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NECESSARY OPTIMALITY CONDITIONS FOR A CLASS OF SEMILINEAR ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH PURE STATE CONSTRAINTS AND MIXED POINTWISE CONSTRAINTS

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Abstract. This paper deals with first- and second-order necessary optimality conditions for a class of optimal control problems governed by semilinear elliptic equations with pure state constraints and mixed state-control constraints. The second-order necessary optimality conditions are established by using tools of variational analysis.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and the boundary Γ of class C^1. We consider the optimal control problem of finding a control function $u \in L^p(\Omega)$ with $1 < p < +\infty$ and a corresponding state $y \in W_0^{1,r}(\Omega)$ such that

$$J(y,u) = \int_\Omega L(x, y(x), u(x))dx \to \inf \tag{1.1}$$

subject to

$$-\sum_{i,j=1}^{N} D_j(a_{ij}(x)D_i y) + h(x,y) = u \quad \text{in} \quad \Omega, \quad y|_\Gamma = 0, \tag{1.2}$$

$$\alpha(x) \leq f(x, y(x)) + u(x) \leq \beta(x) \quad \text{a.e.} \quad x \in \Omega \tag{1.3}$$

$$g(x, y(x)) \leq 0 \quad \text{for all} \quad x \in \overline{\Omega}, \tag{1.4}$$

where $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous mappings, $\alpha, \beta \in L^p(\Omega)$.

Problem (1.1)-(1.4) is quite general. When $f = 0$ and $g = 0$, it becomes a problem with control constraint $\alpha \leq u \leq \beta$. A prototype of this problem is an optimal heat source problem in the domain Ω, where Ω is heated by microwave and the integrand has the form

$$L(x, y, u) = \frac{1}{2}|y - y_\Omega(x)|^2 + \frac{\gamma}{2}|u|^2$$

with $y_\Omega \in L^2(\Omega)$ and $\gamma > 0$. Necessary and sufficient optimality conditions for this problem have been studied in details in [8] and [38]. When $f = 0$, problem (1.1)-(1.4) becomes a problem with pure state constraints and control constraints. Second-order sufficient optimality conditions and necessary optimality conditions

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for the class of this problem have been studied by [9] and [25], respectively. When $g = 0$, then (1.1)-(1.4) becomes a problem with mixed pointwise constraints. Under regularity conditions, second-order necessary optimality conditions and second-order sufficient optimality conditions for this problem have been studied recently by [24] and [26], respectively. For more information on optimal control problems governed by elliptic equations which have close connections to the present work, we refer the readers to [2, 7, 10, 33, 34, 38] and references therein.

Recently, in [4], J. F. Bonnans and A. Hermant have dealt with second-order necessary and sufficient conditions for optimal control problems governed by ordinary differential equations with pure state constraints and mixed control-state constraints. Particularly, [4] gave second-order sufficient conditions which have no gap with second-order necessary conditions under assumptions that the control is continuous and satisfies the strengthened Legendre-Clebsch condition. However, to our best knowledge, the difficult issue of necessary and sufficient optimality conditions for optimal control problems governed by semilinear elliptic equations with pure constraints and mixed pointwise constraints has not yet been studied. In this paper we will address this problem. Namely, we shall derive necessary optimality conditions for the problem under the so-called regularity condition which generalizes the Robinson constraint qualification condition.

To tackle the problem we first derive necessary optimality conditions for an abstract optimal control problem and then apply the obtained result to problem (1.1)-(1.4). For this we assume that Y, U and E are either reflexive Banach spaces or separable Banach spaces, Π is a Banach space and $Z = Y \times U$. We shall denote by Y^*, U^*, E^* and Π^* the dual spaces of Y, U, E and Π, respectively. Suppose that $Y \hookrightarrow C(\overline{\Omega})$ is continuous. We consider the abstract problem of finding $y \in Y$ and $u \in U$ such that

$$J(y,u) \longrightarrow \inf \tag{1.5}$$

subject to

$$\Phi(y,u) = 0 \tag{1.6}$$

$$F(y,u) \in D \tag{1.7}$$

$$G(y) \in Q, \tag{1.8}$$

where $\Phi : Y \times U \to \Pi$, $F : Y \times U \to E$ are given mappings, $G : Y \to C(\overline{\Omega})$ is a mapping which is defined by $G(y) = g(\cdot, y(\cdot))$ with $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a differentiable function; D is a closed convex set in E and Q is a closed convex set in $C(\overline{\Omega})$, which is given by

$$Q = \{\varphi \in C(\overline{\Omega}) \mid \varphi(x) \le 0 \text{ for all } x \in \overline{\Omega}\}.$$

Obviously, problem (1.1)-(1.4) is a special case of problem (1.5)-(1.8) whenever $Y = W_0^{1,r}(\Omega), U = L^p(\Omega)$ and

$$D = \{\phi \in L^p(\Omega) \mid \alpha(x) \le \phi(x) \le \beta(x) \text{ a.e.}\}. \tag{1.9}$$

To deal with first- and second-order necessary optimality conditions for problem (1.5)-(1.8), we often require that D has a nonempty interior. Hence we need $D \subset L^\infty(\Omega)$ and so the control variables u belong to $L^\infty(\Omega)$. For this approach, we refer the readers to Páles and Zeidan [30, 31, 32] on the second-order necessary optimality conditions for optimal control problems governed by ordinary differential equations, Casas et al [8] and [10] for sufficient second-order optimality conditions, Rösch et al [35] on regularity of solution for problems with semilinear elliptic equations and mixed pointwise constraints.

When control variable $u \in L^\infty(\Omega)$, then the mixed pointwise constraints (1.7) as well as (1.3) are easy to deal with. However, in this case, when the constraint set is unbounded, problem (1.1)-(1.4) may not have solution in $W_0^{1,r}(\Omega) \times L^\infty(\Omega)$. The reason is that the object function does not satisfy coercivity conditions. Besides, multipliers belong to dual space $L^\infty(\Omega)^*$ which are measures rather than functions in $L^1(\Omega)$. Therefore we would like to consider the problem when variables $(y, u) \in W_0^{1,r}(\Omega) \times L^p(\Omega)$ with $1 < p < +\infty$. But in this setting, we have $D \subset L^p(\Omega)$ with $1 \leq p < +\infty$ and the interior of D is empty (see [24, Example 1.1]). Therefore the results of [30, 31, 32] fail to apply for problem (1.1)-(1.4) when the control variable $u \in L^p(\Omega)$ with $1 < p < +\infty$. Note that the approach of [30, 31, 32] is based on a method which is due to Dubovitskii and Milyutin in [15], where the second-order variation sets of constraint sets are built under assumptions that they have nonempty interiors.

To overcome this difficulty, Bonnans and Shapiro [3], Cominetti [12] and Jourani [20] and [21] (see also Zowe et al [27] and [39]) gave necessary optimality conditions for the problem under regularity conditions which generalize the Robinson constraint qualification condition [37]. The results in [3] and [12] are for the problem, where the sets A, D and Q are required to be convex meanwhile results in [20] and [21] are proved by using Clarke tangent cones. Recently Kien et al [24]-[26], have obtained some results on necessary and sufficient optimality conditions for mathematical programming problems which extended results in [12, 20, 21] and then derived optimality conditions for a class of semilinear elliptic optimal control problems with mixed pointwise constraints and for a class of semilinear elliptic optimal control problems with pure state constraints, respectively.

In this paper, we continue to develop the results in [24] and [25] to derive optimality conditions for (1.1)-(1.4) under the regularity conditions. It is noted that the appearance of pure state constraint (1.8) and mixed pointwise constraint (1.7) causes the problem more complicated because the state does not link to control by a functional relation. In this case the results in [24] and [25] fail to apply for the current problem.

To deal with the class of these problems, we shall use tools of variational analysis and some techniques which were given in [25] and [30, 31] for establishing optimality conditions of problem (1.5)-(1.8) and then apply the obtained results to derive the first- and second-order optimality conditions for problem (1.1)-(1.4). Although the obtained result is modest, the contribution here is the approach which brings together two areas in which optimal control problems governed by partial differential equations can be considered as mathematical programming problems and vice versa, and a unified theory of first- and second-order optimality conditions. With the results, we hope that, we will be able to derive corresponding second-order sufficient conditions with no gap in our next study in the near future.

The paper is organized as follows. In Section 2, we establish first- and second-order necessary optimality conditions for the abstract optimal control problem. Section 3 is destined for second-order necessary optimality conditions for problem (1.1)-(1.4).

2. Necessary optimality conditions for the abstract optimal control problem

In this section, we derive optimality conditions for problem (1.5)-(1.8).

