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WEAK AND STRONG CONVERGENCE OF ALGORITHMS FOR THE SUM OF TWO ACCRETIVE OPERATORS WITH APPLICATIONS

BUTHINAH A. BIN DEHAISH, XIAOLONG QIN*, ABDUL LATIF,
AND HUDA O. BAKODAH

Abstract. Zeros of sums of two accretive operators are investigated. Weak and strong convergence theorems are established in real uniformly smooth Banach spaces. An application is also considered in the framework of Banach spaces.

1. Introduction

Given a nonempty closed and convex subset C of a Hilbert space H and a maximal monotone operator $T : C \to 2^H$, the corresponding zero problem of the operator T is to find $\bar{x} \in C$ such that $0 \in T\bar{x}$. A classical method for solving the problem is the proximal point algorithm, proposed by Martinet [20,21] and generalized by Rockafellar [30,31]. In the case of $T = A + B$, where A and B are monotone operators, the problem is reduced to as follows:

$$\text{find } \bar{x} \in C \text{ such that } 0 \in (A + B)\bar{x}. \tag{1.1}$$

The solution set of (1.1) is denoted by $(A + B)^{-1}(0)$. In this paper, we will focus our attention on problem (1.1), which is very general in the sense that it includes, as special cases, convexly constrained linear inverse problems, split feasibility problem, convexly constrained minimization problems, fixed point problems, variational inequalities, Nash equilibrium problem in noncooperative games and others; see, for instance, [3,8,11,25,34,36] and the references therein. Because of their importance, forward-backward splitting methods, which were proposed by Lions and Mercier [17], by Passty [24], and, in a dual form for convex programming, by Han and Lou [14], for solving (1.1) have been studied extensively recently; see, for instance, [10,19,23,26,27,33] and the references therein. There is, however, little work in the existing literature in the setting of Banach spaces.

The aim of this paper is to present two forward-backward splitting methods for solving (1.1) in the framework of Banach spaces. Our ideas are mainly inspired by [1,15], where the methods for finding solutions of variational inequalities are constructed in the framework of Banach spaces. In contrast with [1], where only weak

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*Corresponding author.

convergence is obtained, in our results here we give weak and strong convergence of the two algorithms.

The paper is organized in the following way. In Section 2, we present the preliminaries that are needed in our work. In Section 3, we present two algorithms for solving (1.1). Convergence analysis of the algorithms are investigated. As an application of the main results, a fixed point problem of strictly pseudocontractive mappings is investigated in the framework of Banach spaces.

2. Preliminaries

Let E be a real Banach space with the dual E^*. Given of continuous strictly increasing function: $\varphi : R^+ \to R^+$, where R^+ denotes the set of nonnegative real numbers, such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$, we associate with it a (possibly mutivalued) generalized duality map $\mathfrak{J}_\varphi : E \to 2^{E^*}$, defined as $\mathfrak{J}_\varphi(x) : \{x^* \in E^* : x^*(x) = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\}$, $\forall x \in E$. In this paper, we use the generalized duality map associated with the gauge function $\varphi(t) = t^{q-1}$ for $q > 1$,

$$\mathfrak{J}_q : \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

Let $U_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. The norm of E is said to be Fréchet differentiable if for each $x \in U_E$, the limit is attained uniformly for all $y \in U_E$. The norm of E is said to be uniformly Fréchet differentiable if the limit is attained uniformly for all $x, y \in U_E$.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E by

$$\rho_E(t) = \sup\{\frac{\|x + y\| - \|x - y\|}{2} - 1 : x \in U_E, \|y\| \leq t\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$. E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable.

The modulus of convexity of E is the function $\delta_E(\epsilon) : (0, 2] \to [0, 1]$ defined by $\delta_E(\epsilon) = \inf\{1 - \frac{\|x+t\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}$. Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. Let $p > 1$. We say that E is p-uniformly convex if there exists a constant $c_q > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for any $\epsilon \in (0, 2]$.

Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p, where $p > 1$. To be more precise, L_p is mini$\{p, 2\}$-uniformly smooth for every $p > 1$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $\frac{1}{p} + \frac{1}{q} = 1$.

Let D be a nonempty subset of C. Let $Proj_D : C \to D$ be a mapping. $Proj_D : C \to D$ is said to be

(1) contraction if $Proj_D^2 = Proj_D$;

(2) sunny if for each $x \in C$ and $t \in (0,1)$, we have $Proj_D\big(tx+(1-t)Proj_D x\big) = Proj_D x$;
(3) sunny nonexpansive retractction if $Proj_D$ is sunny, nonexpansive and a contraction.

