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APPROXIMATE KKT POINTS FOR SMOOTH VECTOR OPTIMIZATION PROBLEMS IN INFINITE DIMENSIONAL SPACES

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Abstract. In this paper, we establish a Karush-Kuhn-Tucker condition for a weak Pareto solution of a smooth constrained vector optimization problem. We introduce a kind of approximate KKT points for a smooth constrained vector optimization problem and establish the stability results for such approximate KKT points. In particular, under the weaker conditions, we extend and improve some results by Durea, Dutta and Tammer (Optimization, 60, 823-838 (2011)) to the infinite dimensional case.

1. Introduction

Karush-Kuhn-Tucker (KKT for short) condition is a fundamental notion and plays a very important role in optimization theory (cf. [1, 5, 6, 10, 13, 14, 18] and references therein). In the case of scalar optimization, the KKT condition has been extensively studied. For a constrained vector optimization problem, under the convexity assumption, a few results on the KKT condition have been obtained in the literature (cf. [8, 17]); but the study on the KKT condition is quite less for the nonconvex case. In this paper, we concern with the KKT condition for a smooth nonconvex constrained vector optimization problem. Under a weaker constrained qualification than the Mangasarian-Fromovitz condition, we establish the KKT optimality condition for a weak Pareto solution of such a vector optimization problem.

Usually, one uses the KKT condition to structure an iterative algorithm of a constrained optimization problem. However, it is possibly difficult to find exact KKT points for some constrained optimization problems. This gives rise to various approximate KKT points (cf. [3,4,8,9,19]). In this line, it is valuable and interesting to study under which conditions an iterative sequence consisting of approximate KKT points of a constrained optimization problem converges to one of its exact KKT points. Recently, Durea et al. [8] studied approximate KKT points for some constrained vector optimization problems. Let X, Y, Z, W be normed spaces, $K \subset Y$ and $Q \subset Z$ be closed convex cones and let $f : X \to Y$, $g : X \to Z$ and $h : X \to W$ be functions. Durea et al. [8] considered the following constrained optimization problem:

$$(\mathcal{P}) \qquad K - \min f(x) \quad \text{subject to } \; g(x) \in -Q \text{ and } h(x) = 0.$$

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In the case where f, g, h are smooth, Durea et al. [8] defined the notion of an ε-KKT point ($\varepsilon \geq 0$) of $(\mathcal{P})$ in terms of the derivatives ∇f, ∇g and ∇h. Under nonemptyness of both $\mathrm{int}(K)$ and $\mathrm{int}(Q)$ and surjectivity of $\nabla h(\bar{x})$ as well as the so-called Mangasarian-Fromovitz condition, they established a stability result on ε-KKT points of $(\mathcal{P})$. In this paper, motivated by Durea et al. [8], we introduce a different kind of approximate KKT points which are not necessarily feasible points of $(\mathcal{P})$ and can recapture the classical KKT points. Without the nonemptyness assumption on both $\mathrm{int}(K)$ and $\mathrm{int}(Q)$, we provide a stability result on approximate-KKT points of $(\mathcal{P})$. In particular, under the weaker assumptions, we extend and improve the stability result of approximate KKT points for $(\mathcal{P})$ by Durea et al. [8].

2. Preliminaries

Let Y be a normed linear space with the topological dual Y^*. For $y \in Y$ and $\delta > 0$, $B(y, \delta)$ stands for the open ball with center y and radius δ. As usual, we denote by B_Y and S_Y the closed unit ball and sphere of Y, respectively. Let K be a closed convex cone in Y, which specifies an order relation $\leq_K$ in Y as follows:

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in K.$$

We denote by K^+ the dual cone of K, that is,

$$K^+ := \{y^* \in Y^* : 0 \leq \langle y^*, k \rangle, \quad \forall\, k \in K\}.$$

For convenience, let

$$\mathbb{1}_{K^+} := \big\{y^* \in K^+ : \|y^*\| = 1\big\} \quad \text{and} \quad \rho_K := \inf\big\{\|y^*\| : y^* \in \mathrm{w}^*\text{–}\mathrm{cl}\big(\mathbb{1}_{K^+}\big)\big\}, \tag{2.1}$$

where $\mathrm{w}^*\text{–}\mathrm{cl}\big(\mathbb{1}_{K^+}\big)$ denotes the closure of $\mathbb{1}_{K^+}$ with respect to the weak* topology and ρ_K is understood as $+\infty$ if $\mathrm{w}^*\text{–}\mathrm{cl}\big(\mathbb{1}_{K^+}\big) = \emptyset$. Clearly, $\rho_K > 0$ if and only if $0 \notin \mathrm{w}^*\text{–}\mathrm{cl}\big(\mathbb{1}_{K^+}\big)$; moreover $\rho_K = 1$ when Y is finite dimensional. It is known and easy to verify that K has a nonempty interior if and only if there exists $y_0 \in Y$ such that

$$K^+ \subset \big\{y^* \in Y^* : \|y^*\| \leq \langle y^*, y_0 \rangle\big\}.$$

In order to relax the assumption that $\mathrm{int}(K) \neq \emptyset$, Zheng and Ng [20] adopted the so-called dually compact cone. Recall that a closed convex cone K of Y is said to be dually compact if there exists a compact subset C of Y such that

$$K^+ \subset \big\{y^* \in Y^* : \|y^*\| \leq \sup_{y \in C} \langle y^*, y \rangle\big\}. \tag{2.2}$$

It is clear that if Y is finite dimensional then every closed convex cone in Y is dually compact. The following lemma is useful for us and can be found in [17].