For convenience, we define the set

$$A = \{z = (y, u) \in Z \mid \Phi(z) = 0\}$$

and the mapping

$$G_1 : Y \times U \to C(\overline{\Omega}),\ G_1(y, u) = G(y).$$

Then problem (1.5)-(1.8) can be formulated in the following simpler form:

$$J(z) \to \inf \tag{2.1}$$

subject to

$$z \in A \cap F^{-1}(D), \tag{2.2}$$

$$G_1(z) \in Q. \tag{2.3}$$

To deal with optimal conditions we need some facts and concepts of variational analysis which are related to tangent cones. We refer the readers to [3, 13] and [36] on facts of variational analysis.

Let Z_1 be a Banach space and C be a closed set in Z_1. Given a point $z \in C$, the sets

$$\begin{aligned} T^\flat(C, z) &= \Big\{v \in Z_1 |\lim_{t\to 0^+} \frac{d(z + tv, C)}{t} = 0\Big\} \\ &= \{v \in Z_1 | \forall t_n \to 0^+, \exists v_n \to v, z + t_n v_n \in C\} \end{aligned}$$

and

$$\begin{aligned} T(C, z) &= \Big\{v \in Z_1 |\liminf_{t\to 0^+} \frac{d(z + tv, C)}{t} = 0\Big\} \\ &= \{v \in Z_1 | \exists t_n \to 0^+, \exists v_n \to v, z + t_n v_n \in C\} \end{aligned}$$

are called *the adjacent tangent cone* and *contingent cone* to C at z, respectively. It is noted that these cones are closed and we have $T^\flat(C, z) \subset T(C, z)$. We shall denote by $N(C, z)$ the normal cone to C at z, which is defined by

$$N(C, z) = \{z^* \in Z_1^* \mid \langle z^*, z'\rangle \le 0, \forall z' \in T(C, z)\}.$$

It is well known that when C is convex, then

$$T^\flat(C, z) = T(C, z) = \overline{\text{cone}}(C - z)$$

and

$$N(C, z) = \{z^* \in Z_1^* \mid \langle z^*, z' - z\rangle \le 0, \forall z' \in C\}.$$

Given a point $z \in C$ and $v \in Z_1$, the set

$$T^{2\flat}(C, z, v) = \Big\{w \in Z_1 |\lim_{t\to 0^+} \frac{d(z + tv + t^2 w, C)}{t^2} = 0\Big\}$$

$$= \{w \in Z_1 | \forall t_n \to 0^+, \exists w_n \to w, z + t_n v + t_n^2 w_n \in C\}$$

and

$$T^2(C, z, v) = \Big\{w \in Z_1 | \liminf_{t \to 0^+} \frac{d(z + tv + t^2 w, C)}{t^2} = 0\Big\}$$
$$= \{w \in Z_1 | \exists t_n \to 0^+, \exists w_n \to w, z + t_n v + t_n^2 w_n \in C\}$$

are called *the inner* and *outer second order tangent sets*, respectively, to C at z in the direction $v \in Z_1$. These sets are closed and we have $T^{2\flat}(C, z, v) \subset T^2(C, z, v)$. When C is convex, then $T^{2\flat}(C, z, v)$ is convex. Furthermore,

$$T^{2\flat}(C, z, 0) = T^\flat(C, z), \quad T^2(C, z, 0) = T(C, z).$$

Let us denote by $\mathcal{A}_{ad}$ the admissible set of problem (2.1)-(2.3). Fixing an element $\overline{z} = (\overline{y}, \overline{u}) \in \mathcal{A}_{ad}$, we suppose the following hypotheses:

($A1$) the mappings J, Φ, and F are second-order Fréchet differentiable around $\overline{z}$.
($A2$) $\nabla\Phi(\overline{z}) : Z \to \Pi$ is surjective.
($A3$) The function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and second-order differentiable with respect to second variable. Moreover, g has properties that $g(x, 0) < 0$ for all $x \in \Gamma$, and for each $M > 0$, there exists a number $k_M > 0$ such that

$$\big|g_y(x, y_1) - g_y(x, y_2)\big| + \big|g_{yy}(x, y_1) - g_{yy}(x, y_2)\big| \le k_M |y_1 - y_2|$$

for all $x \in \overline{\Omega}$, $y_1, y_2 \in \mathbb{R}$ with $|y_1|, |y_2| \le M$.
($A4$) The regularity condition is satisfied at $\overline{z}$, that is, there exists $\delta > 0$ such that

$$0 \in \text{int} \bigcap_{z \in B_Z(\overline{z}, \delta) \cap A} \big[\nabla F(\overline{z})(T(A, z) \cap B_Z) - (D - F(\overline{z})) \cap B_E\big], \tag{2.4}$$

where B_Z and B_E are closed unit balls in Z and E, respectively, and $B_Z(\overline{z}, \delta)$ is an open ball in Z with center $\overline{z}$ and radius $\delta > 0$.

Problem (1.5)-(1.8) is associated with the following Lagrangian:

$$\mathcal{L}(\lambda, \pi^*, e^*, \mu, z) = \lambda J(y, u) + \langle \pi^*, \Phi(y, u)\rangle + \langle e^*, F(y, u)\rangle + \langle \mu, G(y)\rangle \tag{2.5}$$

with $\lambda \in \mathbb{R}, \pi^* \in \Pi^*, e^* \in E^*$ and $\mu \in \mathcal{M}(\overline{\Omega})$, where $\mathcal{M}(\overline{\Omega})$ is the dual of $C(\overline{\Omega})$ which coincides with the space of finite signed regular Borel measures.

Below we shall formulate optimality conditions in terms of the Lagrangian. Before formulating the second-order optimality conditions, we need the following notion.

Definition 2.1. The set of critical directions to problem (2.1)-(2.3) at $\overline{z} = (\overline{y}, \overline{u})$ is denoted by $\Theta[(\overline{y}, \overline{u})]$, which consists of couples $z = (y, u)$ such that the following conditions are verified:

(i) $J_y(\overline{z})y + J_u(\overline{z})u \le 0$;
(ii) (the linearized equation) $\Phi_y(\overline{z})y + \Phi_u(\overline{z})u = 0$;
(iii) $\nabla F(\overline{z})(y, u) \in T^\flat(D, F(\overline{z}))$;
(iv) $\nabla G(\overline{y})y \in T^\flat(Q, G(\overline{y}))$.

From the definition, we see that $\Theta[(\overline{y},\overline{u})]$ is a closed convex cone, which contains tangent vectors (y,u) to $\mathcal{A}_{ad}$ at $\overline{z}$ such that $J_y(\overline{z})y + J_u(\overline{z})u \le 0$. By [23, Theorem 3.1], condition (iv) is equivalent to the fact that $g_y(x,\overline{y}(x))y(x) \le 0$ whenever $g(x,\overline{y}(x)) = 0$.

Recall that support of a nonnegative measure μ, written as $\mathrm{supp}(\mu)$, is the smallest closed subset of $\overline{\Omega}$ such that $\mu(\overline{\Omega}\setminus \mathrm{supp}(\mu)) = 0$. Below, given a set K in Z and $z^* \in Z^*$, we define $\sigma(z^*,K) = \sup_{z\in K}\langle z^*, z\rangle$.

We are ready to state the main result of this section.

Theorem 2.2. *Suppose that hypotheses* $(A1) - (A4)$ *are fulfilled and* $\overline{z} \in Z$ *is a locally optimal solution of problem* (1.5)-(1.8). *Then for each* $d = (y,u) \in \Theta[\overline{z}]$, *there exist multipliers* $\lambda \ge 0$, $\pi^* \in \Pi^*$, $e^* \in E^*$ *and a nonnegative Borel measure* $\mu \in \mathcal{M}(\overline{\Omega})$ *with* $|\lambda| + \|\mu\| \neq 0$ *such that the following conditions are fulfilled:*

(i) *(the stationary condition)*

$$D_z\mathcal{L}(\lambda,\pi^*,e^*,\mu,\overline{z}) = \lambda\nabla J(\overline{z}) + \nabla\Phi(\overline{z})^*\pi^* + \nabla F(\overline{z})^*e^* + \nabla G(\overline{y})^*\mu = 0;$$

(ii) *(the complementary condition in z)*

$$e^* \in N(D, F(\overline{z}));$$

(iii) *(the complementary condition in y)*

$$\mathrm{supp}(\mu) \subset \{x\in\Omega \mid g(x,\overline{y}(x)) = 0\};$$

(iv) *(the second-order condition)*

$$D^2_{zz}\mathcal{L}(\lambda,\pi^*,e^*,\mu,\overline{z})(d,d) - 2\sigma(e^*, T^{2\flat}(D,F(\overline{z}),\nabla F(\overline{z})d)) - 2\sigma(\mu, T^{2\flat}(Q,G(\overline{y}),\nabla G(\overline{y})y) \ge 0.$$

To prove the theorem, we need to establish some auxiliary lemmas.