D is said to be a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D. The following result, which was established in [6, 13, 29], describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Proj_C : E \to C$ be a retraction and $\mathfrak{J}$ be the normalized duality mapping on E. Then the following are equivalent:
(1) $Proj_C$ is sunny and nonexpansive;
(2) $\|Proj_C x - Proj_C y\|^2 \le \langle x-y, \mathfrak{J}(Proj_C x - Proj_C y)\rangle$, $\forall x, y \in E$;
(3) $\langle x - Proj_C x, \mathfrak{J}(y - Proj_C x)\rangle \le 0$, $\forall x \in E$, $y \in C$.

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction $Proj_C$ is coincident with the metric projection from E onto C. Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$ and let $x_0 \in C$. Then we have from the above that $x_0 = Proj_C x$ if and only if $\langle x - x_0, \mathfrak{J}(y - x_0)\rangle \le 0$ for all $y \in C$, where $Proj_C$ is a sunny nonexpansive retraction from E onto C.

Let $T : C \to C$ be a mapping. The fixed point set of T is denoted by $F(T)$. Recall that T is said to be κ-contractive if there exists a constant $\kappa \in (0,1)$ such that

$$\|Tx - Ty\| \le \kappa\|x, y\|, \quad \forall x, y \in C.$$

T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x, y\|, \quad \forall x, y \in C.$$

T is said to be κ-strictly pseudocontractive if there exists a constant $\kappa \in (0,1)$ such that

$$\langle Tx - Ty, \mathrm{j}_q(x-y)\rangle \le \|x-y\|^q - \kappa\|(x - Tx) - (y - Ty)\|^q, \quad \forall x, y \in C$$

for some $\mathrm{j}_q(x-y) \in \mathfrak{J}_q(x-y)$. It is clear that the above inequality is equivalent to the following

$$\langle (x - Tx) - (y - Ty), \mathrm{j}_q(x-y)\rangle \ge \kappa\|(x - Tx) - (y - Ty)\|^q, \quad \forall x, y \in C.$$

It is known that κ-strictly pseudocontractive mappings are Lipschitz continuous. We also remark here that the class of mapping was first introduced by Browder and Petryshyn [4] in Hilbert spaces. T is said to be pseudocontractive if

$$\langle Tx - Ty, \mathrm{j}_q(x-y)\rangle \le \|x-y\|^q, \quad \forall x, y \in C$$

for some $\mathrm{j}_q(x-y) \in \mathfrak{J}_q(x-y)$.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \ne \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if, for $t > 0$ and $x, y \in D(A)$,

$$\|x - y\| \le \|x - y + t(u - v)\|, \quad \forall u \in Ax, v \in Ay.$$

It follows from Kato [16] that A is accretive if and only if, for $x, y \in D(A)$, there exists $\mathfrak{j}_q(x_1 - x_2)$ such that

$$\langle u - v, \mathfrak{j}_q(x - y)\rangle \geq 0.$$

An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone. In this paper, we use $A^{-1}(0)$ to denote the set of zeros of A.

For an accretive operator A, we can define a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A.

Recall that a single valued operator $A : C \to E$ is said to be α-inverse strongly accretive if there exists a constant $\alpha > 0$ and some $\mathfrak{j}_q(x - y) \in \mathfrak{J}_q(x - y)$ such that

$$\langle Ax - Ay, \mathfrak{j}_q(x - y)\rangle \geq \alpha\|Ax - Ay\|^q, \quad \forall x, y \in C.$$

In order to obtain our main results, we also need the following lemmas.

The following lemmas are trivial.

Lemma 2.1. *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $A : C \to E$ be a single valued operator and let $B : E \to 2^E$ be an m-accretive operator. Then*

$$F(J_a(I - aA)) = (A + B)^{-1}(0),$$

where $J_a(I - aA)$ is the resolvent of B for $a > 0$.

Lemma 2.2. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3 ([35])**.** *Let E be a real q-uniformly smooth Banach space. Then the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, \mathfrak{J}_q(x + y)\rangle$$

and

$$\|x + y\|^q \leq \|x\|^q + q\langle y, \mathfrak{J}_q(x)\rangle + K_q\|y\|^q, \quad \forall x, y \in E,$$

where K_q is some fixed positive constant.

Lemma 2.4 ([35])**.** *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that*

$$\|ax + (1 - a)y\|^p \leq a\|x\|^p + (1 - a)\|y\|^p - \big(a^p(1 - a) + (1 - a)^p a\big)\varphi(\|x - y\|),$$

for all $x, y \in B_r(0) := \{x \in E : \|x\| \leq r\}$ and $a \in [0, 1]$.

Lemma 2.5 ([7])**.** *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex and bounded subset of E. Then there is a strictly increasing and continuous convex function $\psi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that, for* *every Lipschitzian continuous mapping $T : C \to C$ and, for all $x, y \in C$ and $t \in [0,1]$, the following inequality holds:*

$$\|T(tx + (1-t)y) - (tTx + (1-t)Ty)\| \leq L\psi^{-1}\big(\|x-y\| - L^{-1}\|Tx - Ty\|\big),$$

where $L \geq 1$ is the Lipschitz constant of T.