Lemma 2.1. *Let $K \subset Y$ be a closed convex cone. Then K is dually compact if and only if $\rho_K > 0$.*

Let A be a closed subset of X and $\bar{x} \in A$. We denote by $T(A, \bar{x})$ the Clarke tangent cone of A at $\bar{x}$. Then, $T(A, \bar{x})$ is a closed cone, and $u \in T(A, \bar{x})$ if and only if for any sequences $x_n \xrightarrow{A} \bar{x}$ and $t_n \to 0^+$ there exists $u_n \to u$ such that

$$x_n + t_n u_n \in A \quad \forall n \in \mathbb{N},$$

where $x_n \xrightarrow{A} \bar{x}$ means $x_n \to \bar{x}$ and $x_n \in A$ for all $n \in \mathbb{N}$. Let $N(A, \bar{x})$ denote the Clarke normal cone of A at $\bar{x}$, namely

$$N(A, \bar{x}) := \{x^* \in X^* : \langle x^*, u\rangle \leq 0 \text{ for all } u \in T(A, \bar{x})\}.$$

In the case when A is a convex set, it is well known (cf. [6]) that

$$T(A, \bar{x}) = \mathrm{cl}(\cup_{t\geq 0} t(A - \bar{x})) \text{ and } N(A, \bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0 \ \ \forall x \in A\}.$$

Let $F : X \rightrightarrows Y$ be a multifunction and let $\mathrm{gph}(F)$ and $\mathrm{dom}(F)$ denote respectively the graph and domain of F, that is,

$$\mathrm{gph}(F) := \{(x, y) \in X \times Y : y \in F(x)\} \text{ and } \mathrm{dom}(F) := \{x \in X : F(x) \neq \emptyset\}.$$

We say that F is closed (resp. convex) if $\mathrm{gph}(F)$ is a closed (resp. convex) subset of the product space $X \times Y$. Recall that F is metrically regular at $(\bar{x}, \bar{y})$ if there exist $\delta, \tau \in (0, +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)), \quad \forall (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta).$$

The metric regularity is a fundamental notion in optimization and variational analysis and has been well studied (cf. [2, 7, 11, 15, 16] and references therein).

The following lemma (cf. [2, Proposition 2.79 and Theorem 2.81]), known as the Robinson-Ursescu theorem, is useful for us.

Lemma 2.2. *Let X, Y be Banach spaces and $F : X \rightrightarrows Y$ be a closed convex multifunction. Let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$. Then the following statements are equivalent:*

(i) $\bar{y} \in \mathrm{int}(F(X))$, *where* $\mathrm{int}(\cdot)$ *denotes the interior.*

(ii) *There exists $\delta > 0$ such that*

$$B(\bar{y}, \delta t) \subset F(B(\bar{x}, t)), \quad \forall t \in (0, 1).$$

(iii) *F is metrically regular at $(\bar{x}, \bar{y})$.*

3. KKT condition for smooth vecrtor optimization

In this section, we consider constrained vector optimization problem $(\mathcal{P})$ in the smooth case. Throughout the remainder of this paper, let $f : X \to Y$, $g : X \to Z$, $h : X \to W$ be smooth functions, where X, Y, Z, W are normed spaces. The derivatives of f, g and h at $x \in X$ are denoted by $\nabla f(x), \nabla g(x)$ and $\nabla h(x)$, respectively. Let $K \subset Y$ and $Q \subset Z$ be closed convex cones. For convenience, let A denote the feasible set of $(\mathcal{P})$, that is,

$$A := g^{-1}(-Q) \cap h^{-1}(0).$$

In the case when $\mathrm{int}(K) \neq \emptyset$, we recall that a point $\bar{x}$ in A is said to be a weak Pareto solution of $(\mathcal{P})$ if

$$f(A) \cap (f(\bar{x}) - \mathrm{int}(K)) = \emptyset. \tag{3.1}$$

Durea et al. [8] adopted the so-called Mangasarian-Fromovitz condition for $(\mathcal{P})$. Recall (cf. [8]) that $(\mathcal{P})$ satisfies Mangasarian-Fromovitz condition at $\bar{x} \in A$ if $\mathrm{int}(Q) \neq \emptyset$ and there exists $u \in X$ such that

$$\nabla g(\bar{x})(u) \in -\mathrm{int}(Q) \quad \text{and} \quad \nabla h(\bar{x})(u) = 0. \tag{MF}$$

In this paper, we adopt the following weak Robinson qualification:

$$\text{(WRQ)} \qquad (0,0) \in \operatorname{int}\big(\operatorname{cl}\big((\nabla g(\bar{x}), \nabla h(\bar{x}))\big(B_X\big) + Q \times \{0\}\big)\big),$$

where $(\nabla g(\bar{x}), \nabla h(\bar{x}))(B_X) := \{(\nabla g(\bar{x})(x), \nabla h(\bar{x})(x)) : \ x \in B_X\}$. Firstly, we provide a lemma which is useful in the proof of the main result in the section.