Let us fix a couple $(y,u) \in \Theta[(\overline{y},\overline{u})]$ and define the functions on $\overline{\Omega}$

$$a(x) = g(x,\overline{y}(x)), \quad b(x) = g_y(x,\overline{y}(x))y(x),$$

$$\theta_{a,b}(x) = \begin{cases} \liminf_{\substack{s\to x\\ a(s)<0,b(s)>0}} \frac{b^2(s)}{4a(s)} & \text{if} \quad x\in\overline{\Omega}_{a=0,b=0}\cap\partial(\overline{\Omega}_{a<0,b>0}) \\ 0 & \text{if} \quad x\in\overline{\Omega}_{a=0,b=0}\setminus\partial(\overline{\Omega}_{a<0,b>0}) \\ +\infty & \text{otherwise.}\end{cases} \tag{2.6}$$

Here $\overline{\Omega}_{a=0,b=0} = \{x\in\overline{\Omega} \mid a(x) = b(x) = 0\}$ and $\partial(\overline{\Omega}_{a<0,b>0})$ is the boundary of the set $\{x\in\overline{\Omega} \mid a(x) < 0 < b(x)\}$. By Lemma 3.3 in [31], $\theta_{a,b}$ is a lower semicontinuous function on $\overline{\Omega}$. By [23, Theorem 3.2] (see also [31, Corollary 4.2]), we have $w \in T^{2\flat}(Q, g(\cdot,\overline{y}), g_y(\cdot,\overline{y})y)$ if and only if $w(x) \le \theta_{a,b}(x)$ for all $x\in\overline{\Omega}$. Let us define the set

$$T_0^{2\flat}(Q, g(\cdot,\overline{y}), g_y(\cdot,\overline{y})y) = \{\psi\in C(\overline{\Omega}) \mid \psi(x) < \theta_{a,b}(x)\ \forall x\in\overline{\Omega}\}. \tag{2.7}$$

It is clear that

$$\overline{T_0^{2\flat}(Q, g(\cdot,\overline{y}), g_y(\cdot,\overline{y})y)} = T^{2\flat}(Q, g(\cdot,\overline{y}), g_y(\cdot,\overline{y})y). \tag{2.8}$$

We have the following lemma, its proof can be found in the appendix of the paper.

Lemma 2.3. (C.f. [25, Lemma 2.1]) Suppose that hypotheses $(A1)-(A4)$ are fulfilled and $d=(y,u)\in\Theta[(\overline{y},\overline{u})]$. If

$$(\psi,\omega)\in T^{2\flat}(A,\overline{z},d)\cap\Big[\nabla F(\overline{z})^{-1}\big(T^{2\flat}(D,F(\overline{z}),\nabla F(\overline{z})d)-\frac{1}{2}\nabla^2F(\overline{z})d^2\big)\Big]$$

and

$$\nabla G(\overline{y})\psi+\frac{1}{2}\nabla^2G(\overline{y})y^2\in T_0^{2\flat}(Q,G(\overline{y}),\nabla G(\overline{y})y),$$

then for any sequence $t_k\to 0^+$, there exist sequences $\psi_k\to\psi$, $\omega_k\to\omega$ and a number $k_0>0$ such that

$$(\overline{y}+t_ky+t_k^2\psi_k,\overline{u}+t_ku+t_k^2\omega_k)\in\mathcal{A}_{ad},\ \forall k\geq k_0.$$

The following result is a primal form of second-order necessary optimality conditions.

Lemma 2.4. Suppose that assumptions $(A1)-(A4)$ are satisfied and $\overline{z}=(\overline{y},\overline{u})$ is a locally optimal solution of problem (2.1)-(2.3). Then for each $d=(y,u)\in\Theta[(\overline{y},\overline{u})]$ and $(\psi,\omega)\in Z$ satisfying

$$(\psi,\omega)\in T^{2\flat}(A,\overline{z},d)\cap\Big[\nabla F(\overline{z})^{-1}\big(T^{2\flat}(D,F(\overline{z}),\nabla F(\overline{z})d)-\frac{1}{2}\nabla^2F(\overline{z})d^2\big)\Big],$$
$$\nabla G(\overline{y})\psi+\frac{1}{2}\nabla^2G(\overline{y})y^2\in T_0^{2\flat}(Q,G(\overline{y}),\nabla G(\overline{y})y),$$

one has

$$\langle\nabla J(\overline{z}),(\psi,\omega)\rangle+\frac{1}{2}\nabla^2J(\overline{z})(d,d)\geq 0.$$

Proof. The proof is similar to the proof of [25, Lemma 2.2]. However, for convenience of the readers, we provide a short proof here.

By Lemma 2.3, for any sequence $t_k\to 0^+$, there exist sequences $\psi_k\to\psi$, $\omega_k\to\omega$ and a number $k_0>0$ such that

$$(\overline{y}+t_ky+t_k^2\psi_k,\overline{u}+t_ku+t_k^2\omega_k)\in\mathcal{A}_{ad},\ \forall k\geq k_0.$$

Since $(\overline{y},\overline{u})$ is an optimal solution, we have

$$J(\overline{y}+t_ky+t_k^2\psi_k,\overline{u}+t_ku+t_k^2\omega_k)\geq J(\overline{y},\overline{u}).$$

Using a Taylor expansion, we get

$$\begin{aligned}0&\leq\langle\nabla J(\overline{z}),t_k(y,u)+t_k^2(\psi_k,\omega_k)\rangle+\frac{1}{2}\nabla^2J(\overline{z})(t_k(y,u)+t_k^2(\psi_k,\omega_k))^2+o(t_k^2)\\&=t_k\langle\nabla J(\overline{z}),(y,u)\rangle+t_k^2\langle\nabla J(\overline{z}),(\psi_k,\omega_k)\rangle\\&\quad+\frac{t_k^2}{2}\nabla^2J(\overline{z})((y,u)+t_k(\psi_k,\omega_k))^2+o(t_k^2).\end{aligned}$$

Since $\langle\nabla J(\overline{z}),(y,u)\rangle\leq 0$,

$$0\leq t_k^2\langle\nabla J(\overline{z}),(\psi_k,\omega_k)\rangle+\frac{1}{2}t_k^2\nabla^2J(\overline{z})((y,u)+t_k(\psi_k,\omega_k))^2+o(t_k^2).$$

Dividing both sides by t_k^2 and letting $k\to\infty$, we obtain the desired conclusion. □

Proof of Theorem 2.2. Fixing $d = (y, u) \in \Theta[(\overline{y}, \overline{u})]$, we consider two cases:

Case 1. $T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)$ and $T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)$ are nonempty.

In this case we consider in the product space $\mathbb{R} \times E \times C(\overline{\Omega})$ the set

$$\mathcal{K} = \Big\{ \Big(\frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z + r, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z - v, \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z - e\Big) \Big| \ z \in T^{2\flat}(A, \overline{z}, d), v \in T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d), e \in T_0^{2\flat}(Q, G_1(\overline{z}), \nabla G_1(\overline{z})d), r \geq 0 \Big\}.$$

Note that

$$T_0^{2\flat}(Q, G_1(\overline{z}), \nabla G_1(\overline{z})d) = T_0^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y).$$

We now show that $\mathcal{K}$ has the following properties:

- $\mathcal{K}$ is convex. This property follows from convexity of $T^{2\flat}(A, \overline{z}, d)$, $T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)$ and $T_0^{2\flat}(Q, G_1(\overline{z}), \nabla G_1(\overline{z})d)$.
- $\text{int}\mathcal{K} \neq \emptyset$. In fact, by (2.4), there exists $\rho > 0$ such that

$$\begin{aligned} B_E(0, \rho) &\subset \nabla F(\overline{z})(T(A, \overline{z}) \cap B_Z) - T(D, F(\overline{z})) \cap B_E \\ &= \nabla F(\overline{z})(T^\flat(A, \overline{z}) \cap B_Z) - T^\flat(D, F(\overline{z})) \cap B_E. \end{aligned} \tag{2.9}$$

Here $B_E(0, \rho)$ is an open ball in E. From $(A2)$ and [24, Lemma 2.2] (see also [30, Theorem 5]), we have

$$T^\flat(A, \overline{z}) = T(A, \overline{z}) = \{d' \in Z \mid \nabla\Phi(\overline{z})d' = 0\} \tag{2.10}$$

and

$$T^{2\flat}(A, \overline{z}, d) = \Big\{w \in Z \mid \nabla\Phi(\overline{z})w + \frac{1}{2}\nabla^2\Phi(\overline{z})(d, d) = 0\Big\}. \tag{2.11}$$

