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METRIC REGULARITY AND ULAM-HYERS STABILITY RESULTS FOR COINCIDENCE PROBLEMS WITH MULTIVALUED OPERATORS

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Abstract. Open covering and metric regularity are two important properties which play an important role in several topics of modern variational analysis. In this paper, we will present some existence and Ulam-Hyers stability results for coincidence point problems with multivalued operators. The basic hypothesis in these results is the metric regularity.

1. Introduction and preliminaries

Let (X, d) be a metric space and $\mathcal{P}(X)$ be the set of all subsets of X. Consider the following families of subsets of X:

$$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\},\ P_b(X) := \{Y \in P(X) |\ Y \text{ is bounded}\},$$

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\},\ P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\}.$$

If (X, d) is a metric space, then we define

- the gap functional generated by d:

  $$D_d : P(X) \times P(X) \to \mathbb{R}_+,\ D_d(A, B) := \inf\{d(a, b) \mid a \in A,\ b \in B\}.$$

  In particular, if $x_0 \in X$, we put $D_d(x_0, B)$ instead of $D_d(\{x_0\}, B)$.
- the excess functional of A over B generated by d:

  $$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ e_d(A, B) := \sup\{D_d(a, B), a \in A\}.$$
- the Hausdorff-Pompeiu functional generated by d:

  $$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$

Let (X, d) be a metric space. If $F : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for F if and only if $x \in F(x)$. The set $Fix(F) := \{x \in X |\ x \in F(x)\}$ is called the fixed point set of F.

We say that $F : X \to P(Y)$ is onto if and only if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$.

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For the following notions see I.A. Rus [15], I.A. Rus, A. Petruşel, A. Sîntămărian [16] and A. Petruşel [14].

Definition 1.1. Let (X, d) be a metric space, and $F : X \to P_{cl}(X)$ be a multivalued operator. By definition, F is a multivalued weakly Picard (briefly MWP) operator if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:
(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Remark 1.2. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the condition *(*i*)* and *(*ii*)*, in the Definition *1.1* is called a sequence of successive approximations of F starting from $(x, y) \in Graph(F)$.

If $F : X \to P(X)$ is a MWP operator, then we define $F^\infty : Graph(F) \to P(FixF)$ by the formula $F^\infty(x, y) := \{\ z \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x, y) that converges to $z\ \}$.

Definition 1.3. Let (X, d) be a metric space and $F : X \to P(X)$ be a MWP operator. Then, F is called a ψ-multivalued weakly Picard operator (briefly ψ-MWP operator) if and only $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 with $\psi(0) = 0$ and there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x, y)) \le \psi(d(x, y)),\ \text{for all } (x, y) \in Graph(F).$$

If, in particular, ψ has a linear representation (i.e., there exists $c > 0$ such that $\psi(t) = ct$ for all $t \in \mathbb{R}_+$), then F is called a c-MWP operator.

By Covitz-Nadler point principle (see [4], [12]) we get the following example.

Remark 1.4 ([16], [14])**.** Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multivalued k-contraction, i.e., $k \in [0, 1[$ and

$$H_d(F(x), F(u)) \le kd(x, u),\ \text{for all } x, u \in X.$$

Then F is a $\frac{1}{1-k}$-MWP operator.

The concept for metric regularity of multivalued operators appeared at the end of the 1970s when a new branch of analysis later known as "non-smooth" analysis, started the development of non-smooth analysis was mainly stimulated by the needs of optimization theory. But the sources of the concept of metric regularity should be sought in classical theorems of differential calculus and linear analysis.

Metric regularity is a local property, we can obtain new results of fixed points in conditions which characterize theorems of fixed points. In general, metric regularity deals with the study of equation of the type $y \in F(x)$, where $y \in X$ is fixed, for a multivalued operator $F : X \to P(Y)$. Many authors have obtained results in the metric regularity field among whom we remind A. L. Dontchev, A. S. Lewis, R. T. Rockafellar [5], A. L. Dontchev, A. S. Lewis [6], A. D. Ioffe [7], A. D. Ioffe [8], L. A. Lyusternik [9] and others. The norm and the radius of metric regularity characterize this property. A point x is an approximate solution of a generalized equation $y \in F(x)$ if the distance from the point y to the set $F(x)$ is small. The metric regularity of the multivalued operator F means that, locally, a constant multiple of this distance bounds the distance from x to an exact solution. The smallest such constant is the modulus of regularity and it is a measure of the sensitivity or conditioning of the generalized equation.

