VVLI problems:

(I) generalized Minty vector variational-like inequality problem (GMVVLIP): find $x \in M$ such that for any $y \in M$, there exist $\xi_i \in \partial_c f_i(y)$ with $i \in J = \{1, \ldots, p\}$ satisfying

$$(\langle \xi_1, \eta(x,y) \rangle_y, \ldots, \langle \xi_p, \eta(x,y) \rangle_y) \notin R^p_+ \backslash \{0\}.$$

(II) generalized Stampacchia vector variational-like inequality problem (GSVVLIP): find $x \in M$ and $\xi_i \in \partial_c f_i(x)$ with $i \in J = \{1, \ldots, p\}$ such that, for all $y \in M$,

$$(\langle \xi_1, \eta(y,x)\rangle_x, \ldots, \langle \xi_p, \eta(y,x)\rangle_x) \notin -R^p_+ \backslash \{0\}.$$

Theorem 4.2. *Let M be a finite dimensional Riemannian manifold. For each $i \in J$, let f_i be locally Lipschitz on M. Suppose that η is integrable and $\partial_c f_i$ is invariant pseudomonotone w.r.t. η on M for all $i \in J$. If $x \in M$ is a solution of (GMVVLIP), then it is an efficient solution for (VOP). Moreover, if $\partial_c f_i$ is strictly invariant pseudomonotone with respect to η on M for all $i \in J$, and x is a weakly efficient solution of (VOP), then it is also a solution of (GMVVLIP).*

Proof. We show taht x is an efficient solution of (VOP). Suppose not. Then there exists $y \in M$ such that

$$f(y) - f(x) = (f_1(y) - f_1(x), \ldots, f_p(y) - f_p(x)) \in -R^p_+ \setminus \{0\},$$

that is,

$$f_i(x) \geq f_i(y), \quad \forall i \in J$$

with strict inequality holding for some $k \in J$. Since η is integrable, there exists a geodesic γ possessing property (P) such that $\gamma(0) = x$ and $\gamma(1) = y$. It follows from Theorem 3.5 that

$$f_i(\gamma(t)) \leq f_i(x), \quad \forall t \in [0,1] \tag{4.1}$$

with strict inequality holding for some $k \in J$. For any given $t \in [0,1]$, define $\beta : [0,1] \to M$ as

$$\beta(s) = \gamma(st), \quad \forall\, s \in [0,1].$$

Then Lemma 2.3 shows that there exist $l_i \in (0,t)$ and $\xi_i \in \partial_c f_i(\beta(l_i))$ such that

$$f_i(\gamma(t)) - f_i(x) = \langle \xi_i, \beta'(l_i)\rangle_{\beta(l_i)} = t\langle \xi_i, \gamma'(a_i)\rangle_{z_i}, \tag{4.2}$$

where $a_i = l_i t < t$ and $z_i = \gamma(a_i)$. It follows from (4.1) and (4.2) that

$$\langle \xi_i, \gamma'(a_i)\rangle_{z_i} \leq 0, \quad i \in J,\ \xi_i \in \partial_c f_i(z_i)) \tag{4.3}$$

with strict inequality holding for some $k \in J$. Choosing $t_0 \in (0,1)$ such that $t_0 < a_i$ for all $i \in J$. By property (P), one has

$$\eta(\gamma(t_0), \gamma(a_i)) = (t_0 - a_i)\gamma'(a_i) \tag{4.4}$$

and

$$\eta(\gamma(a_i), \gamma(t_0)) = (a_i - t_0)\gamma'(t_0) \tag{4.5}$$

By (4.3) and (4.4), for all $i \in J$, we deduce that

$$\langle \xi_i, \eta(\gamma(t_0), \gamma(a_i))\rangle_{z_i} \geq 0, \quad \xi_i \in \partial_c f_i(z_i),\ i \in J\backslash\{k\}$$

and

$$\langle \xi_k, \eta(\gamma(t_0), \gamma(a_k))\rangle_{z_k} > 0 \tag{4.6}$$

with some $\xi_k \in \partial_c f_k(\gamma(a_k))$. Since each $\partial_c f_i\,(i \in J)$ is invaraint pseudomonotone with respect to η, we have

$$\langle \zeta_i, \eta(\gamma(a_i), \gamma(t_0))\rangle_{\gamma(t_0)} \leq 0, \quad \forall \zeta_i \in \partial_c f_i(\gamma(t_0)),\ i \in J\backslash\{k\} \tag{4.7}$$

and

$$\langle \zeta_k, \eta(\gamma(a_k), \gamma(t_0))\rangle_{\gamma(t_0)} < 0, \quad \forall \zeta_k \in \partial_c f_k((\gamma(t_0)). \tag{4.8}$$