From this, we can show that

$$T^\flat(A, \overline{z}) \pm T^\flat(A, \overline{z}) \subset T^\flat(A, \overline{z}) \tag{2.12}$$

and

$$T^\flat(A, \overline{z}) + w \subset T^{2\flat}(A, \overline{z}, d) \tag{2.13}$$

for any $w \in T^{2\flat}(A, \overline{z}, d)$. Also, since D is convex, [12, Proposition 3.1] implies that

(2.14)

$$\phi + T^\flat(T^\flat(D, F(\overline{z})), \nabla F(\overline{z})d) \subset T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d), \ \forall \phi \in T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d).$$

Since $\nabla\Phi(\overline{z})$ is surjective, $T^{2\flat}(A, \overline{z}, d) \neq \emptyset$. Take

$$\hat{z} \in T^{2\flat}(A, \overline{z}, d), \ \hat{\phi} \in T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)$$

and define the open set

$$V = \nabla F(\overline{z})\hat{z} - \hat{\phi} + \nabla F(\overline{z})d + B_E(0, \rho).$$

From this and (2.9)-(2.14), we have

$$\begin{aligned} V &\subset \nabla F(\overline{z})[(T^\flat(A, \overline{z}) \cap B_Z) + \hat{z}] - \hat{\phi} - [T^\flat(D, F(\overline{z})) \cap B_E - \nabla F(\overline{z})d] \\ &\subset \nabla F(\overline{z})[(\hat{z} + T^\flat(A, \overline{z})) \cap B_Z(0, \varrho_1)] - \hat{\phi} - [(T^\flat(D, F(\overline{z})) - \nabla F(\overline{z})d) \cap B_E(0, \varrho_2)] \end{aligned}$$

$$\subset \nabla F(\overline{z})[T^{2\flat}(A,\overline{z},d)\cap B_Z(0,\varrho_1)] - (\hat{\phi} + T^{\flat}(T^{\flat}(D,F(\overline{z})),\nabla F(\overline{z})d))\cap B_E(0,\varrho_3)$$

$$\subset \nabla F(\overline{z})[T^{2\flat}(A,\overline{z},d)\cap B_Z(0,\varrho_1)] - T^{2\flat}(D,F(\overline{z}),\nabla F(\overline{z})d)\cap B_E(0,\varrho_3),$$

where $\varrho_1 = 1 + \|\hat{z}\|$, $\varrho_2 = 1 + \|\nabla F(\overline{z})d\|$ and $\varrho_3 = \varrho_2 + \|\hat{\phi}\|$. Note that from [24, Theorem 2.5], we have

$$E = \nabla F(\overline{z})(T^{\flat}(A,\overline{z})) - \operatorname{cone}(D - F(\overline{z})).$$

Here cone(V) denotes the cone hull of a set V. By the similar arguments as above, it follows from this and (2.12)-(2.14) that

$$E = \nabla F(\overline{z})T^{2\flat}(A,\overline{z},d) - T^{2\flat}(D,F(\overline{z}),\nabla F(\overline{z})d). \tag{2.15}$$

Put $\hat{V} := \frac{1}{2}\nabla^2 F(\overline{z})d^2 + V$,

$$\gamma = \frac{1}{2}\nabla^2 J(\overline{z})d^2 + \sup\{\nabla J(\overline{z})z \mid z \in T^{2\flat}(A,\overline{z},d)\cap B_Z(0,\varrho_1)\}$$

and

$$M = \frac{1}{2}\|\nabla^2 G_1(\overline{z})d^2\|_0 + \sup\{\|\nabla G_1(\overline{z})z\|_0 \mid z \in T^{2\flat}(A,\overline{z},d)\cap B_Z(0,\varrho_1)\} - \hat{\theta},$$

where $\hat{\theta} = \min_{x\in\overline{\Omega}}\theta_{a,b}(x)$ and $\|\cdot\|_0$ denotes the norm of $C(\overline{\Omega})$. By [31, Lemma 3.3], $\theta_{a,b}$ is lower semicontinuous on $\overline{\Omega}$. Since $T^{2\flat}(Q,G(\overline{y}),\nabla G(\overline{y})y)$ is nonempty, $\theta_{a,b}(x) > -\infty$ for all $x \in \overline{\Omega}$ (see [23, Lemma 3.1]). On the other hand, $\overline{\Omega}$ is a compact set. Hence $\hat{\theta}$ is finite and so is M. Define

$$W = \{\varphi \in C(\overline{\Omega}) \mid \varphi > M\}.$$

Then W is an open set. We now claim that

$$(\gamma,+\infty)\times\hat{V}\times W \subset \mathcal{K}.$$

Indeed, take any $(\xi,v,w)\in(\gamma,+\infty)\times\hat{V}\times W$. Since $v\in\hat{V}$, there exist

$$z_0 \in T^{2\flat}(A,\overline{z},d)\cap B_Z(0,\varrho_1),\ v_0 \in T^{2\flat}(D,F(\overline{z}),\nabla F(\overline{z})d)\cap B_E(0,\varrho_3)$$

such that $v = \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z_0 - v_0$. Since $\xi > \gamma$, we get $\xi > \frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z_0$. Hence there exists $r \geq 0$ such that $\xi = \frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z_0 + r$. Since $w \in W$, we have

$$w > \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \|\nabla G_1(\overline{z})z_0\|_0 - \hat{\theta}.$$

Hence

$$\theta_{a,b} \geq \hat{\theta} > \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z_0 - w.$$

Defining $e_0 = \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z_0 - w$, we get

$$e_0 \in T_0^{2\flat}(Q,G_1(\overline{z}),\nabla G_1(\overline{z})d)$$

and

$$w = \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z_0 - e_0.$$

Thus we have shown that $(\xi,v,w)\in\mathcal{K}$. Consequently, int$\mathcal{K} \neq \emptyset$.

- $(0,0,0)\notin$ int$\mathcal{K}$. Otherwise, there exists $\epsilon > 0$ such that

$$(-\epsilon,\epsilon)\times\{0\}\times\{0\}\subset\mathcal{K}.$$

This implies that there exist $z \in T^{2\flat}(A, \overline{z}, d), v \in T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d), r \geq 0$ and $e \in T_0^{2\flat}(Q, G_1(\overline{z}), \nabla G_1(\overline{z})d)$ such that the following relations hold:

$$\begin{cases} \frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z + r < 0 \\ \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z - v = 0 \\ \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z - e = 0. \end{cases} \tag{2.16}$$

Let $z = (\psi, \omega)$. The second relation of (2.16) implies that

$$z \in T^{2\flat}(A, \overline{z}, d) \cap \nabla F(\overline{z})^{-1}\Big[T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d) - \frac{1}{2}\nabla^2 F(\overline{z})d^2\Big].$$

Meanwhile from the third relation we have

$$\frac{1}{2}\nabla^2 G(\overline{y})y^2 + \nabla G(\overline{y})\psi \in T_0^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y).$$

Combining these with Lemma 2.4, we obtain $\frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z \geq 0$, which contradicts the first relation of (2.16). Our claim is justified.

We now can separate $(0, 0, 0)$ and $\mathcal{K}$ by a hyperplane (see [19, Theorem 1, p.163]). Hence there exists a nonzero functional $(\lambda, e^*, \mu) \in \mathbb{R} \times E^* \times \mathcal{M}(\overline{\Omega})$ such that

$$\begin{aligned} \lambda\Big(\frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z + r\Big) + \Big\langle e^*, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z - v\Big\rangle \\ + \Big\langle \mu, \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z - e\Big\rangle \geq 0 \end{aligned} \tag{2.17}$$

for all $r \geq 0$, $z \in T^{2\flat}(A, \overline{z}, d)$, $v \in T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)$ and $e \in T_0^{2\flat}(Q, G_1(\overline{z}), \nabla G_1(\overline{z})d)$. If $\lambda = 0$ and $\mu = 0$, then we have

$$\Big\langle e^*, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z - v\Big\rangle \geq 0$$

for all $z \in T^{2\flat}(A, \overline{z}, d)$ and $v \in T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)$ or equivalently,

$$\Big\langle e^*, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + w\Big\rangle \geq 0$$

for all $w \in \nabla F(\overline{z})(T^{2\flat}(A, \overline{z}, d)) - T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)$. From this and (2.15), we get $e^* = 0$. Thus we must have $|\lambda| + \|\mu\| \neq 0$. Moreover, we have $\lambda \geq 0$. Otherwise, fixing z, v, e in (2.17) and letting $r \to +\infty$, we obtain a contradiction.