Lemma 3.1. *Let X, Z and W be normed vector spaces and let $\bar{x} \in X$ and $r \in (0, +\infty)$. Then,*

$$r(B_Z \times B_W) \subset \operatorname{cl}\big(\big(\nabla g(\bar{x}), \nabla h(\bar{x})\big)\big(B_X\big) - Q \times \{0\}\big) \tag{3.2}$$

if and only if

$$r(\|z^*\| + \|w^*\|) \leq \|z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x})\|, \quad \forall\ (z^*, w^*) \in Q^+ \times W^*. \tag{3.3}$$

Consequently, (WRQ) holds if and only if there exists $r > 0$ such that (3.3) holds.

Proof. First suppose that (3.2) holds. Let $z \in B_Z$ and $w \in B_W$. Then, there exist $\{x_n\} \subset B_X$ and $\{q_n\} \subset Q$ such that

$$\big(\nabla g(\bar{x})(x_n) - q_n, \nabla h(\bar{x})(x_n)\big) \longrightarrow r(z, w).$$

Let $(z^*, w^*) \in Q^+ \times W^*$. Then $0 \leq \langle z^*, q_n\rangle$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}
\langle z^*, rz\rangle + \langle w^*, rw\rangle &= \lim_{n\to\infty} (\langle z^*, \nabla g(\bar{x})(x_n) - q_n\rangle + \langle w^*, \nabla h(\bar{x})(x_n)\rangle) \\
&\leq \liminf_{n\to\infty} \langle z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x}), x_n\rangle \\
&\leq \|z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x})\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z^*\ \| + \|\ w^*\| &= \sup\{\langle z^*, z\rangle + \langle w^*, w\rangle : (z, w) \in B_Z \times B_W\} \\
&\leq \frac{1}{r} \|z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x})\|.
\end{aligned}$$

This shows that (3.3) holds.

Conversely, suppose that (3.3) holds. To prove (3.2), suppose to the contrary that there exists

$$(z_0, w_0) \in r(B_Z \times B_W) \setminus \operatorname{cl}\big(\big(\nabla g(\bar{x}), \nabla h(\bar{x})\big)\big(B_X\big) + Q \times \{0\}\big). \tag{3.4}$$

Thus, by the separation theorem, there exist $(z^*, w^*) \in Z^* \times W^*$ and $\alpha \in \mathbb{R}$ such that

$$\sup\big\{\langle z^*, \nabla g(\bar{x})(x) + q\rangle + \langle w^*, \nabla h(\bar{x})(x)\rangle : \ (x, q) \in B_X \times Q\big\} < \alpha < \langle z^*, z_0\rangle + \langle w^*, w_0\rangle. \tag{3.5}$$

Hence,

$$-z^* \in Q^+ \ \text{ and } \ \sup\big\{\langle z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x}), x\rangle : \ x \in B_X\big\} < \alpha.$$

This implies that

$$\|z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x})\| < \alpha. \tag{3.6}$$

On the other hand, by (3.4), one has

$$\langle z^*, z_0\rangle + \langle w^*, w_0\rangle \leq \|z^*\| \cdot \|z_0\| + \|w^*\| \cdot \|w_0\| \leq r(\|z^*\| + \|w^*\|).$$

This, together with (3.5) and (3.6), implies that

$$\|z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x})\| < r(\|z^*\| + \|w^*\|),$$

contradicting (3.3) (by $-z^* \in Q^+$). The proof is completed. □

Next we provide some characterizations of the weak Robinson qualification. To do this, recall that a subset A of a Banach space X is called to be CS-closed if $\sum_{n=1}^{\infty} t_n a_n \in A$ for any $\{a_n\} \subset A$ and $\{t_n\} \subset [0,\ 1]$ with $\sum_{n=1}^{\infty} t_n = 1$. The following fact is useful and well-known in functional analysis (cf. [12, p.183, Theorem A.1]).

Fact CS. *Let A be a CS-closed subset of a Banach space X. Then* $\text{int}(A) = \text{int}(\text{cl}(A))$.

Proposition 3.2. *Let X, Z, W be Banach spaces and $\bar{x}$ be a feasible point of ($\mathcal{P}$) and consider following statements:*

(i) *(WRQ) holds.*
(ii) $(0,0) \in \text{int}\{(\nabla g(\bar{x}), \nabla h(\bar{x}))(B_X) + Q \times \{0\}\}$.
(iii) $(\nabla g(\bar{x}), \nabla h(\bar{x}))(X) + Q \times \{0\} = Z \times W$.
(iv) Φ *is metrically regular at* $(\bar{x}, (g(\bar{x}), h(\bar{x})))$, *where* $\Phi(x) := (g(x), h(x)) + Q \times \{0\}$ *for all* $x \in X$.
(v) Φ *is metrically regular at* $(\bar{x}, (0,0))$.

Then (i)⇔(ii)⇔(iii)⇔(iv)⇔(v).