Lemma 2.6 ([5]). *Let E be a real uniformly convex Banach space, C a nonempty closed, and convex subset of E and $T : C \to C$ a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2.7 ([12]). *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property. Suppose that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} \|ax_n + (1-a)p_1 - p_2\|$ exists for all $a \in [0,1]$ and $p_1, p_2 \in \omega_w(x_n)$, where $\omega_w(x_n) : \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$ Then $\omega_w(x_n)$ is a singleton.*

Lemma 2.8 ([2]). *Let E be a real Banach space, and A an m-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big),$$

where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.

Lemma 2.9 ([32]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Banach space E and let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Let $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.10 ([28]). *Let E be a real uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $f : C \to C$ be a contractive mapping and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set. For each $t \in (0,1)$, let x_t be the unique solution of the equation $x = tf(x) + (1-t)Tx$. Then $\{x_t\}$ converges strongly to a fixed point $\bar{x} = Q_{F(T)}f(\bar{x})$, where $Q_{F(T)}$ is the unique sunny nonexpansive retraction from C onto $F(T)$, as $t \to 0$.*

Lemma 2.11 ([22, Lemma 2.11]). *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{a^q}{q} + \frac{q-1}{q}b^{\frac{q}{q-1}},$$

for arbitrary positive real numbers a and b.

Lemma 2.12 ([18]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-t_n)a_n + b_n + c_n$, $\forall n \geq 0$, where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence of real numbers. Assume that*

(a) $\sum_{n=0}^\infty t_n = \infty$ *and* $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$;
(b) $\sum_{n=0}^\infty c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

First, we give the weak convergence theorem.

Theorem 3.1. *Let E be a real uniformly convex and q-uniformly smooth Banach space with the constant K_q and let C be a nonempty closed and convex subset of E. Let $A : C \to E$ be an α-inverse strongly accretive operator and let $B : E \to 2^E$ be an m-accretive operator such that $D(B) \subset C$. Assume that $(A+B)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a positive number sequence and let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{r_n}(x_n - r_n A x_n + e_n), \quad \forall n \geq 0,$$

where $J_{r_n} = (I + r_n B)^{-1}$ and $\{e_n\}$ is sequence in E. Assume that the sequence $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(1) $\limsup_{n\to\infty} \alpha_n < 1$;
(2) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$;
(3) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ converges weakly to a zero of $A+B$.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. In view of Lemma 2.3, we find that

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^q &\leq \|x-y\|^q - q r_n \langle Ax - Ay, \mathfrak{J}_q(x-y)\rangle \\ &\quad + K_q r_n^q \|Ax - Ay\|^q \\ &\leq \|x-y\|^q - q r_n \alpha \|Ax - Ay\|^q + K_q r_n^q \|Ax - Ay\|^q \\ &= \|x-y\|^q - (\alpha q - K_q r_n^{q-1}) r_n \|Ax - Ay\|^q. \end{aligned}$$

From the restriction (2), we find that $I - r_n A$ is nonexpansive. Fixing $p \in (A + B)^{-1}(0)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|J_{r_n}(x_n - r_n A x_n + e_n) - p\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|(x_n - r_n A x_n + e_n) - (p - r_n A)p\| \\ &\leq \|x_n - p\| + \|e_n\|. \end{aligned}$$

In view of Lemma 2.2, we obtain that $\lim_{n\to\infty} \|x_n - p\|$ exists, in particular, $\{x_n\}$ is bounded. Using Lemma 2.3, we find that

$$\begin{aligned} \|(I - r_n A)x_n - (I - r_n A)p + e_n\|^q \quad & + q\langle e_n, \mathfrak{J}_q((I - r_n A)x_n - (I - r_n A)p + e_n)\rangle \\ &\leq \|x_n - p\|^q - q r_n \langle Ax_n - Ap, \mathfrak{J}_q(x_n - p)\rangle \\ &\quad + K_q r_n^q \|Ax_n - Ap\|^q \\ &\quad + q\|e_n\| \|(I - r_n A)x_n - (I - r_n A)p + e_n\|^{q-1} \\ &\leq \|x_n - p\|^q - (\alpha q - K_q r_n^{q-1}) r_n \|Ax_n - Ap\|^q \\ &\quad + q\|e_n\| \|(I - r_n A)x_n - (I - r_n A)p + e_n\|^{q-1}. \end{aligned} \tag{3.1}$$