Substituting $r = 0$ into (2.17), yields

$$\begin{aligned} &\lambda\Big(\frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z\Big) + \Big\langle e^*, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z\Big\rangle \\ &\qquad + \Big\langle \mu, \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z\Big\rangle \\ &\geq \sigma(e^*, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) + \sigma(\mu, T_0^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)), \ \forall z \in T^{2\flat}(A, \overline{z}, d). \end{aligned}$$

Since $\sigma(\mu, T_0^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)) = \sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y))$, we get

$$\begin{aligned} &\lambda\Big(\frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z\Big) + \Big\langle e^*, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z\Big\rangle \\ &\qquad + \Big\langle \mu, \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z\Big\rangle \end{aligned}$$

$$\geq \sigma(e^*, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) + \sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)) \tag{2.18}$$

for all $z \in T^{2\flat}(A, \overline{z}, d)$. From (2.10) and (2.11), we get

$$T^{2\flat}(A, \overline{z}, d) = T^{\flat}(A, \overline{z}) + T^{2\flat}(A, \overline{z}, d).$$

Hence we can rewrite (2.18) in the form

$$\begin{aligned}\lambda\Big(\frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})(z_1 + z_2)\Big) + \Big\langle e^*, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})(z_1 + z_2)\Big\rangle \\ + \Big\langle \mu, \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})(z_1 + z_2)\Big\rangle \\ \geq \sigma(e^*, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) + \sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y))\end{aligned}$$

for all $z_1 \in T^{\flat}(A, \overline{z})$ and $z_2 \in T^{2\flat}(A, \overline{z}, d)$. It follows that

$$\begin{aligned}\lambda J(\overline{z})z_1 + \langle e^*, \nabla F(\overline{z})z_1\rangle + \langle \mu, \nabla G_1(\overline{z})z_1\rangle + \lambda\Big(\frac{1}{2}\nabla^2 J(\overline{z})d^2 + \nabla J(\overline{z})z_2\Big) \\ + \Big\langle e^*, \frac{1}{2}\nabla^2 F(\overline{z})d^2 + \nabla F(\overline{z})z_2\Big\rangle + \Big\langle \mu, \frac{1}{2}\nabla^2 G_1(\overline{z})d^2 + \nabla G_1(\overline{z})z_2\Big\rangle \\ \geq \sigma(e^*, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) + \sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y))\end{aligned} \tag{2.19}$$

for all $z_1 \in T^{\flat}(A, \overline{z})$ and $z_2 \in T^{2\flat}(A, \overline{z}, d)$. This implies that

$$\langle \lambda\nabla J(\overline{z}) + \nabla F(\overline{z})^* e^* + \nabla G_1(\overline{z})^*\mu, z_1\rangle \geq 0$$

for all $z_1 \in T^{\flat}(A, \overline{z})$. Hence

$$-\lambda\nabla J(\overline{z}) - \nabla F(\overline{z})^* e^* - \nabla G_1(\overline{z})^*\mu \in N(A, \overline{z}).$$

Since $T(A, \overline{z}) = T^{\flat}(A, \overline{z}) = \{v \in Z | \nabla\Phi(\overline{z})v = 0\}$, we have

$$N(A, \overline{z}) = \{\nabla\Phi(\overline{z})^*\pi^* | \pi^* \in \Pi^*\}.$$

Hence we can find some $\pi^* \in \Pi^*$ such that

$$-\lambda\nabla J(\overline{z}) - \nabla F(\overline{z})^* e^* - \nabla G_1(\overline{z})^*\mu = \nabla\Phi(\overline{z})^*\pi^*.$$

This is equivalent to $D_z\mathcal{L}(\lambda, \pi^*, e^*, \mu, \overline{z}) = 0$, which is assertion (i) of the theorem. Substituting this expression and $z_1 = 0$ into (2.19) we get

$$\begin{aligned}\langle -\nabla\Phi(\overline{z})^*\pi^*, z_2\rangle + \frac{1}{2}\nabla^2 J(\overline{z})d^2 + \frac{1}{2}\langle e^*, \nabla^2 F(\overline{z})d^2\rangle + \frac{1}{2}\langle \mu, \nabla G_1(\overline{z})d^2\rangle \\ \geq \sigma(e^*, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) + \sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y))\end{aligned}$$

for all $z_2 \in T^{2\flat}(A, \overline{z}, d)$. By (2.11), we have $\nabla\Phi(\overline{z})z_2 + \frac{1}{2}\nabla^2\Phi(\overline{z})d^2 = 0$. Hence we get

$$\begin{aligned}&\frac{1}{2}D^2_{zz}\mathcal{L}(\lambda, \pi^*, e^*, \mu, \overline{z}) \\ &= \frac{1}{2}\big[\langle \pi^*, \nabla^2\Phi(\overline{z})d^2\rangle + \lambda\nabla^2 J(\overline{z})d^2 + \langle e^*, \nabla^2 F(\overline{z})d^2\rangle + \langle \mu, \nabla^2 G_1(\overline{z})d^2\rangle\big] \\ &\geq \sigma(e^*, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) + \sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)).\end{aligned}$$

We obtain assertion (iv) of Theorem 2.2. From the above inequality we get

$$\sigma(e^*, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) < +\infty,\ \sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)) < +\infty.$$

Lemma 4 in [12] implies that

$$e^* \in N(D, F(\bar{z})) \text{ and } \mu \in N(Q, G(\bar{y})).$$

By [31, Theorem 2.1], we have

$$\frac{d\mu}{d|\mu|}(x) \in N((-\infty, 0], g(x, \bar{y}(x)) = \begin{cases} 0 & \text{if } g(x, \bar{y}(x)) < 0 \\ \geq 0 & \text{if } g(x, \bar{y}(x)) = 0 \end{cases} \quad |\mu| - \text{a.e.},$$

where $\frac{d\mu}{d|\mu|}$ denotes the Radon-Nikodym derivative of μ with respect to $|\mu|$. It follows that μ is nonnegative and

$$\text{supp}(\mu) \subset \{x \in \overline{\Omega} \mid g(x, \bar{y}(x)) = 0\}.$$

Since $g(x, 0) < 0$ for all $x \in \Gamma$, we get

$$\text{supp}(\mu) \subset \{x \in \Omega \mid g(x, \bar{y}(x)) = 0\}.$$

We obtain assertion (ii) and (iii) of Theorem 2.2.

Case 2. $T^{2\flat}(D, F(\bar{z}), \nabla F(\bar{z})d)$ or $T^{2\flat}(Q, G(\bar{y}), \nabla G(\bar{y})y)$ is empty.

Then assertion (iv) is automatically fulfilled because

$$\sigma(e^*, T^{2\flat}(D, F(\bar{z}), \nabla F(\bar{z})d)) + \sigma(\mu, T^{2\flat}(Q, G(\bar{y}), \nabla G(\bar{y})y)) = -\infty$$

for all $e^* \in E^*$ and $\mu \in \mathcal{M}(\overline{\Omega})$. In this case, instead of considering the set $\mathcal{K}$, we consider the set

$$\mathcal{K}' = \Big\{(\nabla J(\bar{z})z + r, \nabla F(\bar{z})z - v, \nabla G_1(\bar{z})z - e) \mid \\ z \in T^\flat(A, \bar{z}), v \in T^\flat(D, F(\bar{z})), e \in T_0^\flat(Q, G_1(\bar{z})), r \geq 0\Big\},$$

where

$$T_0^\flat(Q, G_1(\bar{z})) = T_0^\flat(Q, G(\bar{y})) := \{\varphi \in C(\overline{\Omega}) \mid \varphi(x) < 0 \text{ whenever } g(x, \bar{y}(x)) = 0\}.$$

Separating $(0, 0, 0)$ and $\mathcal{K}'$ by a hyperplane and using the similar arguments as in Case 1, we obtain assertions (i)-(iii) of Theorem 2.2. The proof is complete. $\square$

Remark 2.5. When A is a closed convex set, by defining the set $K = D \times Q$ and the mapping

$$H : Z \to E \times C(\overline{\Omega}), \ H(z) = (F(z), G_1(z)),$$

the constraint set of problem (2.1)-(2.3) becomes $z \in A$ and $H(z) \in K$. Then under the Robinson constraint qualification condition

$$0 \in \text{int}\big\{\nabla H(\bar{z})(A - \bar{z}) - K + H(\bar{z})\big\}$$

we can obtain Theorem 2.2 from [3, Theorem 3.45]. Unfortunately, since A is not convex, we fail to apply [3, Theorem 3.45] for our problem. Moreover, in comparison with the Robinson constraint qualification condition, in regularity condition $(A4)$ we do not impose on constraint $G_1(z) \in Q$.

