Journal of Nonlinear and Convex Analysis
Volume 16, Number 7, 2015, 1337–1352

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

THE STRONG DUALITY FOR DC OPTIMIZATION PROBLEMS WITH COMPOSITE CONVEX FUNCTIONS

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Abstract. We consider a DC optimization problem with composite functions in locally convex Hausdorff topological vector spaces. By using the epigraph technique, we give some new constraint qualifications, which completely characterize the weak duality, the zero duality, the strong duality and the total duality between the primal problem and its dual problem.

1. Introduction

Let X and Y be real locally convex Hausdorff topological vector spaces, whose dual spaces, X^* and Y^*, are endowed with the weak*- topologies $w^*(X^*, X)$ and $w^*(Y^*, Y)$, respectively. Let Y be partially ordered by a closed convex cone $K \subseteq Y$. Denote $Y^\bullet = Y \cup \{\infty_Y\}$, where ∞_Y is the greatest element with respect to the partial order $\leq_K$. Let $f_2 : X \to Y^\bullet$ and $f_1 : Y \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ with $f_1(\infty_Y) = +\infty$. Since the following composite convex optimization problem

$$(\mathcal{P}_1) \qquad \inf_{x\in X}\{(f_1 \circ f_2)(x)\}$$

offers a unified framework for treating different kinds of optimization problems and many optimization problems generated practical fields (for example, location and transports, economics and finance) involve composed convex functions, it has been received considerable attention, see, for instance, [3–5, 14–17].

In the recent years, the optimization problem with a difference of two convex functions (DC in short) has received extensive attention(cf. [1, 6–11, 13], and the references therein). The reason is, as pointed out in [6], that DC programming problems are very important from both viewpoints of optimization theory and applications. Particularly, the authors in [3, 4] considered the following DC composite convex optimization problem:

$$(\mathcal{P}_2) \qquad \inf_{x\in X}\{(f_1 \circ f_2)(x) - \langle p, x\rangle\},$$

where $p \in X^*$. By using some closed constraint qualifications, Combari et al. established in [4] the strong duality between the problem $(\mathcal{P}_2)$ and its dual problem

$$(\mathcal{D}_2) \qquad \sup_{\lambda\in K^\oplus}\{-f_1^*(\lambda) - (\lambda f_2)^*(p)\}.$$

2010 *Mathematics Subject Classification.* 90C26, 90C46.

Key words and phrases. DC composite optimization problem, strong duality, total duality, constraint qualification.

This author was supported in part by the National Natural Science Foundation of China (grant 11461027) and supported in part by the Scientific Research Fund of Hunan Provincial Education Department (grant 13B095).

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While, in [3], Boţ et al. presented a new closed constraint qualification, which completely characterizes the strong duality between the problem $(\mathcal{P}_2)$ and its dual problem $(\mathcal{D}_2)$. Recently, Zhou et al. considered in [18] the following composite optimization problem:

$$(\mathcal{P}_3) \qquad \inf_{x\in\Omega}\{(f_1\circ f_2)(x)+(h\circ A)(x)-g_1(x)\},$$

where E is a locally convex Hausdorff topological vector space, $\Omega := \{x \in X : p(x) \in -S\}$, $S \subseteq Z$ is a closed convex cone and $p : X \to Z^\bullet$ is a proper, S-convex and S-epi-closed mapping, $h : E \to \bar{\mathbb{R}}$ is a proper convex function, $g_1 : X \to \bar{\mathbb{R}}$ is a proper convex function, and $A : X \to E$ is a linear continuous mapping, and they established the strong duality between the problem $(\mathcal{P}_3)$ and its dual problem $(\mathcal{D}_3)$ via a closedness-type constraint qualification

$$(\mathcal{D}_3) \qquad \inf_{x^*\in X^*}\ \sup_{\lambda\in S^\oplus,\mu\in K^\oplus,e^*\in E^*}\{g_1^*(x^*)-f_1^*(\mu)-h^*(e^*)-(\lambda\circ p+\mu\circ f_2)^*(x^*-A^*e^*)\}.$$

Inspired by the works mentioned above, we consider the following optimization problem

$$(1.1) \qquad (P) \qquad \inf_{x\in X}\{(f_1\circ f_2)(x)-(g_1\circ g_2)(x)\},$$

and define its dual problem by

$$(1.2) \qquad (D) \qquad \inf_{\lambda\in S^\oplus,u^*\in X^*}\ \sup_{\mu\in K^\oplus}\{g_1^*(\lambda)-f_1^*(\mu)-(\mu f_2)^*(u^*)+(\lambda g_2)^*(u^*)\},$$

where $S \subseteq Z$ is a closed convex cone, $f_1 : Y \to \bar{\mathbb{R}}$ is a proper, convex and K-increasing function (not necessary lower semicontinuous (lsc in brief)), $g_1 : Z \to \bar{\mathbb{R}}$ is a proper, convex, S-increasing function (not necessary lower semicontinuous), $f_2 : X \to Y^\bullet$ is a proper, K-convex function (not necessary K-epi-closed), $g_2 : X \to Z^\bullet$ is a proper, S-convex function (not necessary S-epi-closed), and $S^\oplus, K^\oplus$ is the dual cone of S and K, respectively. Here and throughout the whole paper, following [17, Page 39], we adapt the convention that $(+\infty) + (-\infty) = (+\infty) - (+\infty) = +\infty$, $0 \cdot +\infty = +\infty$ and $0 \cdot (-\infty) = 0$. Then, for any two proper convex functions $h_1, h_2 : X \to \overline{\mathbb{R}}$, we have that

$$(1.3) \qquad h_1(x)-h_2(x)\begin{cases}\in\mathbb{R}, & x\in\operatorname{dom}h_1\cap\operatorname{dom}h_2,\\ =-\infty, & x\in\operatorname{dom}h_1\setminus\operatorname{dom}h_2,\\ =+\infty, & x\notin\operatorname{dom}h_1;\end{cases}$$

hence,

$$(1.4) \qquad h_1-h_2 \text{ is proper} \iff \operatorname{dom}h_1\subseteq\operatorname{dom}h_2.$$

Note that, in the case when g_2 is an identity operator on X and $g_1 \in X^*$, then the problem (P) is the same as the problem $(\mathcal{P}_2)$; and in the case when $g_1 \circ g_2 \equiv 0$, then the problem (P) is reduced into the problem $(\mathcal{P}_1)$.

Let $v(P)$ and $v(D)$ denote the optimal values of problem (P) and (D), respectively. Different from the convex case, the weak duality between (P) and (D) does not necessary hold as shown in Example 3.1 in Section 3, that is, we may have $v(P) < v(D)$. Our main aim in the present paper is to use multiply functions to give some new regularity conditions, which completely characterize the weak duality, the zero duality and the strong duality between (P) and (D). In general, the functions f_1, f_2, g_1, g_2 are not necessarily lsc. Most results obtained in this paper seem new and are proper extensions of the known results in [3, 18]. In particular, our Theorem 4.10 improves the corresponding result in [3, Theorem 5.1].