Proof. Since X, Z, W are Banach spaces, it is known and easy to verify that $(\nabla g(\bar{x}), \nabla h(\bar{x}))(\varepsilon B_X) + Q \times \{0\}$ is a CS-closed set for any $\varepsilon > 0$. This and Fact CS imply that

$$\text{int}\big(\text{cl}\big((\nabla g(\bar{x}), \nabla h(\bar{x}))(\varepsilon B_X) + Q \times \{0\}\big)\big) = \text{int}\big(\nabla g(\bar{x}), \nabla h(\bar{x}))(\varepsilon B_X) + Q \times \{0\}\big), \quad \forall \varepsilon > 0.$$

Hence (i) and (ii) are equivalent. Let $F : X \rightrightarrows Z \times W$ be such that

$$F(x) = (\nabla g(\bar{x})(x - \bar{x}), \nabla h(\bar{x})(x - \bar{x})) + Q \times \{0\}, \quad \forall x \in X.$$

Then F is a closed convex multifunction. Thus, by Lemma 2.2, (ii) and (iii) are equivalent. Since $g(\bar{x}) \in -Q$ and $h(\bar{x}) = 0$, $\{(0,0), (-g(\bar{x}), -h(\bar{x}))\} \subset F(\bar{x})$. This and Lemma 2.2 imply that (iii) is equivalent to anyone of the following (1) an (2):
(1) F is metrically regular at $(\bar{x}, (0,0))$.
(2) F is metrically regular at $(\bar{x}, (-g(\bar{x}), -h(\bar{x})))$.
Let

$$\tilde{g}(x) := (g(x), h(x)) - (\nabla g(\bar{x})(x - \bar{x}), \nabla h(\bar{x})(x - \bar{x})), \quad \forall x \in X.$$

Since g and h are continuously differentiable, it is easy from the mean-value theorem to prove that for any $\varepsilon > 0$ there exists $r > 0$ such that

$$\|\tilde{g}(x_1) - \tilde{g}(x_2)\| \leq \varepsilon \|x_1 - x_2\|, \quad \forall x \in B(\bar{x}, r).$$

Noting that $\Phi(x) = F(x) + \tilde{g}(x)$ for all $x \in X$, it follows from [16, Theorem 10.3.6] that (1) is equivalent to (iv) while (2) is equivalent to (v). The proof is complete. □

Proposition 3.3. *Let X, Z and W be Banach spaces. Suppose that* (WRQ) *holds. Then,*

$$N(A,\bar{x}) = \{z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x}) :\ (z^*, w^*) \in Q^+ \times W^* \text{ and } \langle z^*, g(\bar{x})\rangle = 0\}$$

for any $\bar{x} \in A$.

Proof. Given $\bar{x} \in A$, we claim that

$$T(A,\bar{x}) = \nabla g(\bar{x})^{-1}(T(-Q, g(\bar{x}))) \cap \nabla h(\bar{x})^{-1}(0) \tag{3.7}$$

Let $u \in T(A,\bar{x})$. Then there exist $t_n \to 0^+$ and $u_n \to u$ such that $\bar{x} + t_n u_n \in A$ and so $g(\bar{x} + t_n u_n) \in -Q$ and $h(\bar{x} + t_n u_n) = 0$ for all $n \in \mathbb{N}$. Since g and h are smooth and $h(\bar{x}) = 0$, one has

$$g(\bar{x}) + t_n \nabla g(\bar{x})(u_n) + o(t_n) \in -Q \quad \text{and } t_n \nabla h(\bar{x})(u_n) + o(t_n) = 0$$

Noting that $\nabla g(\bar{x})(u_n) + o(t_n)/t_n \to \nabla g(\bar{x})(u)$, it follows from the convexity of Q that $\nabla g(\bar{x})(u) \in T(-Q, g(\bar{x}))$ and $\nabla h(\bar{x})(u) = 0$, that is, $u \in \nabla g(\bar{x})^{-1}(T(-Q, g(\bar{x}))) \cap \nabla h(\bar{x})^{-1}(0)$. This shows that

$$T(A,\bar{x}) \subset \nabla g(\bar{x})^{-1}(T(-Q, g(\bar{x}))) \cap \nabla h(\bar{x})^{-1}(0).$$

Conversely, let $u \in \nabla g(\bar{x})^{-1}(T(-Q, g(\bar{x}))) \cap \nabla h(\bar{x})^{-1}(0)$ and take any sequences $x_n \xrightarrow{A} \bar{x}$ and $t_n \to 0^+$. Then,

$$g(x_n) \xrightarrow{-Q} g(\bar{x}),\ h(x_n) = 0\ (\forall n \in \mathbb{N}),\ \nabla g(\bar{x})(u) \in T(-Q, g(\bar{x})) \text{ and } \nabla h(\bar{x})(u) = 0.$$