Putting $y_n = J_{r_n}(x_n - r_nAx_n + e_n)$, we find from Lemma 2.4 that

$$\begin{aligned}
&\Big\|\frac{1}{2}(y_n - p) + \frac{1}{2}\big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\Big\|^q \\
&\le \frac{1}{2}\|y_n - p\|^q + \frac{1}{2}\|(I - r_nA)x_n + e_n - (I - r_nA)p\|^q \\
&\quad - \frac{1}{2^q}\varphi\Big(\|(y_n - p) - \big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\|\Big) \\
&\le \|(I - r_nA)x_n + e_n - (I - r_nA)p\|^q \\
&\quad - \frac{1}{2^q}\varphi\Big(\|(y_n - p) - \big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\|\Big).
\end{aligned} \tag{3.2}$$

Substituting (3.1) into (3.2), we arrive at

$$\begin{aligned}
&\Big\|\frac{1}{2}(y_n - p) + \frac{1}{2}\big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\Big\|^q \\
&\le \|x_n - p\|^q - (\alpha q - K_q r_n^{q-1})r_n\|Ax_n - Ap\|^q \\
&\quad + q\|e_n\|\|(I - r_nA)x_n - (I - r_nA)p + e_n\|^{q-1} \\
&\quad - \frac{1}{2^q}\varphi\Big(\|(y_n - p) - \big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\|\Big).
\end{aligned} \tag{3.3}$$

In view of the acctiveness of B, we find that

$$\begin{aligned}
\|y_n - p\| &\le \Big\|y_n - p + \frac{r_n}{2}\Big(\frac{x_n - r_nAx_n + e_n - y_n}{r_n} - \frac{(I - r_nA)p - p}{r_n}\Big)\Big\| \\
&= \Big\|\frac{1}{2}(y_n - p) + \frac{1}{2}\big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\Big\|.
\end{aligned} \tag{3.4}$$

Combining (3.3) with (3.4), we see that

$$\begin{aligned}
\|y_n - p\|^q \le\ & \|x_n - p\|^q - (\alpha q - K_q r_n^{q-1})r_n\|Ax_n - Ap\|^q \\
& + q\|e_n\|\|(I - r_nA)x_n - (I - r_nA)p + e_n\|^{q-1} \\
& - \frac{1}{2^q}\varphi\Big(\|(y_n - p) - \big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\|\Big).
\end{aligned} \tag{3.5}$$

Since $\|\cdot\|^q$ is convex, we find that

$$\begin{aligned}
\|x_{n+1} - p\|^q &\le \alpha_n\|x_n - p\|^q + (1 - \alpha_n)\|y_n - p\|^q \\
&\le \|x_n - p\|^q - (\alpha q - K_q r_n^{q-1})r_n(1 - \alpha_n)\|Ax_n - Ap\|^q \\
&\quad + q\|e_n\|\|(I - r_nA)x_n - (I - r_nA)p + e_n\|^{q-1} \\
&\quad - (1 - \alpha_n)\frac{1}{2^q}\varphi\Big(\|(y_n - p) - \big((I - r_nA)x_n + e_n - (I - r_nA)p\big)\|\Big).
\end{aligned}$$

It follows from the restrictions (1), (2) and (3) that

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0 \tag{3.6}$$

and

$$\lim_{n\to\infty}\|y_n - x_n + r_nAx_n - r_nAp - e_n\| = 0. \tag{3.7}$$

Since

$$\begin{aligned}\|y_n - x_n\| &\le \|y_n - x_n + r_nAx_n - r_nAp - e_n\| + \|r_nAx_n - r_nAp - e_n\| \\ &\le \|y_n - x_n + r_nAx_n - r_nAp - e_n\| + r_n\|Ax_n - Ap\| + \|e_n\|,\end{aligned}$$

we find from (3.6) and (3.7) that

$$\lim_{n\to\infty} \|J_{r_n}(x_n - r_nAx_n + e_n) - x_n\| = 0. \tag{3.8}$$

Notice that

$$\begin{aligned}\|J_{r_n}(x_n - r_nAx_n) - x_n\| &\le \|J_{r_n}(x_n - r_nAx_n) - J_{r_n}(x_n - r_nAx_n + e_n)\| \\ &\quad + \|J_{r_n}(x_n - r_nAx_n + e_n) - x_n\| \\ &\le \|e_n\| + \|J_{r_n}(x_n - r_nAx_n + e_n) - x_n\|.\end{aligned}$$

This implies from (3.8) that

$$\lim_{n\to\infty} \|J_{r_n}(x_n - r_nAx_n) - x_n\| = 0. \tag{3.9}$$

In view of restriction (2), without loss of generality, let us assume that there exists a real number a such that $r_n \ge a > 0$ for all $n \ge 1$. Notice that

$$\begin{aligned}&\Big\langle \frac{x_n - J_a(I - aA)x_n}{a} - \frac{x_n - J_{r_n}(I - r_nA)x_n}{r_n}, \\ &\mathfrak{J}_q\big(J_a(I - aA)x_n - J_{r_n}(I - r_nA)x_n\big)\Big\rangle \ge 0.\end{aligned}$$