The paper is organized as follows. The next section contains some necessary notations and preliminary results. In section 3, some new constraint qualifications are introduced to study the weak duality, the zero duality and the strong duality between (P) and (D). In section 4, we give some special cases of our main results, which improve several known results.

2. Notations and preliminary results

The notations used in the present paper are standard (cf. [17]). In particular, we assume throughout the whole paper that X and Y are real locally convex Hausdorff topological vector spaces, and let X^* denote the dual space of X, endowed with the weak*-topology $w^*(X^*, X)$. By $\langle x^*, x\rangle$, we shall denote the value of the functional $x^* \in X^*$ at $x \in X$; i.e., $\langle x^*, x\rangle = x^*(x)$. For a set Z in X, the interior, closure, convex hull, and the convex cone hull of Z are denoted by $\operatorname{int} Z$, $\operatorname{cl} Z$, $\operatorname{co} Z$, and $\operatorname{cone} Z$, respectively. If $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak*-closure of W. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

Let $f : X \to \overline{\mathbb{R}}$ be a extended real-valued function. The classical conjugate function of f (the Fenchel-Moreau conjugate) is

$$f^*(x^*) := \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\} \quad \text{for each } x^* \in X^*.$$

By definition, the Young-Fenchel inequality below holds:

$$f(x) + f^*(x^*) \geq \langle x, x^*\rangle \text{ for each pair } (x, x^*) \in X \times X^*. \tag{2.1}$$

Let $x \in \operatorname{dom} f$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \leq f(y) \text{ for all } y \in X\}.$$

Then, by definition,

$$0 \in \partial f(x) \Leftrightarrow x \text{ is a minimizer of } f. \tag{2.2}$$

Moreover, by [17, Theorem 2.4.2(iii)], the Young equality holds:

$$f(x) + f^*(x^*) = \langle x^*, x\rangle \Leftrightarrow x^* \in \partial f(x). \tag{2.3}$$

The indicator function $\delta_D : X \to \overline{\mathbb{R}}$ of the nonempty set $D \subseteq X$ is defined by

$$\delta_D(x) := \begin{cases} 0, & x \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

For the sake of convenience, we write μf_2 instead of $\mu \circ f_2$ for any $\mu \in K^{\oplus}$,

$$(\mu f_2)(x) := \begin{cases} \langle \mu, f_2(x)\rangle, & \text{if } x \in \operatorname{dom} f_2, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $K \subseteq Y$ be a closed convex cone. Its dual cone $K^{\oplus}$ is defined by

$$K^{\oplus} = \{y^* \in Y^* : y^*(y) \geq 0 \quad \text{for each } y \in K\}.$$

Denote by $\leq_K$ the partial order on Y induced by K,

$$y_1 \leq_K y_2 \Leftrightarrow y_2 - y_1 \in K \qquad \text{for each } y_1, y_2 \in Y.$$

There are notions given for functions with extended real values.

For a function $f : Y \to \bar{\mathbb{R}}$, one has

• the effective domain:

$$\operatorname{dom} f = \{y \in Y : f(y) < +\infty\},$$

• the epigraph:

$$\operatorname{epi} f = \{(y, r) \in Y \times \mathbb{R} : f(y) \leq r\},$$

• f is proper: $\operatorname{dom} f \neq \emptyset$ and $f(y) \neq -\infty$, $\forall y \in Y$.

• f is K-increasing: for any $y_1, y_2 \in Y$ such that $y_1 \leq_K y_2$ one has $f(y_1) \leq f(y_2)$.

For a function $h : X \to Y^\bullet$ one has

• the effective domain:

$$\operatorname{dom} h = \{x \in X : h(x) \in Y\},$$

• h is proper: $\operatorname{dom} h \neq \emptyset$,

• the K-epigraph:

$$\operatorname{epi}_K h = \{(x, y) \in X \times Y : y \in h(x) + K\},$$

• h is K-epi-closed: if $\operatorname{epi}_K h$ is closed,

• h is K-convex: for any $x_1, x_2 \in X$ and any $t \in [0, 1]$,

$$h(tx_1 + (1-t)x_2) \leq_K th(x_1) + (1-t)h(x_2).$$

Furthermore, if $f, h : X \to \bar{\mathbb{R}}$ are proper convex functions, and f is convex and lsc on $\operatorname{dom} h$, then, by [13, Lemma 2.3],

$$\operatorname{epi}(h - f)^* = \bigcap_{x^* \in \operatorname{dom} f^*} (\operatorname{epi} h^* - (x^*, f^*(x^*))). \tag{2.4}$$

The following lemma is known in [12, 17].

Lemma 2.1. *Let $f, h : X \to \bar{\mathbb{R}}$ be proper convex functions satisfying $\operatorname{dom} f \cap \operatorname{dom} h \neq \emptyset$.*

(i) *If f, h are lsc, then*

$$\operatorname{epi}(f + h)^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} h^*).$$

(ii) *If either f or h is continuous at some point of $\operatorname{dom} f \cap \operatorname{dom} h$, then*

$$\operatorname{epi}(f + h)^* = \operatorname{epi} f^* + \operatorname{epi} h^*.$$

3. The strong duality

Let X, Y and Z be locally convex Hausdorff topological vector spaces with the dual spaces X^* , Y^* and Z^*, respectively. Let Y and Z be partially ordered by closed convex cones $K \subseteq Y$ and $S \subseteq Z$, respectively. Denote $Y^\bullet := Y \cup \{\infty_Y\}$ and $Z^\bullet := Z \cup \{\infty_Z\}$, where ∞_Y and ∞_Z are the greatest elements with respect to the partial orders $\leq_K$ and $\leq_S$, respectively. Let $f_1 : Y \to \bar{\mathbb{R}}$ be a proper, convex and K-increasing function, $g_1 : Z \to \bar{\mathbb{R}}$ be a proper, convex, S-increasing function, $f_2 : X \to Y^\bullet$ be a proper, K-convex function, and $g_2 : X \to Z^\bullet$ be a proper, S-convex function such that $f_1 \circ f_2 - g_1 \circ g_2$ is a proper function and

$\text{dom}(f_1 \circ f_2) \cap \text{dom}(g_1 \circ g_2) \neq \emptyset$. Then, by (1.4), we have that $\emptyset \neq \text{dom}(f_1 \circ f_2) \subseteq \text{dom}(g_1 \circ g_2)$. Consider the following problem defined by (1.1), that is,

$$(P) \qquad \inf_{x \in X} \{(f_1 \circ f_2)(x) - (g_1 \circ g_2)(x)\}, \tag{3.1}$$

and its dual problem

$$(D) \qquad \inf_{\lambda \in S^{\oplus}, u^* \in X^*} \sup_{\mu \in K^{\oplus}} \{g_1^*(\lambda) - f_1^*(\mu) - (\mu f_2)^*(u^*) + (\lambda g_2)^*(u^*)\}. \tag{3.2}$$