Let $F : X \to P(Y)$ be a multivalued operator between metric spaces (X, d) and (Y, d), and $U \subseteq X$, $V \subseteq Y$ given subsets. According to A. D. Ioffe [7] and B. S. Mordukhovich [11], F is said to *cover on* (or to be *open at a linear rate*) with respect to $U \times V$ if there exists a positive constant a such that

$$F(B(x,r)) \supseteq B(F(x) \cap V, ar), \text{ for all } x \in U, r > 0 : B(x,r) \subseteq U, \tag{1.1}$$

where $B(x,r) = \{z \in X : d(z,x) < r\}$ denotes the open ball centered at x, with radius r, and $B(Q,r) := \bigcup_{y \in Q} B(y,r)$ the r-neighborhood of the set Q.

The supremum of all constants satisfying inclusion (1.1) is called *modulus of open covering* of F with respect to $U \times V$ and is denoted by $cov_{U \times V} F$. Such property along with the corresponding modulus, clearly relates to metrical aspects of the surjective behavior of F. It captures a phenomenon around which many fundamental issues of modern variational analysis turn out to revolve. In one of its several manifestations, known and widely employed under the name of *metric regularity*, it takes the form of an inequality providing an estimation for haw far a point x is from being a solution to the generalized equation $y \in F(x)$. In the most developed theorems of subdifferential calculus, all qualification conditions appear to be regularity/ open covering conditions for certain multivalued operators, see A. D. Ioffe [7] and B. S. Mordukhovich [11].

The notion of open covering to the global case is the case in which $U = X$ and $V = Y$.

Definition 1.5 (A. D. Ioffe [7])**.** A multivalued operators $F : X \to P(Y)$ between metric spaces (X, d) and (Y, d) is said cover on X (or to be globally open at a linear rate), provided that there exists a constant $a > 0$ such that

$$F(B(x,r)) \supseteq B(F(x), ar), \text{ for all } x \in X, r > 0. \tag{1.2}$$

The supremum over all values a satisfying inclusion (1.2) is called modulus of global covering of F and denoted for short by $cov(F)$ (instead of $cov_{X \times Y} F$).

Notice that, due to the global validity of inclusion (1.2) and to the openness of the involved enlargement, one has

$$F(B(x,r)) \supseteq B(F(x), cov(F)r), \text{ for all } x \in X, r > 0.$$

Remark 1.6 (A. Uderzo [17])**.**

(i) The open covering property of a multivalued operator admits several useful formulation. It is well known that a mapping F fulfils Definition 1.5 if and only if there exists $l > 0$ such that

$$D(x, F^{-1}(y)) \leq l D(y, F(x)), \text{ for all } x \in X, y \in Y. \tag{1.3}$$

The infimum of all values l satisfying inequality (1.3) is called modulus of global metric regularity of F and denoted by $reg(F)$. The following relation between the modulus of global covering and the modulus of global metric regularity is know to hold

$$reg(F) = \frac{1}{cov(F)},$$

where the case $reg(F) = \infty$, coresponding to $cov(F) = 0$, is intended to mean the absence of global open covering/ metric regularity for a given F.

(ii) Another characterization of open covering/ metric regularity can be obtained in terms of Lipschitz behavior of the inverse multivalued operator. In fact F covers on X if and only if F^{-1} is Lipschitz continuous in Y and it holds

$$lip(F^{-1}) = \frac{1}{cov(F)}.$$

2. Main results

Let (X, d) and (Y, ρ) be two metric spaces and $S, T : X \to P(Y)$ be two multivalued operators.