If (4.8) does not hold, then there exists $\zeta_{k_0} \in \partial_c f_k((\gamma(t_0))$ such that

$$\langle \zeta_{k_0}, \eta(\gamma(a_k), \gamma(t_0))\rangle_{\gamma(t_0)} \geq 0.$$

By using the invariant pseudomonotonicity of $\partial_c f_k$ ($k \in J$) again, we get

$$\langle \xi_k, \eta(\gamma(t_0), \gamma(a_k))\rangle_{z_k} \leq 0, \quad \forall \xi_k \in \partial_c f_k((\gamma(a_i)),$$

which contradicts (4.6). This shows that (4.8) is true. Noting that

$$\eta(x, \gamma(t_0)) = (-t_0)\gamma'(t_0),$$

it follows from (4.5), (4.7) and (4.8) that

$$\langle \xi_i, \eta(x, \gamma(t_0))\rangle_{\gamma(t_0)} \geq 0, \quad \forall\, \xi_i \in \partial_c f_i(\gamma(t_0)),\ i \in J\backslash\{k\}$$

with strict inequality holding for some $k \in J$. That is, for all $\xi_i \in \partial_c f_i(\gamma(t_0))$, $i \in J$,

$$(\langle \xi_1, \eta(x, \gamma(t_0))\rangle_{\gamma(t_0)}, \ldots, \langle \xi_p, \eta(x, \gamma(t_0))\rangle_{\gamma(t_0)}) \in R^p_+\backslash\{0\}.$$

This contradicts the fact that x is a solution of (GMVVLIP).

Moreover, assume that x is a weakly efficient solution of (VOP). We show that x is also a solution of (GMVVLIP). Suppose not. Then there exists $y \in K$ such that, for any $\xi_i \in \partial_c f_i(y)$ with $i \in J$,

$$\langle \xi_i, \eta(x, y)\rangle_y \geq 0$$

with strict inequality holding for some $k \in J$. This shows that $x \neq y$. Since $\partial_c f_i$ is strictly invariant pseudomonotone, from Theorem 3.3, we know that, f_i is strictly pseudoinvex for all $i \in J$ and so

$$f_i(x) > f_i(y), \quad \forall i \in J,$$

that is,

$$f(y) - f(x) = (f_1(y) - f_1(x), \ldots, f_p(y) - f_p(x)) \in -intR^p_+,$$

which contradicts the fact that x is a weakly efficient solution of (VOP). This completes the proof. □

Since strictly invariant pseudomonotone implies invariant pseudomonotone and an efficient solution for (VOP) is a weakly efficient solution, from Theorems 3.3 and 4.2, it is easy to have the following result.

Theorem 4.3. *Let M be a finite dimensional Riemannian manifold. For each $i \in J$, assume that f_i is locally Lipschitz on M. Suppose that η is integrable and $\partial_c f_i$ is strictly invariant pseudomonotone (or f_i is strictly pseudoinvex) w.r.t. η on M for each $i \in J$. Then $x \in M$ is a solution of (GMVVLIP) if and only if it is a weakly efficient solution for (VOP).*

Theorem 4.4. *Let M be a Riemannian manifold. For each $i \in J$, assume that f_i is locally Lipschitz on M. Suppose that $\partial_c f_i$ is invariant pseudomonotone w.r.t. η on M for each $i \in J$. If $x \in M$ is a solution of (GSVVLIP), then it is a solution of (GMVVLIP).*