For each $\lambda \in S^{\oplus}$ and $u^* \in X^*$, we define the subproblem of (D) by

$$(D^{(\lambda,u^*)}) \qquad \sup_{\mu \in K^{\oplus}} \{g_1^*(\lambda) - f_1^*(\mu) - (\mu f_2)^*(u^*) + (\lambda g_2)^*(u^*)\}.$$

Let $\lambda \in S^{\oplus}$ and $u^* \in X^*$. We use $v(P)$, $v(D)$ and $v(D^{(\lambda,u^*)})$ to devote the optimal values of the problem (P), (D) and $(D^{(\lambda,u^*)})$, respectively, that is,

$$v(P) := \inf_{x \in X} \{(f_1 \circ f_2)(x) - (g_1 \circ g_2)(x)\}, \tag{3.3}$$

$$v(D) := \inf_{\lambda \in S^{\oplus}, u^* \in X^*} \sup_{\mu \in K^{\oplus}} \{g_1^*(\lambda) - f_1^*(\mu) - (\mu f_2)^*(u^*) + (\lambda g_2)^*(u^*)\} \tag{3.4}$$

and

$$v(D^{(\lambda,u^*)}) := \sup_{\mu \in K^{\oplus}} \{g_1^*(\lambda) - f_1^*(\mu) - (\mu f_2)^*(u^*) + (\lambda g_2)^*(u^*)\}. \tag{3.5}$$

Definition 3.1. It is said that

(i) the weak duality holds between (P) and (D) if $v(P) \geq v(D)$;

(ii) the zero duality holds between (P) and (D) if $v(P) = v(D)$;

(iii) the strong duality holds between (P) and (D) if $v(P) = v(D)$ and for each $\lambda \in S^{\oplus}$ and $u^* \in X^*$ satisfying $v(D) = v(D^{(\lambda,u^*)})$, the problem $(D^{(\lambda,u^*)})$ has an optimal solution.

The following example shows that the weak duality does not hold in general.

Example 3.2. Let $X = Y = Z := \mathbb{R}$ and $S := \mathbb{R}_-$. Define $f_1 = f_2 := 0$, $g_1 := \text{Id}_{\mathbb{R}}$ and $g_2 := \delta_{\mathbb{R}_+}$, where $\text{Id}_{\mathbb{R}}$ denotes the identity operator on $\mathbb{R}$. Then, $S^{\oplus} = \mathbb{R}_-$ and $\text{dom}(f_1 \circ f_2) \cap \text{dom}(g_1 \circ g_2) = \mathbb{R}_+$. Hence, $v(P) = \inf_{x \in X}\{-(g_1 \circ g_2)(x)\} = -\infty$. While, for each $x^* \in \mathbb{R}$,

$$g_1^*(x^*) = \begin{cases} 0, & \text{if } x^* = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and $(\lambda g_2)^*(x^*) = +\infty$ for each $\lambda \in \mathbb{R}_-$. Thus,

$$v(D) = \inf_{\lambda \in \mathbb{R}_-, u^* \in X^*} \{g_1^*(\lambda) + (\lambda g_2)^*(u^*)\} = +\infty.$$

Consequently, $v(D) > v(P)$ and the weak duality does not hold.

To consider the dualities between (P) and (D), we introduce some auxiliary functions. Let $F, G_1, G_2 : X \times Y \times Z \to \overline{\mathbb{R}}$ be defined by

$$F(x, y, z) := f_1(y), \tag{3.6}$$

$$G_1(x, y, z) := g_1(z), \tag{3.7}$$

and

$$G_2(x,y,z) := \delta_{\{(x,y)\in X\times Y: f_2(x)-y\in -K\}}(x,y) + \delta_{\{(x,z)\in X\times Z: g_2(x)-z\in S\}}(x,z). \tag{3.8}$$

Then the following lemma holds.

Lemma 3.3. *Let $r\in\mathbb{R}$. The following statements are equivalent:*
(i) $v(P)\geq -r$.
(ii) $(0,r)\in \operatorname{epi}(f_1\circ f_2 - g_1\circ g_2)^*$.
(iii) $(0,0,0,r)\in \operatorname{epi}(F-G_1+G_2)^*$.

Proof. (i) $\Leftrightarrow$ (ii) By the definition of the conjugate function, one has

$$v(P) = -(f_1\circ f_2 - g_1\circ g_2)^*(0).$$

Thus, the result is clear.

(ii) $\Leftrightarrow$ (iii) Since f_1 is K-increasing and g_1 is S-increasing, it follows that for each $x^*\in X^*$,

$$\begin{aligned}(f_1\circ f_2 - g_1\circ g_2)^*(0) &= \sup_{x\in X}\{\langle 0,x\rangle - (f_1\circ f_2)(x) + (g_1\circ g_2)(x)\}\\ &= \sup_{x\in X, y\in Y, z\in Z, f_2(x)-y\in -K, g_2(x)-z\in S}\{-f_1(y)+g_1(z)\}\\ &= \sup_{x\in X, y\in Y, z\in Z}\{-f_1(y)+g_1(z)\\ &\qquad - \delta_{\{(x,y)\in X\times Y: f_2(x)-y\in -K\}}(x,y)\\ &\qquad\quad - \delta_{\{(x,z)\in X\times Z: g_2(x)-z\in S\}}(x,z)\}.\end{aligned}$$

This implies that

$$(f_1\circ f_2 - g_1\circ g_2)^*(0) = (F-G_1+G_2)^*(0,0,0).$$

Thus, the result is clear and the proof is complete. □

Let $r\in\mathbb{R}$. For simplicity, we denote

$$\begin{aligned} K_0 \;:=\; & \bigcap_{\lambda\in S^\oplus, u^*\in X^*}\Bigg(\bigcup_{\mu\in K^\oplus}\Big(\{(u^*,-\mu,0,r) : (u^*,r)\in \operatorname{epi}(\mu f_2)^*\}\\ & +\{(0,\mu,0,r) : (\mu,r)\in \operatorname{epi} f_1^*\}\Big) - (u^*,0,0,g_1^*(\lambda)+(\lambda g_2)^*(u^*))\Bigg),\end{aligned} \tag{3.9}$$

where we adapt the convention $\cap_{t\in\emptyset} S_t = X$. Obviously,

$$K_0 \subseteq \{0\}\times\{0\}\times\{0\}\times\mathbb{R}. \tag{3.10}$$

Lemma 3.4. *Let $r\in\mathbb{R}$. Then, $(0,0,0,r)\in K_0$ if and only if $v(D)\geq -r$ and for each $\lambda\in S^\oplus$ and $u^*\in X^*$, there exists $\mu_0\in K^\oplus$ such that*

$$g_1^*(\lambda) - f_1^*(\mu_0) - (\mu_0 f_2)^*(u^*) + (\lambda g_2)^*(u^*) \geq -r. \tag{3.11}$$