Hence, we find that

$$\begin{aligned}&\|J_a(I - aA)x_n - J_{r_n}(I - r_nA)x_n\|^q \\ &\le \frac{r_n - a}{r_n}\langle x_n - J_{r_n}(I - r_nA)x_n, \mathfrak{J}_q\big(J_a(I - aA)x_n - J_{r_n}(I - r_nA)x_n\big)\rangle \\ &\le \|x_n - J_{r_n}(I - r_nA)x_n\|\|J_a(I - aA)x_n - J_{r_n}(I - r_nA)x_n\|^{q-1}.\end{aligned}$$

This implies that $\|J_a(I - aA)x_n - J_{r_n}(I - r_nA)y_n\| \le \|x_n - J_{r_n}(I - r_nA)x_n\|$. It follows that

$$\begin{aligned}\|J_a(I - aA)x_n - x_n\| &\le \|J_a(I - aA)x_n - J_{r_n}(I - r_nA)x_n\| \\ &\quad + \|J_{r_n}(I - r_nA)x_n - x_n\| \\ &\le 2\|J_{r_n}(I - r_nA)x_n - x_n\|.\end{aligned}$$

From (3.9), we arrive at

$$\lim_{n\to\infty} \|J_a(x_n - aAx_n) - x_n\| = 0. \tag{3.10}$$

Define mappings $T_n : C \to C$ by

$$T_nx := \alpha_nx + (1 - \alpha_n)J_{r_n}((I - r_nA)x + e_n), \quad \forall x \in C.$$

Set

$$S_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_n, \quad \forall n, m \ge 1.$$

Then $S_{n,m}$ is nonexpansive and $S_{n,m}x_n = x_{n+m}$. For all $t \in [0, 1]$ and $n, m \ge 1$, put

$$a_n(t) = \|tx_n + (1 - t)p_1 - p_2\|,$$

and

$$b_{n,m} = \|S_{n,m}(tx_n + (1 - t)p_1) - (tx_{n+m} + (1 - t)p_1)\|,$$

where p_1 and p_2 are zeros of $A+B$. Since $\|T_np-p\| \le \|e_n\|$, we find that

$$\begin{aligned}\|S_{n,m}p-p\| &\le \|T_{n+m-1}T_{n+m-2}\cdots T_np - T_{n+m-1}T_{n+m-2}\cdots T_{n+1}p\| + \cdots \\ &\quad + \|T_{n+m-1}p-p\| \\ &\le \|T_np-p\| + \|T_{n+1}p-p\| + \cdots + \|T_{n+m-1}p-p\| \\ &\le \sum_{i=0}^{m-1} e_{n+i}.\end{aligned} \tag{3.11}$$

Using Lemma 2.5, we find that

$$\begin{aligned}b_{n,m} &\le \psi^{-1}\big(\|x_n-p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\|\big) \\ &= \psi^{-1}\big(\|x_n-p_1\| - \|x_{n+m}-p_1+p_1-S_{n,m}p_1\|\big) \\ &\le \psi^{-1}\big(\|x_n-p_1\| - (\|x_{n+m}-p_1\| - \|p_1-S_{n,m}p_1\|)\big).\end{aligned} \tag{3.12}$$

In view of (3.11), we find that $\{b_{n,m}\}$ converges uniformly to zero as $n\to\infty$ for all $m\ge 1$. It also follows from (3.11) that

$$\begin{aligned}a_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\ &\le b_{n,m} + \|S_{n,m}(tx_n+(1-t)p_1) - p_2\| \\ &\le b_{n,m} + \|S_{n,m}(tx_n+(1-t)p_1) - S_{n,m}p_2\| + \|S_{n,m}p_2-p_2\| \\ &\le b_{n,m} + a_n(t) + \|S_{n,m}p_2 - p_2\| \\ &\le b_{n,m} + a_n(t) + \sum_{i=0}^{m-1} e_{n+i}.\end{aligned} \tag{3.13}$$

Taking lim sup as $m\to\infty$ and then the lim inf as $n\to\infty$, we find that $\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)$. This proves that $\lim_{n\to\infty} a_n(t)$ for any $t\in[0,1]$. In view of Lemma 2.6, we see that $\omega_w(x_n)\subset (A+B)^{-1}(0)$. This implies from Lemma 2.7 that $\omega_w(x_n)$ is singleton. This proves the proof. □

If $\alpha=0$, then Theorem 3.1 is reduced to the following.