Definition 2.1. By definition, a solution of the coincidence problem for S and T is a pair $(x^*, y^*) \in X \times Y$ such that:

$$y^* \in T(x^*) \cap S(x^*).$$

Denote by $CP(S,T) \subset X \times Y$ the set of all solutions of the coincidence problem for S and T.

Let d_Z be a traditional scalar metric on $Z := X \times Y$. Let us consider the following multivalued coincidence problem

$$\text{find } (x, y) \in X \times Y \text{ such that } y \in S(x) \cap T(x). \tag{2.1}$$

Definition 2.2. The multivalued coincidence problem (2.1) is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}^2_+ \to \mathbb{R}_+$ increasing, continuous in 0 and with $\psi(0) = 0$, such that for every $\varepsilon_1, \varepsilon_2 > 0$ and for each solution $w^* := (u^*, v^*) \in X \times Y$ of the following approximative coincidence problem

$$D_\rho(S(u), v) \le \varepsilon_1 \text{ and } D_\rho(T(u), v) \le \varepsilon_2, \tag{2.2}$$

there exists a solution $z^* := (x^*, y^*)$ of (2.1) such that

$$d_Z(z^*, w^*) \le \psi(\varepsilon_1, \varepsilon_2).$$

If there exists $c_1, c_2 > 0$ such that $\psi(t_1, t_2) = c_1 t_1 + c_2 t_2$ for each $t_1, t_2 \in \mathbb{R}_+$, then the multivalued coincidence problem (2.1) is said to be Ulam-Hyers stable.

Let (X, d) and (Y, ρ) be two metric spaces and the following two metrics on $X \times Y$:

$$d^*((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2), \text{ for all } (x_1, y_1), (x_2, y_2) \in X \times Y$$

$$d_*((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \rho(y_1, y_2)\}, \text{ for all } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Denote by H_{d^*} and H_{d_*} the Hausdorff-Pompeiu functionals on $P(X \times Y)$ generated by d^* and d_* respectively.

Lemma 2.3. *Let (X, d) and (Y, ρ) be two metric spaces. Then, we have:*

$$1)\quad D_{d^*}((x,y), A \times B) = D_d(x, A) + D_\rho(y, B)$$
$$2)\quad D_{d_*}((x,y), A \times B) = \max\{D_d(x, A), D_\rho(y, B)\},$$

for each $x \in X$, $y \in Y$, $A \subset X$ and $B \subset Y$.

Proof. Let $x \in X$, $y \in Y$, $A \subset X$ and $B \subset Y$. For 1) we have:

$$D_{d^*}((x,y), A \times B) = \inf_{(a,b)\in A\times B}\{d^*((x,y),(a,b))\} = \inf_{(a,b)\in A\times B}\{d(x,a) + \rho(y,b)\}$$
$$= \inf_{a\in A} d(x,a) + \inf_{b\in B} \rho(y,b) = D_d(x, A) + D_\rho(y, B).$$

In a similar way, we can prove 2). □

Lemma 2.4. *Let us consider X, Y be two nonempty sets and $S, T : X \to P(Y)$ be two multivalued operators. If S is onto and we define $G : X \times Y \to P(X) \times P(Y)$, $G(x, y) = S^{-1}(y) \times T(x)$, then*

$$CP(S, T) = Fix(G).$$

Proof. We successively have the equivalences $(u^*, v^*) \in Fix(G_1) \iff (u^*, v^*) \in G(u^*, v^*) \iff (u^*, v^*) \in S^{-1}(v^*) \times T(u^*) \iff u^* \in S^{-1}(v^*)$ and $v^* \in T(u^*) \iff v^* \in S(u^*)$ and $v^* \in T(u^*) \iff v^* \in T(u^*) \cap S(u^*) \iff (u^*, v^*) \in CP(S, T)$. □

Theorem 2.5 ([19]). *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multivalued operator. If there exists $k \in [0, 1)$ such that*

$$e_d(F(x), F(y)) \le k \cdot d(x, y), \quad \textit{for all } x, y \in X,$$

then there exists at least one fixed point for F.

Our first existence and stability result is the following.