Proof. Let $(0,0,0,r)\in K_0$ and $\lambda\in S^\oplus$, $u^*\in X^*$ be arbitrary. Then, there exist $\mu_0\in K^\oplus$ and $r_1,r_2\in\mathbb{R}$ such that

$$(0,0,0,r) = (u^*,-\mu_0,0,r_1) + (0,\mu_0,0,r_2) - (u^*,0,0,g_1^*(\lambda)+(\lambda g_2)^*(u^*)), \tag{3.12}$$

where

$$(u^*,r_1)\in \operatorname{epi}(\mu_0 f_2)^*,\quad (\mu_0,r_2)\in \operatorname{epi} f_1^* \quad\text{and}\quad r = r_1+r_2-g_1^*(\lambda)-(\lambda g_2)^*(u^*).$$

Thus,

(3.13)
$$-r = -r_1 - r_2 + g_1^*(\lambda) + (\lambda g_2)^*(u^*) \le -(\mu_0 f_2)^*(u^*) - f_1^*(\mu_0) + g_1^*(\lambda) + (\lambda g_2)^*(u^*),$$

and (3.11) is proven. Moreover, by (3.13), we see that

$$-r \le \sup_{\mu \in K^{\oplus}} \{g_1^*(\lambda) - f_1^*(\mu) - (\mu f_2)^*(u^*) + (\lambda g_2)^*(u^*)\}$$

and by the arbitrariness of λ and u^*,

$$-r \le \inf_{\lambda \in S^{\oplus}, u^* \in X^*} \sup_{\mu \in K^{\oplus}} \{g_1^*(\lambda) - f_1^*(\mu) - (\mu f_2)^*(u^*) + (\lambda g_2)^*(u^*)\}.$$

This together with the the definition of $v(D)$ implies that $v(D) \ge -r$.

Conversely, suppose that $v(D) \ge -r$ and for each $\lambda \in S^{\oplus}$ and $u^* \in X^*$, there exists $\mu_0 \in K^{\oplus}$ satisfying (3.11). Let $\lambda \in S^{\oplus}$ and $u^* \in X^*$. Then, there exists $\mu_0 \in K^{\oplus}$ such that (3.11) holds. Denote $r_1 := (\mu_0 f_2)^*(u^*)$ and $r_2 := r + g_1^*(\lambda) + (\lambda g_2)^*(u^*) - r_1$. Then, $(u^*, r_1) \in \operatorname{epi}(\mu_0 f_2)^*$ and $(\mu_0, r_2) \in \operatorname{epi} f_1^*$. This implies that

$$\begin{aligned}(0,0,0,r) \in \{(u^*, -\mu_0, 0, r_1) : (u^*, r_1) \in \operatorname{epi}(\mu_0 f_2)^*\} + \{(0, \mu_0, 0, r_2) : (\mu_0, r_2) \in \operatorname{epi} f_1^*\} \\ - (u^*, 0, 0, g_1^*(\lambda) + (\lambda g_2)^*(u^*)),\end{aligned}$$

and

$$\begin{aligned}(0,0,0,r) \in \bigcup_{\mu \in K^{\oplus}} \Big(&\{(u^*, -\mu, 0, r_1) : (u^*, r_1) \in \operatorname{epi}(\mu f_2)^*\} \\ &+ \{(0, \mu, 0, r_2) : (\mu, r_2) \in \operatorname{epi} f_1^*\} \\ &- (u^*, 0, 0, g_1^*(\lambda) + (\lambda g_2)^*(u^*))\Big).\end{aligned}$$

Therefore, by the arbitrariness of λ and u^*, we see that $(0,0,0,r) \in K_0$, which completes the proof. □

The following theorem characterizes completely the weak duality between (P) and (D).

Theorem 3.5. *The weak duality holds between (P) and (D) if and only if the family (F, G_1, G_2) satisfies*

(3.14)
$$K_0 \subseteq \operatorname{epi}(F - G_1 + G_2)^*.$$

Proof. Suppose that the weak duality holds between (P) and (D), that is, $v(P) \ge v(D)$. Let $r \in \mathbb{R}$ and $(0,0,0,r) \in K_0$. Then, by Lemma 3.4, $v(D) \ge -r$ and $v(P) \ge -r$ by the weak duality between (P) and (D). Thus, by Lemma 3.3, one sees that $(0,0,0,r) \in \operatorname{epi}(F - G_1 + G_2)^*$. Therefore, (3.14) holds.

Conversely, suppose that (3.14) holds. To show $v(P) \ge v(D)$, suppose on the contrary that $v(P) < v(D)$. Then, there exists $r \in \mathbb{R}$ such that $v(P) < -r < v(D)$. By the definition of $v(D)$, we have that for each $\lambda \in S^{\oplus}$ and $u^* \in X^*$, there exists $\mu_0 \in K^{\oplus}$ such that (3.11) holds. Thus, by Lemma 3.4, one sees that $(0,0,0,r) \in K_0$, and then $(0,0,0,r) \in \operatorname{epi}(F - G_1 + G_2)^*$ (thanks to (3.14)). This together with Lemma 3.3 implies that $-r \le v(P)$, which contradicts with $v(P) < -r$. Consequently, we have $v(P) \ge v(D)$ and the proof is complete. □

The following theorem provides a characterization for the zero duality to hold between (P) and (D).

Theorem 3.6. *The zero duality holds between (P) and (D) if and only if the family (F, G_1, G_2) satisfies*

$$\operatorname{cl} K_0 = \operatorname{epi}(F - G_1 + G_2)^* \cap (\{(0,0,0)\} \times \mathbb{R}). \tag{3.15}$$

Proof. Suppose that the zero duality holds between (P) and (D), that is, $v(P) = v(D)$. Then, by Theorem 3.5, (3.14) holds and hence

$$\operatorname{cl} K_0 \subseteq \operatorname{epi}(F - G_1 + G_2)^*, \tag{3.16}$$

since $\operatorname{epi}(F - G_1 + G_2)^*$ is w*-closed. This together with (3.10) implies that

$$\operatorname{cl} K_0 \subseteq \operatorname{epi}(F - G_1 + G_2)^* \cap (\{(0,0,0)\} \times \mathbb{R}). \tag{3.17}$$

To show (3.15), it remains to show that the converse inclusion of (3.17) holds. To do this, let $(0,0,0,r) \in \operatorname{epi}(F - G_1 + G_2)^*$. Then, by Lemma 3.3, $v(P) \geq -r$ and $v(D) \geq -r$ by the zero duality between (P) and (D). Let $\varepsilon > 0$. Then, for each $\lambda \in S^\oplus$ and $u^* \in X^*$, there exists $\mu_0 \in K^\oplus$ such that

$$-r - \varepsilon \leq g_1^*(\lambda) - f_1^*(\mu_0) - (\mu_0 f_2)^*(u^*) + (\lambda g_2)^*(u^*),$$

which implies that $(0,0,0,r+\varepsilon) \in K_0$, thanks to Lemma 3.4. Hence, $(0,0,0,r) \in \operatorname{cl} K_0$ and

$$\operatorname{epi}(F - G_1 + G_2)^* \cap (\{(0,0,0)\} \times \mathbb{R}) \subseteq \operatorname{cl} K_0.$$

This together with (3.17) implies that the (3.15) holds.