3. Necessary optimality conditions for semilinear elliptic optimal control problems

3.1. **Assumptions and statement of main results.** Throughout this section, we assume that the boundary Γ is of class $C^{1,1}$, $1 < p < +\infty$ and

$$\frac{1}{N} > \frac{1}{r} \geq \frac{1}{p} - \frac{1}{N}. \tag{3.1}$$

We shall denote by s and q the conjugate numbers of r and p, respectively. Let $W^{-1,r}(\Omega)$ be the dual space of $W_0^{1,s}(\Omega)$. Recall that given an element $u \in W^{-1,r}(\Omega)$, a function $y \in W_0^{1,r}(\Omega)$ is a solution of (1.2) iff

$$\int_\Omega (\sum_{i,j=1}^N a_{ij}(x) D_i y D_j \varphi) dx + \int_\Omega h(x,y)\varphi dx = \langle u, \varphi \rangle \quad \forall \varphi \in W_0^{1,s}(\Omega).$$

Under assumptions (H_2) and (H_3) below, [8, Theorem 2.4] implies that, for each $u \in W^{-1,r}(\Omega)$ with $r > N$, equation (1.2) has a unique solution $y \in W_0^{1,r}(\Omega)$. By (3.1) and the Sobolev and Rellich theorem (see [14, Theorem 1.6]), we have $L^p(\Omega) \hookrightarrow W^{-1,r}(\Omega)$. Hence for each $u \in L^p(\Omega)$, equation (1.2) has a unique solution $y \in W_0^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ and so constraint (1.4) is well defined.

A pair (y, u) is said to be an admissible couple of problem (1.1)-(1.4) if it satisfies the constraints (1.2)-(1.4). The set of such pairs will be denoted by $\mathcal{A}_{ad}$. An admissible couple $(\overline{y}, \overline{u})$ is called a locally optimal solution of problem (1.1)-(1.4) if there exists a number $\epsilon > 0$ such that the following implication holds:

$$\forall (y,u) \in \mathcal{A}_{ad},\ \|y - \overline{y}\|_{W_0^{1,r}(\Omega)} + \|u - \overline{u}\|_{L^p(\Omega)} \leq \epsilon \Rightarrow J(y,u) \geq J(\overline{y}, \overline{u}).$$

Given a couple $(\overline{y}, \overline{u}) \in \mathcal{A}_{ad}$, the symbols $L[x], L_y[x], g_y[x], f[x]$, etc., stand for

$$L(x, \overline{y}(x), \overline{u}(x)), L_y(x, \overline{y}(x), \overline{u}(x)), g_y(x, \overline{y}(x)), f(x, \overline{y}(x)),$$

etc., respectively. We also assume that $\hat{h}$ is one of the functions h and f, and D is defined by (1.9). Fixing a couple $(\overline{y}, \overline{u}) \in \mathcal{A}_{ad}$, we impose the following hypotheses:

(H_1) $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to $(y, u), L(x, 0, 0) \in L^1(\Omega)$ and for any positive number M, there is a constant $K_{L,M}$ such that

$$|L_y(x,y,u)| + |L_u(x,y,u)| \leq K_{L,M}(|y|^{a_1} + |u|^{a_2}),$$

$$\begin{aligned}\big|L_y(x,y_1,u_1) - L_y(x,y_2,u_2)\big| + \big|L_u(x,y_1,u_1) - L_u(x,y_2,u_2)\big| \\ \leq K_{L,M}(|y_1 - y_2|^{b_1} + |u_1 - u_2|^{b_2}),\end{aligned}$$

$$\begin{aligned}\big|L_{yy}(x,y_1,u_1) - L_{yy}(x,y_2,u_2)\big| + \big|L_{yu}(x,y_1,u_1) - L_{yu}(x,y_2,u_2)\big| \\ + \big|L_{uu}(x,y_1,u_1) - L_{uu}(x,y_2,u_2)\big| \leq K_{L,M}\left(|y_1 - y_2|^{c_1} + m|u_1 - u_2|^{c_2}\right)\end{aligned}$$

for all $u_1, u_2 \in \mathbb{R}$, $|y_i| \leq M$ and a.e. $x \in \Omega$ with $a_1, b_1, c_1 > 0, 0 < a_2, b_2 \leq p - 1, 0 < c_2 \leq p - 2, m = 0$ whenever $1 < p \leq 2$ and $m = 1$ whenver $p > 2$.

(H_2) The functions $a_{ij} : \overline{\Omega} \to \mathbb{R}$ are of class $C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$ and there is a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \forall \xi \in \mathbb{R}^N,\ x \in \overline{\Omega}.$$

(H_3) $\hat{h} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable and for all $y \in \mathbb{R}$ the following condition holds

$$\hat{h}(\cdot, 0) \in L^p(\Omega), \quad \hat{h}_y(x, y) \geq 0 \quad \text{a.e. } x \in \Omega.$$

Moreover, for each $M > 0$, there is a constant $K_{\hat{h},M} > 0$ such that

$$\begin{aligned} \left|\hat{h}_y(x, y)\right| + \left|\hat{h}_{yy}(x, y)\right| &\leq K_{\hat{h},M}, \\ \left|\hat{h}_{yy}(x, y_1) - \hat{h}_{yy}(x, y_2)\right| &\leq K_{\hat{h},M}|y_2 - y_1| \end{aligned}$$

for a.e. $x \in \Omega$ and $|y|, |y_1|, |y_2| \leq M$.
(H_4) The function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies assumption $(A3)$.

Definition 3.1. A pair $d = (y, u) \in W_0^{1,r}(\Omega) \times L^p(\Omega)$ is said to be a critical direction for problem (1.1)-(1.4) at $\overline{z} = (\overline{y}, \overline{u})$ if the following conditions hold:

(i) $\nabla J(\overline{z})d = \int_\Omega (L_y[x]y(x) + L_u[x]u(x))dx \leq 0$;
(ii) $-\sum_{i,j=1}^N D_j(a_{ij}(\cdot)D_i y) + h_y(\cdot, \overline{y})y = u \quad \text{in } \Omega, \quad y|_\Gamma = 0$;
(iii) $f_y[\cdot]y + u \in \overline{\text{cone}}(D - f[\cdot] - \overline{u})$;
(iv) $T^{2\flat}(Q, g[\cdot], g_y[\cdot]y) \neq \emptyset$.

We will denote by $\mathcal{C}[(\overline{y}, \overline{u})]$ the set of such critical directions. It is clear that $\mathcal{C}[(\overline{y}, \overline{u})]$ is a convex cone containing $(0, 0)$. Note that condition (iv) of Definition 3.1 is equivalent to saying that $\theta_{a,b}(x) > -\infty$ for all $x \in \overline{\Omega}$. This condition implies that $g_y[\cdot]y \in T^\flat(Q, g[\cdot])$. By [23, Theorem 3.1], we have $g_y[x]y(x) \leq 0$ whenever $g[x] = 0$.

We are ready to state our main result of this section.

Theorem 3.2. *Suppose that hypotheses $(H_1) - (H_4)$ are satisfied and $(\overline{y}, \overline{u})$ is a locally optimal solution of problem (1.1)-(1.4). Then for each $(y, u) \in \mathcal{C}[(\overline{y}, \overline{u})]$, there exist functions $\phi \in W_0^{1,s}(\Omega)$, $\psi \in L^q(\Omega)$, a number $\lambda \geq 0$ and a nonnegative Borel measure $\mu \in \mathcal{M}(\overline{\Omega})$ with $|\lambda| + \|\mu\| \neq 0$ such that the following conditions are fulfilled:*

(i) *(the adjoint equation)*

$$-\sum_{i,j=1}^{N} D_i(a_{ij}(\cdot)D_j\phi) + h_y[\cdot]\phi = -\lambda L_y[\cdot] - f_y[\cdot]^*\psi - g_y[\cdot]^*\mu \ \textit{in}\ \Omega$$
$$\psi|_\Gamma = 0;$$

(ii) *(the stationary condition in u)*

$$\lambda L_u[x] - \phi(x) + \psi(x) = 0 \quad \textit{a.e.};$$

(iii) *(the complementary condition in z)*

$$\int_\Omega \psi(x)(v(x) - f[x] - \overline{u}(x))dx \le 0 \ \forall \alpha \le v \le \beta;$$

(iv) *(the complementary condition in y)*

$$\text{supp}(\mu) \subset \{x \in \Omega \mid g[x] = 0\};$$

(v) *(the second order condition)*

$$\lambda \int_\Omega (L_{yy}[x]y^2(x) + 2L_{yu}[x]y(x)u(x) + L_{uu}[x]u^2(x))dx + \int_\Omega \phi(x)h_{yy}[x]y(x)^2dx + \int_\Omega \psi(x)f_{yy}[x]y(x)^2dx + \int_{\overline{\Omega}} g_{yy}[x]y^2(x)d\mu \ge 2\int_{\overline{\Omega}} \theta_{a,b}(x)d\mu + 2\sigma(\psi, T^{2\flat}(D, f[\cdot] + \overline{u}, f_y[\cdot]y + u)).$$

Let us give a corollary of Theorem 3.2. For this we shall denote by $\mathcal{C}_0[(\overline{y}, \overline{u})]$ the set of couples (y, u) satisfying conditions (i) and (ii) of Definition 3.1 and the following conditions:

(iii) $f_y[\cdot]y + u \in \text{cone}(D - f[\cdot] - \overline{u})$;

(iv) $g_y[\cdot]y \in \text{cone}(Q - g[\cdot])$.