Theorem 2.6. *Let (X, d) and (Y, ρ) be two complete metric spaces. Let $T, S : X \to P(Y)$ be two multivalued operators, such that S is an onto operator and:*

(i) *$T : X \to P_{cl}(Y)$ is a contraction with constant $k_T < 1$;*
(ii) *$S : X \to P(Y)$ is metrically regular on X with constant $k_S \in (0, 1)$ and $S^{-1}(y)$ is closed for each $y \in Y$.*

Then there exists at least one solution of multivalued coincidence problem (2.1).

If, in addition, S^{-1} and T have compact values then the problem (2.1) *is Ulam-Hyers stable.*

Proof. Let $Z := X \times Y$ and $G_1 : Z \to P(Z)$ defined by $G_1(u, v) = S^{-1}(v) \times T(u)$. For $u := (u_1, u_2) \in Z$, $v := (v_1, v_2) \in Z$, $x := (x_1, x_2) \in G_1(u)$ and used Lemma 2.3 we have:

$$D_{d^*}(x, G_1(v)) = D_{d^*}((x_1, x_2), (G_1(v_1, v_2)) = D_{d^*}((x_1, x_2), (S^{-1}(v_2) \times T(v_1)))$$
$$= D_d(x_1, S^{-1}(v_2)) + D_\rho(x_2, T(v_1)).$$

Taking into account that S is metrically regular on X, T is a contraction and $x \in G_1(u)$ we obtain:

$$D_{d^*}(x, G_1(v)) \le k_s D_\rho(v_2, S(x_1)) + H_\rho(T(u_1), T(v_1))$$
$$\le k_S \cdot \rho(u_2, v_2) + k_T \cdot d(u_1, v_1) \qquad \text{(because } u_2 \in S(x_1))$$

$$\leq \max\{k_T, k_S\}[d(u_1, v_1) + \rho(u_2, v_2)]$$
$$= \max\{k_T, k_S\} \cdot d^*((u_1, u_2), (v_1, v_2)).$$

Denote by $k := \max\{k_T, k_S\}$, we obtain that:

$$D_{d^*}(x, G_1(v)) \leq k \cdot d^*(u, v).$$

So,

$$\begin{aligned}\sup_{x\in G_1(u)} D_{d^*}(x, G_1(v)) &= \sup_{(x_1,x_2)\in G_1(u_1,u_2)} D_{d^*}((x_1, x_2), G_1(v_1, v_2)) \\ &= e_{d^*}(G_1(u_1, u_2), G_1(v_1, v_2)) \\ &\leq k \cdot d^*((u_1, u_2), (v_1, v_2)) = k \cdot d^*(u, v).\end{aligned} \tag{2.3}$$

Similarly, for $y := (y_1, y_2) \in G_1(v)$ and used Lemma 2.3, we have:

$$\begin{aligned}D_{d^*}(y, G_1(u)) &= D_{d^*}((y_1, y_2), (G_1(u_1, u_2)) \\ &= D_{d^*}((y_1, y_2), (S^{-1}(u_2) \times T(u_1))) \\ &= D_d(y_1, S^{-1}(u_2)) + D_\rho(y_2, T(u_1)).\end{aligned}$$

Taking into account that S is metrically regular on X, T is a contraction and $y \in G_(v)$ we obtain:

$$\begin{aligned}D_{d^*}(y, G_1(u)) &\leq k_s \cdot D_\rho(S(y_1), u_2) + H_\rho(T(v_1), T(u_1)) \\ &\leq k_S \cdot \rho(u_2, v_2) + k_T \cdot d(u_1, v_1) \qquad \text{(because } v_2 \in S(y_1)) \\ &\leq \max\{k_T, k_S\}[d(u_1, v_1) + \rho(u_2, v_2)] \\ &= \max\{k_T, k_S\} \cdot d^*((u_1, u_2), (v_1, v_2)).\end{aligned}$$