Conversely, suppose that the family (F, G_1, G_2) satisfies (3.15). Then, the family (F, G_1, G_2) satisfies (3.14) and so $v(P) \geq v(D)$ by Theorem 3.5. To show the converse inequality, suppose on the contrary that $v(D) < v(P)$. Then, there exists $r \in \mathbb{R}$ such that $v(D) < -r < v(P)$. Thus, by Lemma 3.3, $(0,0,0,r) \in \operatorname{epi}(F - G_1 + G_2)^*$. This together with (3.15) implies that $(0,0,0,r) \in \operatorname{cl} K_0$. Therefore, there exists a net $\{(0,0,0,r_n)\} \subseteq K_0$ such that $r_n \to r$. Hence, by Lemma 3.4, for each $\lambda \in S^\oplus$ and $u^* \in X^*$, there exists $\mu_0 \in K^\oplus$ such that

$$g_1^*(\lambda) - f_1^*(\mu_0) - (\mu_0 f_2)^*(u^*) + (\lambda g_2)^*(u^*) \geq -r_n \to -r.$$

This together with the definition of $v(D)$ implies that $v(D) \geq -r$, which contradicts with $v(D) < -r$. Hence, $v(P) = v(D)$ and the proof is complete. □

Theorem 3.7. *The following statements are equivalent:*

(i) *The strong duality holds between (P) and (D).*

(ii) *$v(P) = v(D)$ and for each $\lambda \in S^\oplus$ and $u^* \in X^*$, there exists $\mu_0 \in K^\oplus$ satisfying*

$$g_1^*(\lambda) - f_1^*(\mu_0) - (\mu_0 f_2)^*(u^*) - (\lambda g_2)^*(u^*) \geq v(D). \tag{3.18}$$

(iii) *The family (F, G_1, G_2) satisfies*

$$K_0 = \operatorname{epi}(F - G_1 + G_2)^* \cap (\{(0,0,0)\} \times \mathbb{R}). \tag{3.19}$$

Proof. (i) $\Rightarrow$ (ii) It follows from the definition of the strong duality.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Let $\lambda \in S^{\oplus}$ and $u^* \in X^*$. Then, $v(D) = v(P)$ and there exists $\mu_0 \in K^{\oplus}$ satisfying (3.18). Thus, by Theorem 3.5, (3.14) holds. Therefore, by (3.10), we only need to show that the set on the right-hand side of (3.19) is contained in the set on the left-hand side. To do this, let $(0,0,0,r) \in \operatorname{epi}(F - G_1 + G_2)^*$. Then, by Lemma 3.3, we have $-r \leq v(P)$. Therefore, $-r \leq v(D)$ and $\mu_0 \in K^{\oplus}$ satisfies (3.18). This together with Lemma 3.4 implies that $(0,0,0,r) \in K_0$ as $\lambda \in S^{\oplus}$ and $u^* \in X^*$ are arbitrary. Thus, $\operatorname{epi}(F - G_1 + G_2)^* \cap (\{(0,0,0)\} \times \mathbb{R}) \subseteq K_0$, and this completes the proof of the implication (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) Suppose that the family (F, G_1, G_2) satisfies (3.19). Then, the family (F, G_1, G_2) satisfies the (3.14), and so $v(P) \geq v(D)$ by Theorem 3.5. Thus, to prove the strong duality, by Definition 3.1(iii), it suffices to show that $v(D) \geq v(P)$ and for any $\lambda \in S^{\oplus}$ and $u^* \in X^*$ satisfying $v(D) = v(D^{(\lambda,u^*)})$, there exists $\mu_0 \in K^{\oplus}$ such that μ_0 is the optimal solution of the problem $(D^{(\lambda,u^*)})$. Note that the conclusion holds trivially if $v(P) = -\infty$. Below we consider only in the case when $-r := v(P) \in \mathbb{R}$. By Lemma 3.3, $(0,0,0,r) \in \operatorname{epi}(F - G_1 + G_2)^*$, and hence $(0,0,0,r) \in K_0$ thanks to (3.19). Thus, by Lemma 3.4, we have that $v(D) \geq -r$ and for each $\lambda \in S^{\oplus}$ and $u^* \in X^*$, there exists $\mu_0 \in K^{\oplus}$ satisfying (3.11). Hence, $v(P) = v(D)$ and for any $\lambda \in S^{\oplus}$ and $u^* \in X^*$ satisfying $v(D) = v(D^{(\lambda,u^*)})$, μ_0 is the optimal solution of the problem $(D^{(\lambda,u^*)})$. The proof is complete. $\square$

The remainder of this section is devoted to study the total duality between (P) and (D). For this purpose, let $S(P)$ denote the optimal solution set of (P). It is said that the total duality holds between (P) and (D) if the strong duality holds between (P) and (D) provided that $S(P) \neq \emptyset$.

Theorem 3.8. *Let $x_0 \in S(P)$. Suppose that the weak duality holds, and for each $u^* \in X^*$, there exists $\mu_0 \in \partial f_1(f_2(x_0)) \cap K^{\oplus}$ such that $u^* \in \partial(\mu_0 f_2)(x_0)$. Then, the strong duality holds between (P) and (D).*

Proof. Let $u^* \in X^*$ be arbitrary. Then, there exists $\mu_0 \in \partial f_1(f_2(x_0)) \cap K^{\oplus}$ such that $u^* \in \partial(\mu_0 f_2)(x_0)$. By Young equality (2.3), we have that

$$(\mu_0 f_2)(x_0) + (\mu_0 f_2)^*(u^*) = \langle u^*, x_0 \rangle, \tag{3.20}$$

and

$$f_1(f_2(x_0)) + f_1^*(\mu_0) = \langle \mu_0, f_2(x_0) \rangle. \tag{3.21}$$

Let $\lambda \in S^{\oplus}$ be arbitrary. By the Young-Fenchel inequality (2.1), one has that

$$g_1^*(\lambda) + g_1(g_2(x_0)) \geq \langle \lambda, g_2(x_0) \rangle \quad \text{and} \quad (\lambda g_2)^*(u^*) + \langle \lambda, g_2(x_0) \rangle \geq \langle u^*, x_0 \rangle. \tag{3.22}$$