It is clear that $\mathcal{C}_0[(\overline{y}, \overline{u})]$ is a cone which is contained in $\mathcal{C}[(\overline{y}, \overline{u})]$.

Corollary 3.3. Suppose that hypotheses $(H_1) - (H_4)$ are satisfied and $(\overline{y}, \overline{u})$ is a locally optimal solution of problem (1.1)-(1.4). Then for each $(y, u) \in \mathcal{C}_0[(\overline{y}, \overline{u})]$, there exist functions $\phi \in W_0^{1,s}(\Omega)$, $\psi \in L^q(\Omega)$, a number $\lambda \ge 0$ and a nonnegative Borel measure $\mu \in \mathcal{M}(\overline{\Omega})$ with $|\lambda| + \|\mu\| \ne 0$ such that assertions (i)-(iv) of Theorem 3.2 and the following assertion are fulfilled:

(v′) (the nonnegative second order condition)

$$\lambda \int_\Omega (L_{yy}[x]y^2(x) + 2L_{yu}[x]y(x)u(x) + L_{uu}[x]u^2(x))dx + \int_\Omega \phi(x)h_{yy}[x]y(x)^2dx + \int_\Omega \psi(x)f_{yy}[x]y(x)^2dx + \int_{\overline{\Omega}} g_{yy}[x]y^2(x)d\mu \ge 0.$$

3.2. **Proofs of Theorem 3.2 and Corollary 3.3.** For the proof of Theorem 3.2, we shall reduce problem (1.1)-(1.4) to problem (1.5)-(1.8) and then apply Theorem 2.1. To do this, we put

$$Y = W_0^{1,r}(\Omega),\ U = L^p(\Omega),\ Z = Y \times U,$$
$$\Pi = W^{-1,r}(\Omega),\ E = L^p(\Omega)$$

and define the mappings

$$\Phi : Z \to \Pi,\ \Phi(y, u) = -\sum_{i,j=1}^N D_j(a_{ij}(\cdot)D_iy) + h(\cdot, y) - u$$
$$F : Z \to E,\ F(y, u) = f(\cdot, y) + u$$
$$G_1 : Z \to C(\overline{\Omega}),\ G_1(y, u) = G(y) = g(\cdot, y).$$

Then problem (1.1)-(1.4) is assigned with the following Lagrangian:

$$\mathcal{L}(\lambda,\phi,\psi,\mu,z)=\lambda J(z)+\langle\phi,\Phi(z)\rangle+\langle\psi,F(z)\rangle+\langle\mu,G(y)\rangle \tag{3.2}$$

with $z=(y,u)$, $\lambda\in\mathbb{R}$, $\phi\in\Pi^*,\psi\in E^*$ and $\mu\in\mathcal{M}(\overline{\Omega})$.

In what follows, we shall show that assumptions $(A1)-(A4)$ of Theorem 2.1 are fulfilled.

Lemma 3.4. Under assumptions $(H_1)-(H_4)$, the mappings J, Φ, F and G_1 are of class C^2 around $(\overline{y},\overline{u})$ and their derivatives are given by the following formulae:

$$\nabla J(\overline{z})=(L_y[\cdot],L_u[\cdot]),\ \nabla\Phi(\overline{y},\overline{u})=(\Lambda+h_y[\cdot],-I)$$
$$\nabla F(\overline{z})=(f_y[\cdot],I),\ \nabla G_1(\overline{z})=(g_y[\cdot],0)$$
$$\nabla^2 J(\overline{z})=\begin{bmatrix} L_{yy}[\cdot] & L_{yu}[\cdot]\\ L_{uy}[\cdot] & L_{uu}[\cdot]\end{bmatrix},\nabla^2\Phi(\overline{z})=\begin{bmatrix} h_{yy}[\cdot] & 0\\ 0 & 0\end{bmatrix}$$

and

$$\nabla^2 F(\overline{z})=\begin{bmatrix} f_{yy}[\cdot] & 0\\ 0 & 0\end{bmatrix},\nabla^2 G_1(\overline{z})=\begin{bmatrix} g_{yy}[\cdot] & 0\\ 0 & 0\end{bmatrix}.$$

Here I is the identity mapping and Λ is defined by $\Lambda y=-\sum_{i,j=1}^N D_j(a_{ij}(\cdot)D_i y)$.

Proof. The proof is straightforward, which is based on standard arguments. □

As a consequence, we have the following.

Corollary 3.5. Suppose assumptions $(H_1)-(H_4)$ and $d=(y,u)\in Z$. Then one has

$$D_y\mathcal{L}(\lambda,\phi,\psi,\mu,\overline{z})=\lambda L_y[\cdot]+\Lambda^*\phi+h_y[\cdot]^*\phi+f_y[\cdot]^*\psi+g_y[\cdot]^*\mu$$
$$D_u\mathcal{L}(\lambda,\phi,\psi,\mu,\overline{z})=\lambda L_u[\cdot]-\phi+\psi$$

and

$$\begin{aligned}D^2_{zz}\mathcal{L}(\lambda,\phi,\psi,\mu,\overline{z})(d,d)=&\lambda\int_\Omega\big(L_{yy}[x]y^2+2L_{yu}[x]yu\\&+L_{uu}[x]u^2\big)dx+\int_\Omega h_{yy}[x]\phi y^2dx\\&+\int_\Omega f_{yy}[x]\psi y^2dx+\int_{\overline{\Omega}} g_{yy}[x]y^2d\mu.\end{aligned}$$

Lemma 3.6. Under assumptions (H_2) and (H_3), the following assertions hold:

(i) the mapping $\nabla\Phi(z)$ is surjective for all z around $\overline{z}$;
(ii) The regularity condition (2.4) is valid.

Proof. Assertion (i) follows from [24, Lemma 3.1]. According to [24, Theorem 2.5] (see also [39, Theorem 2.1]), the regularity condition (2.4) is equivalent to the following condition:

$$E=\bigcap_{z\in B_Z(\overline{z},\delta)\cap A}\big[\nabla F(\overline{z})(T(A,z))-\text{cone}(D-F(\overline{z}))\big]\text{ for some }\delta>0. \tag{3.3}$$

Choose $\delta > 0$ small enough and $\hat{z} \in B_Z(\overline{z}, \delta)$. We shall show that

$$E = \nabla F(\overline{z})(T(A, \hat{z})) - \text{cone}(D - F(\overline{z})). \tag{3.4}$$

By (i), $\nabla\Phi(\hat{z})$ is surjective. Hence from [24, Lemma 2.2], we have

$$\begin{aligned} T(A, \hat{z}) = T^\flat(A, \hat{z}) &= \{z = (y, u) \in Z \mid \nabla\Phi(\hat{z})z = 0\} \\ &= \{(y, u) \in Y \times U \mid \Lambda y + h_y(\cdot, \hat{y})y = u\}. \end{aligned}$$

Taking any $v \in E = L^p(\Omega)$, we consider the equation

$$\Lambda y + h_y(\cdot, \hat{y})y + f_y(\cdot, \overline{y})y = v \text{ in } \Omega,\ y|_\Gamma = 0.$$

By (H_3) and [17, Theorem 2.4. 2.5, p. 124], this equation has a unique solution $y \in W^{2,p}(\Omega)$. By (3.1) and the embedding theorem (see [6, Corollary 9.14]), the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous. Note that $W^{2,p}(\Omega) = \{y \in W^{1,p}(\Omega) \mid Dy \in W^{1,p}(\Omega)\}$. Hence the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ is continuous. Since $r > N$, we have $y \in C(\overline{\Omega})$. Hence $y \in W_0^{1,r}(\Omega)$. Putting $\Lambda y + h_y(\cdot, \hat{y})y = u$, we see that $(y, u) \in T(A, \hat{z})$. Thus we have $f_y(\cdot, \hat{y})y + u = v$. This is equivalent to

$$\nabla F(\hat{z})(y, u) - (F(\overline{z}) - F(\overline{z})) = v.$$

Consequently, (3.4) is obtained. The proof of the lemma is complete. □

Proof of Theorem 3.2. From Lemmas 3.4-3.6, we see that assumptions $(A1) - (A4)$ are fulfilled. Take any $d = (y, u) \in \mathcal{C}[(\overline{y}, \overline{u})]$. According to Theorem 2.2, there exist a nonnegative number λ, functions $\phi \in \Pi^* = W_0^{1,s}(\Omega)$, $\psi \in E^* = L^q(\Omega)$ and a nonnegative Borel measure $\mu \in \mathcal{M}(\overline{\Omega})$ with $|\lambda| + \|\mu\| \neq 0$ such that the following conditions are fulfilled:

(i′) (the adjoint equation) $D_z\mathcal{L}(\lambda, \phi, \psi, \mu, \overline{z}) = 0$;

(ii′)] (the complementary condition in z) $\psi \in N(D, F(\overline{z}))$;

(iii′) (the complementary conditions in y)

$$\text{supp}(\mu) \subset \{x \in \Omega \mid g[x] = 0\};$$

(iv′) (the second-order condition)

$$\begin{aligned} D_{zz}\mathcal{L}(\lambda, \phi, \psi, \mu, \overline{z})(d, d) \geq & 2\sigma(\psi, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)) \\ & + 2\sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)). \end{aligned}$$

From Corollary 3.5 and (i'), we obtain assertions (i) and (ii) of Theorem 3.2. Assertions (ii′) and (iii′) imply (iii) and (iv), respectively.

By definition of $\mathcal{C}[(\overline{y}, \overline{u})]$, we have $T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y) \neq \emptyset$. Using similar arguments as in the proof of [25, Theorem 3.1] (see also the proof of [23, Lemma 4.4]), we have

$$\sigma(\mu, T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)) = \int_{\overline{\Omega}} \theta_{a,b}(x) d\mu,$$

where $a(x) := g[x]$ and $b(x) := g_y[x]y(x)$ and $\theta_{a,b}(x)$ is defined by (2.6). Combining this with (iv′) and Corollary 3.5, we obtain assertion (v) of Theorem 3.2. The proof is complete. □

Proof of Corollary 3.3. Since $\mathcal{C}_0[(\overline{y}, \overline{u})] \subset \mathcal{C}[(\overline{y}, \overline{u})]$, for each $(y, u) \in \mathcal{C}_0[(\overline{y}, \overline{u})]$, there exist multipliers λ, ϕ, ψ and μ such that the assertions (i)-(v) of Theorem 3.2 are valid. Let us show that the implication (v) $\Rightarrow$ (v)$'$ holds. Indeed, from $\nabla G(\overline{y})y \in \text{cone}(Q - G(\overline{y}))$, there exists $s > 0$ such that $\nabla G(\overline{y})y = s(v - G(\overline{y}))$ for some $v \in Q$. By convexity of Q, for any sequence $t_n \to 0^+$, we have

$$t_n \nabla G(\overline{y})y = t_n sv + (1 - st_n)G(\overline{y}) - G(\overline{y}) \in Q - G(\overline{y}).$$

This implies that $G(\overline{y}) + t_n \nabla G(\overline{y})y \in Q$. Hence $0 \in T^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)$. By [23, Theorem 3.2] (see also [31, Corollary 4.2]), we have $\theta_{a,b}(x) \geq 0$ for all $x \in \overline{\Omega}$. Hence $\int_{\overline{\Omega}} \theta_{a,b}(x) d\mu \geq 0$. Similarly, we have $0 \in T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)$. It follows that

$$2\sigma\big(\psi, T^{2\flat}(D, f[\cdot] + \overline{u}, f_y[\cdot]y + u)\big) = \sigma\big(\psi, T^{2\flat}(D, F(\overline{z}), \nabla F(\overline{z})d)\big) \geq 0.$$

Hence we obtain assertion (v$'$) of the corollary. The proof is complete. $\square$

Appendix

In this appendix, we shall present the proof of Lemma 2.3, which is similar to the proof of Lemma 2.1 in [25]. *Proof of Lemma 2.3.* From $(A4)$ and [24, Theorem

2.3], we have $(\psi, \omega) \in T^{2\flat}(A \cap F^{-1}(D), \overline{z}, d)$. Hence for any sequence $t_k \to 0^+$, there exists $(\psi_k, \omega_k) \to (\psi, \omega)$ such that

$$(\overline{y}, \overline{u}) + t_k(y, u) + t_k^2(\psi_k, \omega_k) \in A \cap F^{-1}(D).$$

To show that there exists $k_0 > 0$ such that

$$(\overline{y} + t_k y + t_k^2 \psi_k, \overline{u} + t_k u + t_k^2 \omega_k) \in \mathcal{A}_{ad}, \ \forall k \geq k_0$$

we need to prove that, there exists $k_0 > 0$ such that for all $k \geq k_0$ one has

$$g(x, \overline{y}(x) + t_k y(x) + t_k^2 \psi_k(x)) \leq 0 \quad \forall x \in \overline{\Omega}. \tag{3.5}$$

By a Taylor expansion, we have

$$\begin{aligned}
G(\overline{y} + t_k y + t_k^2 \psi_k) =& G(\overline{y}) + t_k(\nabla G(\overline{y})y) \\
&+ t_k^2\Big(\nabla G(\overline{y})\psi_k + \frac{1}{2}\nabla^2 G(\overline{y})(y + t_k\psi_k)^2\Big) + o(t_k^2) \\
=& G(\overline{y}) + t_k(\nabla G(\overline{y})y) + t_k^2 \nabla G(\overline{y})\psi \\
&+ t_k^2 \nabla G(\overline{y})(\psi_k - \psi) + \frac{1}{2} t_k^2 \nabla^2 G(\overline{y})(y + t_k\psi_k)^2 + o(t_k^2) \\
=& G(\overline{y}) + t_k \nabla G(\overline{y})y + t_k^2\Big(\nabla G(\overline{y})\psi + \frac{1}{2}\nabla^2 G(\overline{y})y^2\Big) + o(t_k^2).
\end{aligned}$$

Thus we have

(3.6)

$$\begin{aligned}
g(x, \overline{y}(x) + t_k y(x) + t_k^2 \psi_k(x)) &= g(x, \overline{y}(x)) + t_k g_y(x, \overline{y}(x))y(x) \\
&+ t_k^2(g_y(x, \overline{y}(x))\psi(x) + \frac{1}{2} g_{yy}(x, \overline{y}(x))y^2(x)) + o(t_k^2), \qquad (3.7)
\end{aligned}$$

where $o(t_k^2)$ is independent of x. Let us define functions

$$a(x) = g(x, \overline{y}(x)) \text{ and } b(x) = g_y(x, \overline{y}(x))y(x).$$

Since $(\overline{y}, \overline{u})$ is an admissible couple, we have $a(x) \le 0$ for all $x \in \overline{\Omega}$. Since $d = (y, u) \in \Theta[(\overline{y}, \overline{u})]$, we have $\nabla G(\overline{y})y \in T^\flat(Q, G(\overline{y}))$. By [23, Theorem 3.1], we have

$$b(x) = g_y(x, \overline{y}(x))y(x) \le 0 \text{ whenever } a(x) = g(\overline{y}(x)) = 0.$$

Since $\nabla G(\overline{y})\psi + \frac{1}{2}\nabla^2 G(\overline{y})y^2 \in T_0^{2\flat}(Q, G(\overline{y}), \nabla G(\overline{y})y)$, we have

$$g_y(x, \overline{y}(x))\psi(x) + \frac{1}{2}g_{yy}(x, \overline{y}(x))y^2(x) < \theta_{a,b}(x) \ \forall x \in \overline{\Omega}.$$

Here $\theta_{a,b}$ is defined by (2.6). By [31, Lemma 3.3], there exists $\epsilon_0 > 0$ such that

$$g(x, \overline{y}(x)) + \epsilon g_y(x, \overline{y}(x))y(x) + \epsilon^2\Big((g_y(x, \overline{y}(x))\psi(x) + \frac{1}{2}g_{yy}(x, \overline{y}(x))y^2(x)) + \epsilon_0\Big) \le 0 \tag{3.8}$$

for all $\epsilon \in (0, \epsilon_0)$ and for all $x \in \overline{\Omega}$. Let us choose k_0 such that $o(t_k^2)/t_k^2 \le \epsilon_0$ and $t_k \in (0, \epsilon_0)$ for all $k \ge k_0$. Then from (3.6) and (3.8), we obtain

$$g(x, \overline{y}(x) + t_k y(x) + t_k^2\psi_k(x)) \le -\epsilon_0 t_k^2 + \epsilon_0 t_k^2 = 0$$

for all $x \in \overline{\Omega}$ and for all $k \ge k_0$. The proof is complete. □

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