Hence, there exist $z_n \to \nabla g(\bar{x})(u)$ such that $g(x_n) + t_n z_n \in -Q$ for all $n \in \mathbb{N}$. From the smoothness of g and h, it is easy to verify that there exist $v_n \xrightarrow{Z} 0$ and $e_n \xrightarrow{W} 0$ such that

$$(g(x_n + t_n u), h(x_n + t_n u)) + t_n(-v_n, -e_n) \in -Q \times \{0\} \quad \forall n \in \mathbb{N},$$

namely $t_n(v_n, e_n) \in \Phi(x_n + t_n u)$ for all $n \in \mathbb{N}$. By Proposition 3.2,

$$d(x_n + t_n u, A) = d(x_n + t_n u, \Phi^{-1}((0,0))) \le \tau d((0,0), \Phi(x_n + t_n u)) \le \tau t_n(\|v_n\| + \|e_n\|)$$

for all sufficiently large n, where τ is a positive constant. Hence there exists $a_n \in A$ such that $\|x_n + t_n u - a_n\| \le 2\tau t_n(\|v_n\| + \|e_n\|)$. Let $u_n = \frac{a_n - x_n}{t_n}$. Then $x_n + t_n u_n = a_n \in A$ and $\|u_n - u\| \le 2\tau(\|v_n\| + \|e_n\|) \to 0$. This shows that $u \in T(A,\bar{x})$. Therefore, (3.7) holds and

$$\begin{aligned} N(A,\bar{x}) &= N\big(\nabla g(\bar{x})^{-1}(T(-Q, g(\bar{x}))) \cap \nabla h(\bar{x})^{-1}(0), 0\big) \\ &= N\big((\nabla g(\bar{x}), \nabla h(\bar{x}))^{-1}(T(-Q, g(\bar{x})) \times \{0\}), 0\big) \\ &= N\big((\nabla g(\bar{x}), \nabla h(\bar{x}))^{-1}(-T(Q, -g(\bar{x})) \times \{0\}), 0\big). \end{aligned}$$

Since Q is a closed convex cone, $Q \subset T(Q, -g(\bar{x}))$. It follows from Proposition 3.2 that $(\nabla g(\bar{x}), \nabla h(\bar{x}))(X) + T(Q, -g(\bar{x})) \times \{0\} = Z \times W$. Thus, by [17, Theorem 3.1],

$$\begin{aligned} &N(A, \bar{x}) \\ &= N\big((\nabla g(\bar{x}), \nabla h(\bar{x}))^{-1}(-T(Q, -g(\bar{x})) \times \{0\}), 0\big) \\ &= \big\{\partial((z^*, w^*) \circ (\nabla g(\bar{x}), \nabla h(\bar{x})))(0) : \ (z^*, w^*) \in N\big(-T(Q, -g(\bar{x})) \times \{0\}, (0,0)\big)\big\} \\ &= \big\{\partial((z^*, w^*) \circ (\nabla g(\bar{x}), \nabla h(\bar{x})))(0) : \ (z^*, w^*) \in N\big(-Q, g(\bar{x})\big) \times W^*\big\} \\ &= \big\{z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x}) : \ (z^*, w^*) \in Q^+ \times W^* \text{ and } \langle z^*, g(\bar{x})\rangle = 0\big\}. \end{aligned}$$

The proof is complete. □

Theorem 3.4. *Let X, Y, Z and W be normed vector spaces. Let* $\operatorname{int}(K) \neq \emptyset$ *and $\bar{x}$ be a weak Pareto solution of ($\mathcal{P}$). Suppose that that* (WRQ) *holds. Then, there exist $y^* \in \mathbb{1}_{K^+}$, $z^* \in Q^+$ and $w^* \in W^*$ such that*

$$y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}) + w^* \circ \nabla h(\bar{x}) = 0 \ \text{ and } \ \langle z^*, g(\bar{x})\rangle = 0. \tag{3.8}$$

Proof. Since $\bar{x}$ is a weak Pareto solution of ($\mathcal{P}$), (3.1) holds. We claim that

$$\nabla f(\bar{x})\big[T(A, \bar{x})\big] \cap \operatorname{int}(-K) = \emptyset. \tag{3.9}$$

Granting this, by the separation theorem, there exists $y^* \in \mathbb{1}_{K^+}$ such that

$$\inf\{\langle y^*, \nabla f(\bar{x})(u)\rangle : \ u \in T(A, \bar{x})\} \geq 0.$$

It follows that

$$0 \in y^* \circ \nabla f(\bar{x}) + N(A, \bar{x}).$$

This and Proposition 3.3 imply that there exist $z^* \in Q^+$ and $w^* \in W^*$ such that (3.8) holds. It remains to show that (3.9) holds. Let $u \in T(A, \bar{x})$. Then there exists $u_n \to u$ such that $\bar{x} + \frac{u_n}{n} \in A$ for all $n \in \mathbb{N}$. Hence, by (3.1),

$$f\Big(\bar{x} + \frac{u_n}{n}\Big) = f(\bar{x}) + \frac{1}{n}\nabla f(\bar{x})(u_n) + o\Big(\frac{1}{n}\Big) \notin f(\bar{x}) - \operatorname{int}(K) \quad \forall n \in \mathbb{N}$$

and so $\nabla f(\bar{x})(u_n) + no(\frac{1}{n}) \notin -\operatorname{int}(K)$ for all $n \in \mathbb{N}$. Therefore, $\nabla f(\bar{x})(u) \notin -\operatorname{int}(K)$. This shows that (3.9) holds. The proof is complete. □

4. Approximate KKT point

Durea et al. [8] introduced the following notion of approximate KKT points for ($\mathcal{P}$).

Definition 4.1. Let $\varepsilon \geq 0$. A feasible point $\bar{x}$ of ($\mathcal{P}$) is said to be an ε-KKT point of ($\mathcal{P}$) if there exist $y^* \in \mathbb{1}_{K^+}$, $z^* \in Q^+$ and $p^* \in W^*$ such that

$$\|y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}) + p^* \circ \nabla h(\bar{x})\| \leq \varepsilon. \tag{4.1}$$

Let $T(\varepsilon)$ denote the set of all ε-KKT points of ($\mathcal{P}$).