Combing this with (3.20), (3.21) and (3.22), we have that

$$\begin{aligned}
& g_1^*(\lambda) - f_1^*(\mu_0) - (\mu_0 f_2)^*(u^*) + (\lambda g_2)^*(u^*) \\
= \ & g_1^*(\lambda) - f_1^*(\mu_0) + (\mu_0 f_2)(x_0) - \langle u^*, x_0 \rangle + (\lambda g_2)^*(u^*) \\
= \ & g_1^*(\lambda) + (f_1 \circ f_2)(x_0) - \langle u^*, x_0 \rangle + (\lambda g_2)^*(u^*) \\
\geq \ & \langle \lambda, g_2(x_0) \rangle - g_1(g_2(x_0)) + f_1(f_2(x_0)) - \langle u^*, x_0 \rangle + (\lambda g_2)^*(u^*) \\
\geq \ & f_1(f_2(x_0)) - g_1(g_2(x_0)) \\
= \ & v(P),
\end{aligned}$$

where the last equality holds because of $x_0 \in S(P)$. Thus, by the definition of $v(D)$, we see that $v(D) \geq v(P)$. This together with the weak duality between (P) and (D) implies that $v(D) = v(P)$ and for each $\lambda \in S^{\oplus}$ and $u^* \in X^*$, there exists $\mu_0 \in K^{\oplus}$ such that (3.18) holds. It follows from Theorem 3.7 that the strong duality holds and the proof is complete. □

4. The special cases

In this section, we will give some special cases of our general results. Recall that Id_X denotes the identity operator on X. As before, we assume that f_1, f_2, K, S, g_1, g_2 are the same as in Section 3, that is, $K \subseteq Y$ is a closed convex cone, $f_1 : Y \to \bar{\mathbb{R}}$ is a proper, convex and K-increasing function, $f_2 : X \to Y^{\bullet}$ is a proper, K-convex function, $S \subseteq Z$ is a closed convex cone, $g_1 : Z \to \bar{\mathbb{R}}$ is a proper, convex, S-increasing function, and $g_2 : X \to Z^{\bullet}$ is a proper, S-convex function such that $f_1 \circ f_2 - g_1 \circ g_2$ is proper.

4.1. **The case $g_2 = \mathrm{Id}_X$.** Let $X = Z$ and $g_2 = \mathrm{Id}_X$. Then, the problem defined by (3.1) reduced into the following optimization problem:

$$(P_1) \qquad \inf_{x \in X} \{(f_1 \circ f_2)(x) - g_1(x)\}.$$

Note that for each $u^* \in X^*$ and $\lambda \in S^{\oplus}$,

$$(\lambda g_2)^*(u^*) = \begin{cases} 0, & \lambda = u^*, \\ +\infty, & \text{otherwise}. \end{cases}$$

Then, the dual problem defined by (3.2) becomes

$$(D_1) \qquad \inf_{u^* \in X^*} \sup_{\mu \in K^{\oplus}} \{-f_1^*(\mu) - (\mu f_2)^*(u^*) + g_1^*(u^*)\}.$$

Moreover, the corresponding set defined by (3.9) can be expressed as

$$\begin{aligned} K_1 \ := \ & \bigcap_{u^* \in X^*} \bigcup_{\mu \in K^{\oplus}} \Big(\{(u^*, -\mu, r) : (u^*, r) \in \operatorname{epi}(\mu f_2)^*\} \\ & + \{(0, \mu, r) : (\mu, r) \in \operatorname{epi} f_1^*\} - (u^*, 0, g_1^*(u^*)) \Big). \end{aligned}$$

Let $\tilde{F}, \tilde{G}_1, \tilde{G}_2 : X \times Y \to \bar{\mathbb{R}}$ be defined by

$$\tilde{F}(x, y) := f_1(y), \tag{4.1}$$

$$\tilde{G}_1(x, y) := g_1(x), \tag{4.2}$$

and

$$\tilde{G}_2(x, y) := \delta_{\{(x,y) \in X \times Y : f_2(x) - y \in -K\}}(x, y). \tag{4.3}$$

Then, by Theorems 3.5,3.6 and 3.7, we can get the following theorems straightforwardly.

Theorem 4.1. *The weak duality holds between (P_1) and (D_1) if and only if the family $(\tilde{F}, \tilde{G}_1, \tilde{G}_2)$ satisfies*

$$K_1 \subseteq \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2)^*. \tag{4.4}$$

Theorem 4.2. *The zero duality holds between* (P_1) *and* (D_1) *if and only if the family* $(\tilde{F}, \tilde{G}_1, \tilde{G}_2)$ *satisfies*

$$\operatorname{cl} K_1 = \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2)^* \cap (\{(0,0)\} \times \mathbb{R}).$$

Theorem 4.3. *The strong duality holds between* (P_1) *and* (D_1) *if and only if the family* $(\tilde{F}, \tilde{G}_1, \tilde{G}_2)$ *satisfies*

$$K_1 = \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2)^* \cap (\{(0,0)\} \times \mathbb{R}). \tag{4.5}$$

Furthermore, we consider the following composite optimization problem:

$$(P_2) \qquad \inf_{x \in X} \{(f_1 \circ f_2)(x) + (h \circ A)(x) - g_1(x)\},$$

where E is a locally convex Hausdorff topological vector space with E^* is its dual space, $h : E \to \bar{\mathbb{R}}$ is a proper convex function and $A : X \to E$ is a linear continuous mapping. Assume that $A(\operatorname{dom}(f_1 \circ f_2) \cap \operatorname{dom} g_1) \cap \operatorname{dom} h \neq \emptyset$. Following [18], we define the dual problem of (P_2) by

$$(D_2) \qquad \inf_{u^* \in X^*} \sup_{e^* \in E^*, \mu \in K^\oplus} \{g_1^*(u^*) - f_1^*(\mu) - (\mu f_2)^*(u^* - A^* e^*) - h^*(e^*)\}.$$

To discuss the dualities between (P_2) and its dual problem (D_2), we need to introduce some new regularity conditions. To this end, we shall consider the identify operator $\mathrm{Id}_{\mathbb{R}}$ on $\mathbb{R}$, and the image set $(A^* \times \mathrm{Id}_{\mathbb{R}})(Z)$ of a set $Z \subseteq E^* \times \mathbb{R}$ through the map $A^* \times \mathrm{Id}_{\mathbb{R}} : E^* \times \mathbb{R} \to X^* \times \mathbb{R}$, that is,

$$(x^*, r) \in (A^* \times \mathrm{Id}_{\mathbb{R}})(Z) \Leftrightarrow \begin{cases} \exists\ e^* \in E^* \text{ such that } (e^*, r) \in Z \\ \text{and } A^* e^* = x^*. \end{cases}$$