Proof. Let $x \in M$ be a solution of (GSVVLIP). We show that x is also a solution of (GMVVLIP). Suppose not. Then there exists $y \in M$ such that, for all $\xi_i \in \partial_c f_i(y)$ with $i \in J$,

$$(\langle \xi_1, \eta(x,y)\rangle_y, \ldots, \langle \xi_p, \eta(x,y)\rangle_y) \in R^p_+ \backslash \{0\},$$

or equivalently,

$$\langle \xi_i, \eta(x,y)\rangle_y \geq 0, \quad i \in J$$

with strict inequality holds for some $k \in J$. Since $\partial_c f_i$ is invariant pseudomonotone w.r.t. η on M for each $i \in J$, we have

$$\langle \zeta_i, \eta(y,x)\rangle_x \leq 0, \quad \forall\, \zeta_i \in \partial_c f_i(x),\, i \in J$$

with strict inequality holds for some $k \in J$. It follows that

$$(\langle \zeta_1, \eta(y,x)\rangle_x, \ldots, \langle \zeta_p, \eta(y,x)\rangle_x) \in -R^p_+ \backslash \{0\},$$

which contradicts the fact that x is a solution of (GSVVLIP). This completes the proof. □

Next we consider the following weak forms of (GSVVLIP) and (GMVVLIP).

(I) generalized weak Minty vector variational-like inequality problem (GWMVVLIP): find $x \in M$ such that, for any $y \in M$, there exist $\xi_i \in \partial_c f_i(y)$ with $i \in J$ satisfying

$$(\langle \xi_1, \eta(x,y)\rangle_y, \ldots, \langle \xi_p, \eta(x,y)\rangle_y) \notin intR^p_+.$$

(II) generalized weak Stampacchia vector variational-like inequality problem (GWSVVLIP): find $x \in M$ and $\xi_i \in \partial_c f_i(x)$ with $i \in J$ such that, for all $y \in M$,

$$(\langle \xi_1, \eta(y,x)\rangle_x, \ldots, \langle \xi_p, \eta(y,x)\rangle_x) \notin -intR^p_+.$$

Theorem 4.5. *Let M be a Riemannian manifold. For each $i \in J$, assume that $\partial_c f_i$ is strictly invariant pseudomonotone w.r.t. η on M. If $x \in M$ is a solution of (GWSVVLIP), then it is a solution of (GWMVVLIP).*

Proof. Let $x \in M$ be a solution of (GWSVVLIP). If x is not a solution of (GWMVVLIP), then there exists $y \in M$ such that, for any $\xi_i \in \partial_c f_i(y)$ with $i \in J$,

$$(\langle \xi_1, \eta(x,y)\rangle_y, \ldots, \langle \xi_p, \eta(x,y)\rangle_y) \in intR^p_+,$$

or equivalently,

$$\langle \xi_i, \eta(x,y)\rangle_y > 0, \quad \forall i \in J.$$

Since $\partial_c f_i$ is strictly invariant pseudomonotone w.r.t. η on M, one has

$$\langle \zeta_i, \eta(y,x)\rangle_x < 0, \quad \forall \zeta_i \in \partial_c f_i(x),\, i \in J,$$

which contradicts the fact that x is a solution of (GWSVVLIP). This completes the proof. □

Theorem 4.6. *Let M be a Riemannian manifold. For each $i \in J$, assume that f_i is locally Lipschitz and pseudoinvex w.r.t. η on M. If $x \in M$ is a solution of (GWSVVLIP), then it is a weakly solution of (VOP).*

Proof. Suppose that $x \in M$ is a solution of (GWSVVLIP). We show that x is a weakly efficient solution of (VOP). Suppose not. Then there exists $y \in M$ such that

$$(f_1(y) - f_1(x), \ldots, f_p(y) - f_p(x)) \in -intR^p_+,$$

or equivalently,

$$f_i(y) < f_i(x), \quad \forall i \in J. \tag{4.9}$$

Now the pseudoinvexity of f_i with $i \in J$ implies that

$$\langle \xi_i, \eta(y,x)\rangle_x < 0, \quad \forall \xi_i \in \partial_c f_i(x). \tag{4.10}$$