Corollary 3.2. *Let E be a real uniformly convex and q-uniformly smooth Banach space with the constant K_q and let C be a nonempty closed and convex subset of E. Let $A: C\to E$ be an α-inverse strongly accretive operator and let $B: E\to 2^E$ be an m-accretive operator such that $D(B)\subset C$. Assume that $(A+B)^{-1}(0)\ne\emptyset$. Let $\{r_n\}$ be a positive number sequence. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0\in C$ and*

$$x_{n+1} = J_{r_n}(x_n - r_nAx_n + e_n), \quad \forall n\ge 0,$$

where $J_{r_n} = (I+r_nB)^{-1}$ and $\{e_n\}$ is sequence in E. Assume that the sequence $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(1) $0<\liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$;
(2) $\sum_{n=0}^{\infty}\|e_n\|<\infty$.

Then $\{x_n\}$ converges weakly to a zero of $A+B$.

Next, we give the strong convergence theorem.

Theorem 3.3. *Let E be a real q-uniformly smooth Banach space with the constant K_q and let C be a nonempty closed and convex subset of E. Let $A : C \to E$ be an α-inverse strongly accretive operator and let $B : E \to 2^E$ be an m-accretive operator such that $D(B) \subset C$. Assume that $(A + B)^{-1}(0) \neq \emptyset$. Let $f : C \to C$ be a fixed κ-contraction. Let $\{r_n\}$ be a positive number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n}(x_n - r_n A x_n + e_n), \quad \forall n \geq 0,$$

where $J_{r_n} = (I + r_n B)^{-1}$ and $\{e_n\}$ is sequence in E. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(4) $\liminf_{n\to\infty} r_n > 0$, $r_n \leq (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$, $\lim_{n\to\infty} |r_n - r_{n-1}| = 0$;
(5) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = Proj_{(A+B)^{-1}(0)} f(\bar{x})$, where $Proj_{(A+B)^{-1}(0)}$ is the unique sunny nonexpansive retraction of C onto $(A+B)^{-1}(0)$.

Proof. As proved in Theorem 3.1, we see that $I - r_n A$ is nonexpansive. It follows from Lemma 2.1 that $(A + B)^{-1}$ is closed and convex. Fixing $p \in (A + B)^{-1}(0)$, we find that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|J_{r_n}(x_n - r_n A x_n + e_n) - p\| \\
&\leq \alpha_n \kappa \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| \\
&\quad + \gamma_n \|(x_n - r_n A x_n + e_n) - (I - r_n A)p\| \\
&\leq \big(1 - \alpha_n(1 - \kappa)\big)\|x_n - p\| + \alpha_n \|f(p) - p\| + \|e_n\| \\
&\leq \max\Big\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \kappa}\Big\} + \|e_n\| \\
&\leq \max\Big\{\|x_{n-1} - p\|, \frac{\|f(p) - p\|}{1 - \kappa}\Big\} + \|e_{n-1}\| + \|e_n\| \\
&\ \ \vdots \\
&\leq \max\Big\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \kappa}\Big\} + \sum_{i=0}^{n} \|e_i\| \\
&\leq \max\Big\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \kappa}\Big\} + \sum_{i=0}^{\infty} \|e_i\| < \infty.
\end{aligned}$$

This proves that the sequence $\{x_n\}$ is bounded. Putting $z_n = x_n - r_n A x_n + e_n$ and $m_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$, we have

$$x_{n+1} = (1 - \beta_n) m_n + \beta_n x_n. \tag{3.14}$$

In light of

$$\begin{aligned} m_{n+1} - m_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} J_{r_{n+1}} z_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n J_{r_n} z_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - J_{r_{n+1}} z_{n+1}) - \frac{\alpha_n}{1-\beta_n}(f(x_n) - J_{r_n} z_n) \\ &\quad + J_{r_{n+1}} z_{n+1} - J_{r_n} z_n, \end{aligned}$$

we find that

$$\begin{aligned} \|m_{n+1} - m_n\| \le & \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - J_{r_{n+1}} z_{n+1}\| \\ & + \frac{\alpha_n}{1-\beta_n} \|f(x_n) - J_{r_n} z_n\| \\ & + \|J_{r_{n+1}} z_{n+1} - J_{r_n} z_n\|. \end{aligned} \tag{3.15}$$

Notice that

$$\|z_n - z_{n+1}\| \le \|x_n - x_{n+1}\| + \|r_n - r_{n+1}\| \|Ax_{n+1}\| + \|e_n\| + \|e_{n+1}\|.$$