Let $r \in \mathbb{R}$, we can also denote a set K_2 by

$$\begin{aligned} K_2 := \bigcap_{u^* \in X^*} \Big(\bigcup_{\mu \in K^\oplus} &(\{(u^*, -\mu, r) : (u^*, r) \in \operatorname{epi}(\mu f_2)^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} h^*)\} \\ &+ \{(0, \mu, r) : (\mu, r) \in \operatorname{epi} f_1^*\}) - (u^*, 0, g_1^*(u^*)) \Big), \end{aligned}$$

and let $\tilde{G}_3 : X \times Y \to \bar{\mathbb{R}}$ be defined by

$$\tilde{G}_3(x, y) := (h \circ A)(x).$$

Recall that $\tilde{F}, \tilde{G}_1,$ and $\tilde{G}_2$ are the same as (4.1), (4.2) and (4.3), respectively. Then, for each $r \in \mathbb{R}$, we get that

$$(0, r) \in \operatorname{epi}(f_1 \circ f_2 + h \circ A - g_1)^* \Longleftrightarrow (0, 0, r) \in \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3)^*.$$

Similar to the proof of the Theorems 3.5,3.6 and 3.7, we can obtain the following theorems.

Theorem 4.4. *The weak duality holds between* (P_2) *and* (D_2) *if and only if the family* $(\tilde{F}, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3)$ *satisfies*

$$K_2 \subseteq \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3)^*. \tag{4.6}$$

Theorem 4.5. *The zero duality holds between* (P_2) *and* (D_2) *if and only if the family* $(\tilde{F}, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3)$ *satisfies*

$$\operatorname{cl} K_2 = \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3)^* \cap (\{(0,0)\} \times \mathbb{R}).$$

Theorem 4.6. *The strong duality holds between* (P_2) *and* (D_2) *if and only if the family* $(\tilde{F}, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3)$ *satisfies*

$$K_2 = \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3)^* \cap (\{(0,0)\} \times \mathbb{R}). \tag{4.7}$$

Remark 4.7. In [18], the authors concerned with the following composite optimization problem:

$$(P_3) \quad \inf_{x \in \Omega} \{(f_1 \circ f_2) + (h \circ A)(x) - g_1(x)\},$$

where $\Omega := \{x \in X : p(x) \in -S\}$, $S \subseteq Z$ is a closed convex cone and $p : X \to Z^\bullet$ is a proper, S-convex and S-epi-closed mapping. Under the assumption that

$$f_1, h, g_1 \text{ are lsc and } f_2 \ \text{ is } K\text{-epi-closed}, \tag{4.8}$$

and the following closure condition:

$$(CQ) \quad \bigcup_{\lambda \in S^\oplus, \mu \in K^\oplus} \{(u^*, -\mu, r) : (\lambda p + \mu f_2)^*(u^*) \leq r\} + \{0\} \times \operatorname{epi}(f_1^*)$$
$$+ \{(u^*, 0, r) : (u^*, r) \in (A^* \times \mathrm{Id}_\mathbb{R})(\operatorname{epi} h^*)\} \text{ is closed regarding } X^* \times \{0\} \times \mathbb{R},$$

they established in [18, Theorem 4.1] the strong duality between (P_3) and its dual problem

$$(D_3) \quad \inf_{u^* \in X^*} \sup_{\lambda \in S^\oplus, \mu \in K^\oplus, e^* \in E^*} \{g_1^*(u^*) - f_1^*(\mu) - (\lambda \circ p + \mu \circ f_2)^*(u^* - A^* e^*) - h^*(e^*)\}.$$

Note that in the case when $\Omega = X$, (P_3) is reduced to (P_2), (D_3) is reduced to (D_2) and (CQ) is reduced to

$$(CQ1) \quad \bigcup_{\mu \in K^\oplus} \{(u^*, -\mu, r) : (\mu f_2)^*(u^*) \leq r\} + \{(u^*, 0, r) : (u^*, r) \in (A^* \times \mathrm{Id}_\mathbb{R})(\operatorname{epi} h^*)\}$$
$$+ \{0\} \times \operatorname{epi}(f_1^*) \text{ is closed regarding } X^* \times \{0\} \times \mathbb{R}.$$

For simplicity, we denote

$$\begin{aligned} K_3 \ := \ & \bigcup_{\mu \in K^\oplus} \{(u^*, -\mu, r) : (\mu f_2)^*(u^*) \leq r\} \\ & + \{(u^*, 0, r) : (u^*, r) \in (A^* \times \mathrm{Id}_\mathbb{R})(\operatorname{epi} h^*)\} \\ & + \{0\} \times \operatorname{epi}(f_1^*). \end{aligned}$$

The following proposition shows that our Theorem 4.6 improves the corresponding result in [18, Theorem 4.1] in the case when $\Omega = X$.

Proposition 4.8. *If* (4.8) *holds, then the following implication holds:*

$$(CQ1) \Longrightarrow (4.7).$$

Proof. Suppose that (4.8) and (CQ_1) hold. Since f_2 is proper convex and K-epi-closed, f_1, h, g_1 are proper convex and lsc, it follows that $\tilde{F}, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3$ are proper, convex and lsc, and

$$\operatorname{epi} \tilde{F}^* = \{0\} \times \operatorname{epi} f_1^*, \tag{4.9}$$

$$\operatorname{epi} \tilde{G}_2^* = \operatorname{cl} \bigcup_{\mu \in K^\oplus} \{(u^*, -\mu, r) : (\mu f_2)^*(u^*) \le r\}, \tag{4.10}$$

$$\operatorname{epi} \tilde{G}_3^* = \operatorname{cl}\{(u^*, 0, r) : (u^*, r) \in (A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} h^*)\},$$

where (4.10) follows from [3, Proposition 3.1] and the last equality holds by [2, Lemma 1]. Then, by Lemma 2.1(i), we get that

$$\begin{aligned} \operatorname{epi}(\tilde{F} + \tilde{G}_2 + \tilde{G}_3)^* &= \operatorname{cl}(\operatorname{epi} \tilde{F}^* + \operatorname{epi} \tilde{G}_2^* + \operatorname{epi} \tilde{G}_3^*) \\ &= \operatorname{cl} K_3 \\ &= K_3, \end{aligned}$$

where the last equality follows from $(CQ1)$. Note that $\tilde{G}_1$ is lsc, it follows from (2.4) that

$$\begin{aligned} &\operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3)^* \bigcap (\{(0,0)\} \times \mathbb{R}) \\ &= \bigcap_{u^* \in X^*} \Big(\operatorname{epi}(\tilde{F} + \tilde{G}_2 + \tilde{G}_3)^* - (u^*, 0, g_1^*(u^*))\Big) \bigcap (\{(0,0)\} \times \mathbb{R}) \\ &= \bigcap_{u^* \in X^*} \Big(K_3 - (u^*, 0, g_1^*(u^*))\Big) \bigcap (\{(0,0)\} \times \mathbb{R}) \\ &= K_2 \bigcap (\{(0,0)\} \times \mathbb{R}). \end{aligned}$$