In fact, if there exist $i_0 \in J$ and $\xi_{i_0} \in \partial_c f_{i_0}(x)$ such that $\langle \xi_{i_0}, \eta(y,x)\rangle_x \geq 0$, then the pseudoinvexity of f_{i_0} implies that

$$f_{i_0}(y) \geq f_{i_0}(x),$$

which is a contradiction with (4.9). Thus, we know that (4.10) is true and so

$$(\langle \xi_1, \eta(y,x)\rangle_x, \ldots, \langle \xi_p, \eta(y,x)\rangle_x) \in -intR^p_+,$$

which contradicts the fact that x is a solution of (GWSVVLIP). This completes the proof. □

Theorem 4.7. *Let M be a Riemannian manifold. For each $i \in J$, assume that f_i is locally Lipschitz and strictly pseudoinvex w.r.t. η on M. If $x \in M$ is a weakly solution of (VOP), then it is a solution of (GWMVVLIP).*

Proof. We show that $x \in M$ is a solution of (GWMVVLIP). Suppose not. Then there exists $y \in M$ such that, for any $\xi_i \in \partial_C f_i(y)$ with $i \in J$,

$$(\langle \xi_i, \eta(x,y)\rangle_y, \ldots, \langle \xi_p, \eta(x,y)\rangle_y) \in intR^p_+,$$

or equivalently,

$$\langle \xi_i, \eta(x,y)\rangle_y > 0, \quad \forall i \in J.$$

This shows that $x \neq y$. Since f_i is strictly pseudoinvex w.r.t. η on M, we know that $f_i(x) > f_i(y)$ and so

$$(f_1(y) - f_1(x), \ldots, f_p(y) - f_p(x)) \in -intR^p_+,$$

which contradicts that x is a weakly efficient solution of (VOP). This completes the proof. □

Acknowledgements

The authors are grateful to the editor and the referees for their valuable comments and suggestions.

References

[1] S. Al-Homidan and Q. H. Ansari, *Generalized Minty vector variational-like inqequalities and vector optimization problems*, J. Optim. Theory Appl. **144** (2010), 1–11.

[2] Q. H. Ansari and J. C. Yao (eds.), *Recent Developments in Vector Optimization*, Springer-Verlag, Berlin, New York, Heidelberg, 2012.

[3] Q. H. Ansari and M. Rezaei, *Generalized vector variational-like inequalities and vector optimization in Asplund spaces*, Optimization **62** (2013), 721–734.

[4] D. Azagra, J. Ferrera and F. Lopez-Mesas, *Nonsmooth analysis and Hamilton-Jacobi equations on Riemannian manifolds*, J. Funct. Anal. **220** (2005), 304–361.

[5] A. Barani, *Generalized monotonicity and convexity for locally Lipschitz functions on Hadamard manifolds*, *Differ. Geom. Dyn. Syst.*, **15** (2013), 26–37.

[6] A. Barani and M. R. Pouryayevali, *Invariant monotone vector fields on Riemannian manifolds*, Nonlinear Anal. TMA **70** (2009), 1850–1861.

[7] I. Chavel, *Riemannian Geometry-A Modern Introduction*, Cambridge University Press, 1993.

[8] S. L. Chen, N. J. Huang and D. O'Regan, *Geodesic B-preinvex functions and multiobjective optimization problems on Riemannian manifolds*, J. Appl. Math. **2014** (2014), Article ID 524698, 12 pages.

[9] F. H. Clarke, Yu. S. Ledyaev, R. G. Stern and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Grad. texts in Math. 178, Springer, 1998.

[10] J. X. Da Cruze Neto, O. P. Ferreira, L. R. Lucambio Pérez, *Monotone point-to-set vector fields*, Balkan J. Geom. Appl. **5** (2000), 69–79.

[11] R. J. Dwilewicz, *A short history of convexity*, Diff. Geom. Dyn. Syst. **11** (2009), 112–129.

[12] L. Fan, S. Liu and S. Gao, *Generalized monotonicity and convexity of nondifferentiable functions*, J. Math. Anal. Appl. **279** (2003), 276–289.

[13] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981), 545–550.