In the case when $Y = \mathbb{R}$, $Z = \mathbb{R}^m$, $W = \mathbb{R}^n$, $K = \mathbb{R}_+$ and $Q = \mathbb{R}^m_+$, ($\mathcal{P}$) reduces to the following standard scalar constrained problem

($\tilde{\mathcal{P}}$) $\quad \min f(x)$ subject to $g_i(x) \leq 0$ $(i = 1, \ldots, m)$ and $h_1(x) = \cdots = h_n(x) = 0$.

The KKT point (or KKT condition) for $(\tilde{\mathcal{P}})$ is a classical notion in optimization and can be stated as follows: a feasible point $\bar{x}$ of $(\tilde{\mathcal{P}})$ is a KKT point of $(\tilde{\mathcal{P}})$ if there exist $\lambda_i \in \mathbb{R}_+$ $(i = 1, \dots, m)$ and $\mu_j \in \mathbb{R}$ $(j = 1, \dots, n)$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{n} \mu_j \nabla h_i(\bar{x}) = 0 \text{ and } \lambda_i g_i(\bar{x}) = 0 \ (i = 1, \dots, m).$$

In this case, a point in $T(0)$ is not necessarily a KKT point of $(\tilde{\mathcal{P}})$ (because (4.1) does not imply that $\lambda_i g_i(\bar{x}) = 0$ $(i = 1, \dots, m)$ even when $\varepsilon = 0$). This motivates us to introduce the following kind of approximate KKT points.

Definition 4.2. Let $\varepsilon \geq 0$. A point $\bar{x}$ in X is said to be a normal ε-KKT point of $(\mathcal{P})$ if there exist $y^* \in \mathbb{1}_{K^+}$, $z^* \in Q^+$ and $p^* \in W^*$ such that (4.1) holds and

$$d(g(\bar{x}), -Q) \leq \varepsilon, \ \|h(\bar{x})\| \leq \varepsilon, \ |\langle z^*, g(\bar{x}) \rangle| \leq \varepsilon. \tag{4.2}$$

Let $\hat{T}(\varepsilon)$ denote the set of all normal ε-KKT points of $(\mathcal{P})$.

In contrast with the KKT point notion by Durea et al., a normal ε-KKT point of $(\mathcal{P})$ reduces to the classical KKT point in the special case when $\varepsilon = 0$, $Y = \mathbb{R}$, $Z = \mathbb{R}^m$, $W = \mathbb{R}^n$, $K = \mathbb{R}_+$ and $Q = \mathbb{R}^m_+$; moreover a normal ε-KKT point is not necessarily a feasible point of $(\mathcal{P})$. Thus, the normal ε-KKT points may be useful from the computational point of view.

Under the Mangasarian-Fromovitz condition, Durea et al. [8] proved the following theorem.

Theorem A. *Let X, Y, Z be Banach spaces and W be finite dimensional, and let* $\operatorname{int}(K) \neq \emptyset$ *and* $\operatorname{int}(Q) \neq \emptyset$. *Let $\bar{x}$ be a feasible point of $(\mathcal{P})$ such that* (MF) *is satisfied and $\nabla h(\bar{x})(X) = W$. Then the following statements hold*:

(i) *If $\bar{x} \in \limsup_{\mu \to 0^+} T(\mu)$ then $\bar{x} \in T(0)$.*

(ii) *Suppose that Y is finite dimensional and $\varepsilon > 0$. If $\bar{x} \in \limsup_{\mu \to \varepsilon^+} T(\mu)$, then $\bar{x} \in T(\varepsilon)$.*

Relaxing the assumptions that $\operatorname{int}(K)$ and $\operatorname{int}(Q)$ are nonempty and that (MF) holds, we extend Theorem A to the case when W and Y are infinite dimensional.

Theorem 4.3. *Let X, Y, Z, W be normed vector spaces. Let $\bar{x}$ be a feasible point of $(\mathcal{P})$ and $\varepsilon \geq 0$. Suppose that the ordering cone K is dually compact and that* (WRQ) *holds. Then, $\bar{x} \in \limsup_{\mu \to \varepsilon^+} \hat{T}(\mu) \Rightarrow \bar{x} \in \hat{T}(\frac{\varepsilon}{\rho_K})$, where ρ_K is as in (2.1). Consequently, $\bar{x} \in \limsup_{\mu \to 0^+} \hat{T}(\mu) \Rightarrow \bar{x} \in \hat{T}(0)$.*

Proof. Suppose that $\bar{x} \in \limsup_{\mu \to \varepsilon^+} \hat{T}(\mu)$. Then, there exist $\mu_n \to \varepsilon^+$ and $x_n \to \bar{x}$ such that each x_n is a normal μ_n-KKT point of $(\mathcal{P})$. Hence

$$d(g(x_n), -Q) \leq \mu_n \text{ and } \|h(x_n)\| \leq \mu_n. \tag{4.3}$$

Take $y_n^* \in \mathbb{1}_{K^+}$ and $(z_n^*, w_n^*) \in Q^+ \times W^*$ such that

$$\||\langle z_n^*, g(x_n) \rangle| \leq \mu_n \text{ and } \|y_n^* \circ \nabla f(x_n) + z_n^* \circ \nabla g(x_n) + w_n^* \circ \nabla h(x_n)\| \leq \mu_n. \tag{4.4}$$