It follows from Lemma 2.8 that

$$\begin{aligned} &\|J_{r_n} z_n - J_{r_{n+1}} z_{n+1}\| \\ &= \left\| J_{r_{n+1}} \Big(\frac{r_{n+1}}{r_n} z_n + \Big(1 - \frac{r_{n+1}}{r_n}\Big) J_{r_n} z_n \Big) - J_{r_{n+1}} z_{n+1} \right\| \\ &\le \left\| \frac{r_{n+1}}{r_n}(z_n - z_{n+1}) + \Big(1 - \frac{r_{n+1}}{r_n}\Big)(J_{r_n} z_n - z_{n+1}) \right\| \\ &\le \|z_n - z_{n+1}\| + \frac{|r_n - r_{n+1}|}{r_n} \|J_{r_n} z_n - z_n\| \\ &\le \|x_n - x_{n+1}\| + \|r_n - r_{n+1}\| \Big(\|Ax_{n+1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n} \Big) \\ &\quad + \|e_n\| + \|e_{n+1}\|. \end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15), we arrive at

$$\begin{aligned} \|m_{n+1} - m_n\| - \|x_n - x_{n+1}\| \le & \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - J_{r_{n+1}} z_{n+1}\| \\ & + \frac{\alpha_n}{1-\beta_n} \|f(x_n) - J_{r_n} x_n\| \\ & + \|r_n - r_{n+1}\| \Big(\|Ax_{n+1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n} \Big) \\ & + \|e_n\| + \|e_{n+1}\|. \end{aligned}$$

In view of conditions (2), (3), (4) and (5), we get that

$$\limsup_{n\to\infty} (\|m_{n+1} - m_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Using Lemma 2.9, we find that $\lim_{n\to\infty} \|m_n - x_n\| = 0$. In view of (3.14), we find that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Notice that

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + (1-\alpha_n)(J_{r_n}(x_n - r_nAx_n + e_n) - x_n).$$

In view of (3.17), we find from the restriction (2) that $\lim_{n\to\infty} \|J_{r_n}(x_n - r_nAx_n + e_n) - x_n\| = 0$. As proved in Theorem 3.1, we find that

$$\lim_{n\to\infty} \|J_a(x_n - aAx_n) - x_n\| = 0. \tag{3.18}$$

Let x_t be the unique solution to the fixed point equation $x_t = tf(x_t) + (1-t)J_a(I - aA)x_t$, $\forall t \in (0,1)$. Putting $\bar{x} = \lim_{t\to 0} x_t$, one has $\bar{x} = Proj_{(A+B)^{-1}(0)} f(\bar{x})$, where $Proj_{(A+B)^{-1}(0)}$ is the unique sunny nonexpansive retraction of C onto $(A+B)^{-1}(0)$.

Now, we are in a position to claim that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_n - \bar{x})\rangle \le 0$. It follows that

$$\begin{aligned}
\|x_t - x_n\|^q &\le t\langle f(x_t) - x_n, \mathfrak{J}_q(x_t - x_n)\rangle \\
&\quad + (1-t)\langle J_a(I - aA)x_t - x_n, \mathfrak{J}_q(x_t - x_n)\rangle \\
&\le t\langle f(x_t) - x_t, \mathfrak{J}_q(x_t - x_n)\rangle + t\langle x_t - x_n, \mathfrak{J}_q(x_t - x_n)\rangle \\
&\quad + (1-t)\big(\langle J_a(I - aA)x_t - J_a(I - aA)x_n, \mathfrak{J}_q(x_t - x_n)\rangle \\
&\quad + \langle J_a(I - aA)x_n - x_n, \mathfrak{J}_q(x_t - x_n)\rangle\big) \\
&\le t\langle f(x_t) - x_t, \mathfrak{J}_q(x_t - x_n)\rangle + \|x_t - x_n\|^q \\
&\quad + \|J_a(I - aA)x_n - x_n\|\|x_t - x_n\|^{q-1},
\end{aligned}$$

which implies that

$$\langle f(x_t) - x_t, J_q(x_n - x_t)\rangle \le \frac{1}{t}\|J_a(I - aA)x_n - x_n\|\|x_t - x_n\|^{q-1}.$$

Fixing t and letting $n \to \infty$, we find from (3.18) that

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, \mathfrak{J}_q(x_n - x_t)\rangle \le 0. \tag{3.19}$$

In view of the fact that the duality map $\mathfrak{J}_q$ is single valued and strong-weak* uniformly continuous on bounded sets of a Banach space E with a uniformly Gâteaux differentiable norm, we get that

$$\begin{aligned}
&|\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_n - \bar{x})\rangle - \langle f(x_t) - x_t, \mathfrak{J}_q(x_n - x_t)\rangle| \\
&= |\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_n - \bar{x}) - \mathfrak{J}_q(x_n - x_t)\rangle \\
&\quad + \langle f(\bar{x}) - \bar{x} - (f(x_t) - x_t), \mathfrak{J}_q(x_n - x_t)\rangle| \\
&\le |\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_n - \bar{x}) - \mathfrak{J}_q(x_n - x_t)\rangle| \\
&\quad + \|f(\bar{x}) - \bar{x} - (f(x_t) - x_t)\|\|x_n - x_t\|^{q-1}.
\end{aligned}$$

Hence, $\forall\epsilon > 0$, $\exists\delta > 0$ such that $t \in (0,\delta)$, we have that

$$\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_n - \bar{x})\rangle \le \langle f(x_t) - x_t, \mathfrak{J}_q(x_n - x_t)\rangle + \epsilon.$$