[14] S. Hosseini and M. R. Pouryayevali, *Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds*, Nonlinear Anal. TMA **74** (2011), 3884–3895.

[15] T. Jabarootian and J. Zafarani, *Generalized invariant monotonicity and invexity of nondifferentiable functions*, J. Global Optim. **36** (2006), 537–564.

[16] S. Karamadrian and S. Schaible, *Seven kinds of monotone maps*, J. Optim. Theory Appl. **66** (1990), 37–46.

[17] R. N. Kaul and S. Kaur, *Optimality criteria in nonlinear programming involving nonconvex functions*, J. Math. Anal. Appl. **105** (1985), 104–112.

[18] W. Klingenberg, *A Course in Differential Geometry*, Springer-verlag, New York- Heidelberg, 1978.

[19] Y. S. Ledyaev and Q. J. Zhu, *Nonsmooth Analysis on Smooth Manifolds*, Transactions American Mathematical Society, Providence, RI, 2007.

[20] C. Li, G. López and V. Martín-Márquez, Monotone vector fields and the proximal point algorithm on Hadamard manifolds, *J. London Math. Soc.* **79** (2009), 663–683.

[21] C. Li, B. S. Mordukhovich, J. H. Wang and J. C. Yao, *Weak sharp minima on Riemannian manifolds*, SIAM J. Optim. **21** (2011), 1523–1560.

[22] C. Li and J. C. Yao, *Variational inequalities for set-valued vector fields on Riemannian manifolds: convexity of the solution set and the proximal point algorithm*, SIAM J. Control Optim. **50** (2012) 2486–2514.

[23] D. Luc, *On generalized convex nonsmooth functions*, Bull. Austral. Math. Soc. **49** (1994), 139-149.

[24] S. K. Mishra and S. Y. Wang, *Vector variational-like inequalities and nonsmooth vector optimization problems*, Nonlinear Anal. **64** (2006), 1939–1945.

[25] S. Mititelu, *Generalized invexity and vector optimization on differential manifolds*, Differ. Geom. Dyn. Syst. **3** (2001), 21–31.

[26] S. R. Mohan and S. K. Neogy, *On invex sets and preinvex functions*, J. Math. Anal. Appl. **189** (1995), 901–908.

[27] S. Z. Németh, *Monotone vector fields*, Publicationes Mathematics **54** (1999), 437–449.

[28] J. P. Penot and P. H. Quang, *Generalized convexity of functions and generalized monotonicity of set-valued maps*, J. Optim. Theory Appl. **92** (1997), 343–356.
[29] R. Pini, *Convexity along curves and invexity*, Optimization **29** (1994), 301–309.
[30] R. Pini, *Invexity and generalized convexity*, Optimization **22** (1991), 513–525.
[31] R. Pini and C. Singh, *Generalized convexity and generalized monotonicity*, J. Inf. Optim. Sci. **20** (1999), 215–233.
[32] R. Pini and C. Singh, *A survey of recent (1985-1995) advances in generalized convexity with applications to duality theory and optimality conditions*, Optimization **39** (1997), 311–360.
[33] T. Rapscák, *Smooth Nonlinear Optimization in* R^n, Kluwer Academic Publishers, Dordrecht, 1997.
[34] G. J. Tang, L. W. Zhou and N. J. Huang, *The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds*, Optim. Lett. **7** (2013), 779–790.
[35] G. J. Tang and N. J. Huang, *An inexact proximal point algorithm for maximal monotone vector fields on Hadamard manifolds*, Oper. Res. Lett. **41** (2013), 586–591.
[36] W. Thámelt, *Directional derivatives and generalized gradients on manifolds*, Optimization **25** (1992), 97–115.
[37] C. Udriste, *Convex Functions and Optimization Methods on Riemannian Manifolds*, in Mathematics and its Applications, vol. 297, Kluwer Academic Publishers, 1994.

Manuscript received September 15, 2014
revised November 20, 2014

Sheng-lan Chen
School of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, P. R. China
E-mail address: chensl@cqupt.edu.cn

Nan-jing Huang
Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P. R. China
E-mail address: nanjinghuang@hotmail.com