By (WRQ) and Lemma 3.1, there exists $r > 0$ such that

$$r(\|z_n^*\| + \|w_n^*\|) \le \|z_n^* \circ \nabla g(\bar{x}) + w_n^* \circ \nabla h(\bar{x})\|, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Since $x_n \to \bar{x}$ and g, h are continuously differentiable on X, without loss of generality, we can assume that

$$\max\{\|\nabla g(x_n) - \nabla g(\bar{x})\|, \|\nabla h(x_n) - \nabla h(\bar{x})\|\} \le \frac{r}{2}, \quad \forall n \in \mathbb{N}.$$

It follows from the second inequality of (4.4) that, for any $n \in \mathbb{N}$,

$$\begin{aligned} \|z_n^* \circ \nabla g(\bar{x}) + w_n^* \circ \nabla h(\bar{x})\| &\le \|z_n^* \circ (\nabla g(\bar{x}) - \nabla g(x_n)) + w_n^* \circ (\nabla h(\bar{x}) - \nabla h(x_n))\| \\ &\quad + \|y_n^* \circ \nabla f(x_n)\| \\ &\quad + \|y_n^* \circ \nabla f(x_n) + z_n^* \circ \nabla g(x_n) + w_n^* \circ \nabla h(x_n)\| \\ &\le \frac{r}{2}(\|z_n^*\| + \|w_n^*\|) + \|\nabla f(x_n)\| + \mu_n. \end{aligned}$$

Thus, by (4.5), one has

$$\frac{r}{2}(\|z_n^*\| + \|w_n^*\|) \le \mu_n + \|\nabla f(x_n)\|, \quad \forall n \in \mathbb{N}.$$

Noting that $\mu_n \to \varepsilon$ and $\|\nabla f(x_n)\| \to \|\nabla f(\bar{x})\|$, this implies that $\{\|z_n^*\|\}$ and $\{\|w_n^*\|\}$ are bounded. Without loss of generality, we can assume that $z_n^* \xrightarrow{w^*} z^*$, $w_n^* \xrightarrow{w^*} p^*$, $y_n^* \xrightarrow{w^*} y^*$ (taking subnets if necessary). Then, $z^* \in Q^+$, $y^* \in \text{w}^*\text{-cl}(\mathbb{1}_{K^+})$ and

$$y_n^* \circ \nabla f(\bar{x}) + z_n^* \circ \nabla g(\bar{x}) + w_n^* \circ \nabla h(\bar{x}) \xrightarrow{w^*} y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}) + p^* \circ \nabla h(\bar{x}).$$

Since $\|\nabla f(x_n) - \nabla f(\bar{x})\| \to 0$, $\|\nabla g(x_n) - \nabla g(\bar{x})\| \to 0$ and $\|\nabla h(x_n) - \nabla h(\bar{x})\| \to 0$, it follows that

$$y_n^* \circ \nabla f(x_n) + z_n^* \circ \nabla g(x_n) + w_n^* \circ \nabla h(x_n) \xrightarrow{w^*} y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}) + p^* \circ \nabla h(\bar{x}).$$

Thus, by (4.4), one has $\|y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}) + p^* \circ \nabla h(\bar{x})\| \le \varepsilon$. Since K is dually compact and $y^* \in \text{w}^*\text{-cl}(\mathbb{1}_{K^+})$, it follows from Lemma 2.1 and the definition of ρ_K that $0 < \rho_K \le \|y^*\| \le 1$ and so

$$\left\| \frac{y^*}{\|y^*\|} \circ \nabla f(\bar{x}) + \frac{z^*}{\|y^*\|} \circ \nabla g(\bar{x}) + \frac{p^*}{\|y^*\|} \circ \nabla h(\bar{x}) \right\| \le \frac{\varepsilon}{\rho_K}. \tag{4.6}$$

On the other hand, by (4.3) and the first inequality of (4.4), one also has

$$d(g(\bar{x}) - Q) \le \varepsilon \le \frac{\varepsilon}{\rho_K}, \; \|h(\bar{x})\| \le \varepsilon \le \frac{\varepsilon}{\rho_K} \; \text{ and } \; \left\langle \frac{z^*}{\|y^*\|}, g(\bar{x}) \right\rangle \le \frac{\varepsilon}{\|y^*\|} \le \frac{\varepsilon}{\rho_K}.$$

This and (4.6) imply that $\bar{x} \in \hat{T}(\frac{\varepsilon}{\rho_K})$. The proof is completed. □

Corollary 4.4. *Let X, Z, W be Banach spaces and Y be finite dimensional. Let $\bar{x} \in X$ and $\varepsilon \ge 0$. Suppose that*

$$\big(\nabla g(\bar{x}), \nabla h(\bar{x})\big)\big(X\big) + Q \times \{0\} = Z \times W.$$

Then, $\bar{x} \in \limsup_{\mu \to \varepsilon^+} \hat{T}(\mu) \Longrightarrow \bar{x} \in \hat{T}(\varepsilon)$.