It follows from (3.19) that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_n - \bar{x})\rangle \le 0. \tag{3.20}$$

Finally, we prove that $x_n \to \bar{x}$ as $n \to \infty$. In view of Lemma 2.11, we find that

$$\|x_{n+1} - \bar{x}\|^q \le \alpha_n\langle f(x_n) - f(\bar{x}), \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle$$

$$
\begin{aligned}
&+ (1-\alpha_n)\|J_{r_n}(x_n - r_nAx_n + e_n) - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1} \\
&+ \alpha_n\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle \\
\leq& \big(1 - \alpha_n(1-\kappa)\big)\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1} \\
&+ \alpha_n\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle + e_n\|x_{n+1} - \bar{x}\|^{q-1} \\
\leq& \big(1 - \alpha_n(1-\kappa)\big)\Big(\frac{1}{q}\|x_n - \bar{x}\|^q + \frac{q-1}{q}\|x_{n+1} - \bar{x}\|^q\Big) \\
&+ \alpha_n\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle + e_n\|x_{n+1} - \bar{x}\|^{q-1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - \bar{x}\|^q \leq& \big(1 - \alpha_n(1-\kappa)\big)\|x_n - \bar{x}\|^q + q\alpha_n\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle \\
&+ qe_n\|x_{n+1} - \bar{x}\|^{q-1}.
\end{aligned}
$$

In view of (3.20), we find from Lemma 2.12 that $\{x_n\}$ converges strongly to $\bar{x}$. This proves the proof. □

Remark 3.4. The framework of the spaces in Both Theorem 3.1 and Theorem 3.3 can be applicable to L_p, where $p > 1$.

4. Applications

In this section, we consider a fixed point problem of κ-strictly pseudocontractive mappings.

Theorem 4.1. *Let E be a real uniformly convex and q-uniformly smooth Banach space with the constant K_q and let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be an α-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. Let $\{r_n\}$ be a positive number sequence and let $\{\alpha_n\}$ be a real number sequence in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$x_{n+1} = \big(1 - r_n(1-\alpha_n)\big)x_n + r_n(1-\alpha_n)Tx_n, \quad \forall n \geq 0.$$

Assume that the sequence $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(1) $\limsup_{n\to\infty} \alpha_n < 1$;
(2) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Putting $A = I - T$, we find that A is α-inverse strongly accretive and $F(T) = A^{-1}(0)$. Notice that

$$
\begin{aligned}
x_{n+1} &= \big(1 - r_n(1-\alpha_n)\big)x_n + r_n(1-\alpha_n)Tx_n \\
&= \alpha_n x_n + (1-\alpha_n)\big((1-r_n)x_n + r_nTx_n\big) \\
&= \alpha_n x_n + (1-\alpha_n)(x_n - r_n(I-T)x_n) \\
&= \alpha_n x_n + (1-\alpha_n)(x_n - r_nAx_n).
\end{aligned}
$$

Using Theorem 3.1, we find the desired conclusion immediately. □

Theorem 4.2. *Let E be a real q-uniformly smooth Banach space with the constant K_q and let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be an α-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a fixed κ-contraction. Let $\{r_n\}$ be a positive number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(1-r_n)x_n + r_n(1-\alpha_n)Tx_n, \quad \forall n \geq 0.$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(4) $\liminf_{n\to\infty} r_n > 0$, $r_n \leq (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$, $\lim_{n\to\infty} |r_n - r_{n-1}| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = Proj_{F(T)} f(\bar{x})$, where $Proj_{F(T)}$ is the unique sunny nonexpansive retraction of C onto $F(T)$.

Proof. Putting $A = I - T$, we find that A is α-inverse strongly accretive and $F(T) = A^{-1}(0)$. Notice that

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)(1-r_n)x_n + r_n(1-\alpha_n)Tx_n \\ &= \alpha_n f(x_n) + (1-\alpha_n)\big((1-r_n)x_n + r_n Tx_n\big) \\ &= \alpha_n f(x_n) + (1-\alpha_n)(x_n - r_n(I-T)x_n) \\ &= \alpha_n f(x_n) + (1-\alpha_n)(x_n - r_n Ax_n). \end{aligned}$$

Using Theorem 3.3, we find the desired conclusion immediately. □

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Buthinah A. Bin Dehaish
Department of Mathematics, Faculty of Science for Girls, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: bbendehaish@kau.edu.sa

Xiaolong Qin
School of Science, Wuhan University of Technology, Wuhan, China; Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: qxlxajh@163.com

Abdul Latif
Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: alatif@kau.edu.sa

Huda O. Bakodah
Department of Mathematics, Faculty of Science for Girls, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: hbakodah@kau.edu.sa