This together with the fact $K_2 \subseteq \{0\} \times \{0\} \times \mathbb{R}$ implies that (4.7) holds. The proof is complete. □

4.2. **The case $g_1 \in X^*$ and $g_2 = \mathrm{Id}_X$.** In the case when $g_2 = \mathrm{Id}_X$ and $g_1 := p \in X^*$, then, the problem defined by (3.1) is reduced into the following composite optimization problem:

$$(P_4) \qquad \inf_{x \in X} \{(f_1 \circ f_2)(x) - \langle p, x \rangle\}.$$

Note that for each $u^* \in X^*$ and $\lambda \in S^\oplus$,

$$g_1^*(\lambda) = \begin{cases} 0, & \lambda = p, \\ +\infty, & \text{otherwise}, \end{cases} \quad \text{and} \quad (\lambda g_2)^*(u^*) = \begin{cases} 0, & u^* = \lambda, \\ +\infty, & \text{otherwise}. \end{cases}$$

Then, the dual problem defined by (3.2) becomes

$$(D_4) \qquad \sup_{\mu \in K^\oplus} \{-f_1^*(\mu) - (\mu f_2)^*(p)\},$$

and the corresponding set defined by (3.9) can be expressed as

$$K_4 := \bigcup_{\mu \in K^\oplus} \Big(\{(p, -\mu, r) : (p, r) \in \operatorname{epi}(\mu f_2)^*\} + \{(0, \mu, r) : (\mu, r) \in \operatorname{epi} f_1^*\} - (p, 0, 0)\Big).$$

Note by (2.1) that for each $p \in X^*$, $x \in X$ and $\mu \in K^{\oplus}$,

$$-f_1^*(\mu) - (\mu f_2)^*(p) \le f_1(f_2(x)) - \langle \mu, f_2(x) \rangle + (\mu f_2)(x) - \langle p, x \rangle = f_1(f_2(x)) - \langle p, x \rangle.$$

It follows that $v(D_4) \le v(P_4)$, that is, the weak duality holds between (P_4) and (D_4). Define $\tilde{G}_1 : X \times Y \to \bar{\mathbb{R}}$ by

$$\tilde{G}_1(x, y) = p(x)$$

and recall that $\tilde{F}, \tilde{G}_2$ are defined respectively by (4.1) and (4.3). Then, by Theorems 4.2 and 4.3, we get the following theorems directly.

Theorem 4.9. *The zero duality holds between* (P_4) *and* (D_4) *if and only if the family* $(\tilde{F}, \tilde{G}_1, \tilde{G}_2)$ *satisfies*

$$\operatorname{cl} K_4 = \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2)^* \cap (\{(0,0)\} \times \mathbb{R}).$$

Theorem 4.10. *The strong duality holds between* (P_4) *and* (D_4) *if and only if the family* $(\tilde{F}, \tilde{G}_1, \tilde{G}_2)$ *satisfies*

$$K_4 = \operatorname{epi}(\tilde{F} - \tilde{G}_1 + \tilde{G}_2)^* \cap (\{(0,0)\} \times \mathbb{R}). \tag{4.11}$$

Remark 4.11. Under the assumption that

$$f_1 \text{ is lsc and } f_2 \text{ is } K\text{-epi-closed, ,} \tag{4.12}$$

the authors in [3, Theorem 5.1] established the strong duality between (P_4) and (D_4) via the following closure condition

$$\text{(CQ2)} \quad \bigcup_{\mu \in K^{\oplus}} \{(x^*, -\mu, r) : (x^*, r) \in \operatorname{epi}(\mu f_2)^*\} + \{0\} \times \operatorname{epi} f_1^* \text{ is closed regarding the subspace } X^* \times \{0\} \times \mathbb{R}.$$

Then, by the following Proposition 4.12, one sees that our Theorem 4.10 improves the corresponding result in [3].

Proposition 4.12. *If* (4.12) *holds, then*

$$(CQ2) \Rightarrow (4.11).$$

Proof. Suppose that (4.12) and $(CQ2)$ hold. It is easy to see that the equalities (4.9) and (4.10) are also hold. Since $\tilde{F}, \tilde{G}_2$ are proper, convex and lsc functions, it follows from Lemma 2.1(i) that

$$\begin{aligned} \operatorname{epi}(\tilde{F} + \tilde{G}_2)^* &= \operatorname{cl}(\operatorname{epi} \tilde{F}^* + \operatorname{epi} \tilde{G}_2^*) \\ &= \operatorname{cl}\Big(\bigcup_{\mu \in K^{\oplus}} \{(u^*, -\mu, r) : (\mu f_2)^*(u^*) \le r\} + \{0\} \times \operatorname{epi}(f_1^*) \Big) \\ &= \bigcup_{\mu \in K^{\oplus}} \{(u^*, -\mu, r) : (\mu f_2)^*(u^*) \le r\} + \{0\} \times \operatorname{epi} f_1^*, \end{aligned}$$

where the last equality follows from $(CQ2)$. Note that $\tilde{G}_1$ is lsc and

$$\tilde{G}_1^*(x^*, y^*) = \begin{cases} 0, & \text{if } (x^*, y^*) = (p, 0), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, by (2.4), we have that

$$\begin{aligned}
&\operatorname{epi}(\tilde{F}-\tilde{G}_1+\tilde{G}_2)^* \bigcap(\{(0,0)\}\times\mathbb{R})\\
&= \bigcap_{x^*\in X^*, y^*\in Y^*}\Big(\operatorname{epi}(\tilde{F}+\tilde{G}_2)^* - (x^*, y^*, \tilde{G}_1^*(x^*,y^*))\Big)\bigcap(\{(0,0)\}\times\mathbb{R})\\
&= \Big(\bigcup_{\mu\in K^{\oplus}}\{(u^*,-\mu,r) : (\mu f_2)^*(u^*)\le r\} + \{0\}\times\operatorname{epi}(f_1^*) - (p,0,0)\Big)\bigcap(\{(0,0)\}\times\mathbb{R})\\
&= K_4\bigcap(\{(0,0)\}\times\mathbb{R}).
\end{aligned}$$

This together with the fact $K_4 \subseteq \{0\}\times\{0\}\times\mathbb{R}$ implies that (4.11) holds. The proof is complete. □

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Manuscript received September 6, 2014
revised December 10, 2014

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