Proof. Since Y is finite dimensional, the ordering cone K in Y is dually compact and the weak* topology is identical with the norm topology in Y^*. Hence $\mathbb{1}_{K^+}$ is closed with respect to the weak* topology and so $\rho_K = 1$. Thus, the proof is completed by Theorem 3.4 and Proposition 3.4. $\square$

With $\hat{T}(\mu)$ replaced by $T(\mu)$, one can similarly prove the following result (in fact its proof is simpler than that of Theorem 4.3).

Theorem 4.5. *Let X, Y, Z, W be normed vector spaces. Let $\bar{x}$ be a feasible point of $(\mathcal{P})$ and $\varepsilon \geq 0$. Suppose that the ordering cone K is dually compact and that* (WRQ) *holds. Then, $\bar{x} \in \limsup_{\mu \to \varepsilon^+} T(\mu) \Rightarrow \bar{x} \in T(\frac{\varepsilon}{\rho_K})$, where ρ_K is as in (2.1). Consequently, $\bar{x} \in \limsup_{\mu \to 0^+} T(\mu) \Rightarrow \bar{x} \in T(0)$.*

Remark 4.6. It is possibly interesting to compare the assumptions in Theorems 4.3 and 4.5 with the corresponding ones in Theorem A. Recall that Theorem A requires the following assumptions:

(A1) X, Y, Z are Banach spaces and W is finite dimensional,
(A2) the ordering cones K and Q have nonempty interiors, and
(A3) $\nabla h(\bar{x})(X) = W$ and (MF) is satisfied.

In Theorems 4.3 and 4.5, (A1) is relaxed to the assumption that X, Y, Z and W are normed spaces and (A2) is relaxed to the assumption that K is dually compact. Now, in the case when X, Z, W are Banach spaces, we show that (A3) is stronger than (WRQ) in Theorems 4.3 and 4.5. To do this, we claim that if (MF) holds, then

$$(\nabla g(\bar{x}), \nabla h(\bar{x}))(X) + Q \times \{0\} = Z \times \nabla h(\bar{x})(X). \tag{4.7}$$

Indeed, by (MF), there exists $\bar{u} \in X$ such that $\nabla g(\bar{x})(\bar{u}) \in -\text{int}Q$ and $\nabla h(\bar{x})(\bar{u}) = 0$. Therefore, there exists $\delta > 0$ such that $\delta B_Z \subset \nabla g(\bar{x})(\bar{u}) + Q$ and so

$$Z = \mathbb{R}_+ \delta B_Z \subset \nabla g(\bar{x})(\mathbb{R}_+ \bar{u}) + \mathbb{R}_+ Q = \nabla g(\bar{x})(\mathbb{R}_+ \bar{u}) + Q.$$

It follows that

$$Z = Z + \nabla g(\bar{x})(x) = \nabla g(\bar{x})(\mathbb{R}_+ \bar{u} + x) + Q, \quad \forall x \in X.$$

Hence

$$Z \times \{\nabla h(\bar{x})(x)\} = (\nabla g(\bar{x}), \nabla h(\bar{x}))(\mathbb{R}_+ \bar{u} + x) + Q \times \{0\}, \quad \forall x \in X.$$

This implies that (4.7) holds. Thus, in the case when X, Z, W are Banach spaces, Proposition 3.2 and (4.7) imply that

$$[\text{(MF) and } \nabla h(\bar{x})(X) = W] \Longrightarrow \text{(WRQ)}.$$

Therefore, Theorem 4.5 extend and improve Theorem A.

We conclude the section with an example which shows that the assumptions of Theorems 4.3 and 4.5 are fulfilled but the assumptions of Theorem A are not satisfied. Let

$$X = \mathbb{R}^3, \ Y = Z = \mathbb{R}^2, \ W = \mathbb{R}, \ K = Q = \mathbb{R}_+ \times \{0\},$$

and let $f, g : \mathbb{R}^3 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$f(u_1, u_2, u_3) = (u_1^2, 0),\ g(u_1, u_2, u_3) = (u_1 + u_2, u_2 + u_3)$ and $h(u_1, u_2, u_3) = u_1 + u_3$ for all $(u_1, u_2, u_3) \in \mathbb{R}^3$. Then K is dually compact. Noting that g and h are linear, it is easy to verify that

$$\big(\nabla g(x), \nabla h(x)\big)\Big(\frac{s_1 - s_2 + s_3}{2}, \frac{s_1 + s_2 - s_3}{2}, \frac{s_2 + s_3 - s_1}{2}\Big) = (s_1, s_2, s_3)$$

for all $x \in \mathbb{R}^3$ and $(s_1, s_2, s_3) \in \mathbb{R}^3$, and so

$$\big(\nabla g(x), \nabla h(x)\big)(X) + Q \times \{0\} = \mathbb{R}^2 \times \mathbb{R}, \quad \forall x \in \mathbb{R}^3.$$

Let $\bar{x} = (0, 0, 0)$. Then $\bar{x}$ is a feasible point of $(\mathcal{P})$ and the assumptions of Theorem 4.3 are fulfilled at $\bar{x}$. But, $\mathrm{int}(K) = \mathrm{int}(Q) = \emptyset$; hence either (MF) or the nonemptyness of $\mathrm{int}(K)$ is not satisfied.

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