We obtain that: $D_{d^*}(y, G_1(u)) \leq k \cdot d^*(u, v)$. So,

$$\begin{aligned}\sup_{y\in G_1(v)} D_{d^*}(y, G_1(u)) &= \sup_{(y_1,y_2)\in G_1(v_1,v_2)} D_{d^*}((y_1, y_2), G_1(u_1, u_2)) \\ &= e_{d^*}(G_1(v_1, v_2), G_1(u_1, u_2)) \\ &\leq k \cdot d^*((u_1, u_2), (v_1, v_2)) = k \cdot d^*(u, v).\end{aligned} \tag{2.4}$$

From relations (2.3) and (2.4) we obtain:

$$H_{d^*}(G_1(u), G_1(v)) = \max\{e_{d^*}(G_1(u), G_1(v)), e_{d^*}(G_1(v), G_1(u))\} \leq k \cdot d^*(u, v),$$

for all $(u, v) \in Z \times Z$. So, we deduce that $G_1 : Z \to P(Z)$ is a multivalued contraction. Moreover, G_1 has nonempty and closed values. Thus, we can apply Covitz-Nadler's fixed point theorem for G_1 and we obtain that there exists $z^* \in Z$ such that $z^* \in G_1(z^*)$. Thus, by Lemma 2.4 (a), the multivalued coincidence problem (2.1) has at least one solution.

For the second conclusion, let $\varepsilon_1, \varepsilon_2 > 0$ and $w^* := (u^*, v^*) \in X \times Y$ a solution of the approximative coincidence problem, i.e.,

$$D_\rho(S(u^*), v^*) \leq \varepsilon_1 \text{ and } D_\rho(T(u^*), v^*) \leq \varepsilon_2 \tag{2.5}$$

By the first part of our proof, we get that $z^* := (x^*, y^*) \in X \times Y$ is a solution of the multivalued coincidence problem (2.1). Moreover, we apply Remark 1.4 for G_1

and we get that G_1 is $\frac{1}{1-k}$-MWP operator, i.e.,

$$d^*(w^*, z^*) \leq \frac{1}{1-k} d^*(w^*, t), \text{ for each } t \in G(w^*).$$

Then, for every $(t_1, t_2) \in S^{-1}(v^*) \times T(u^*)$ we have that

$$d^*(w^*, z^*) \leq \frac{1}{1-k}[d(u^*, t_1) + \rho(v^*, t_2)].$$

By (2.5) and the compactness of the values of T, there exists $t_2^* \in T(u^*)$ such that $D_\rho(T(u^*), v^*) = \rho(t_2^*, v^*)$. Since $S^{-1}(v^*)$ is compact, using again (2.5) we get that there exists $t_1^* \in S^{-1}(v^*)$ such that $D_d(u^*, S^{-1}(v^*)) = d(u^*, t_1^*)$. Then, taking into account that S is metrically regular on X we get that

$$d(u^*, t_1^*) = D_d(u^*, S^{-1}(v^*)) \leq D_\rho(S(u^*), v^*).$$

In conclusion

$$d^*(z^*, w^*) \leq \frac{1}{1-k}[d(u^*, t_1^*) + \rho(v^*, t_2^*)] \leq \frac{1}{1-k}(\varepsilon_1 + \varepsilon_2),$$

proving that the multivalued coincidence problem (2.1) is Ulam-Hyers stable. □

Theorem 2.7. *Let (X, d) and (Y, ρ) be two complete metric spaces. Let $T, S : X \to P(Y)$ be two onto multivalued operators, such that:*

(i) *$T : X \to P_{cl}(Y)$ is metrically regular on X with constant $k_T \in (0, 1)$;*
(ii) *$S^{-1} : Y \to P(X)$ is metrically regular on Y with constant $k_S \in (0, 1)$ and $S^{-1}(y)$ is closed, for all $y \in Y$;*

Then there exists at least one solution of the multivalued coincidence problem (2.1).

If, in addition, S^{-1} and T have compact values then the problem (2.1) is Ulam-Hyers stable.

Proof. Let $Z := X \times Y$. We define $G_1, G_2 : Z \to P(Z)$ by $G_1(u, v) = S^{-1}(v) \times T(u)$ and $G_2(u', v') = T^{-1}(v') \times S(u')$. We can observe that $G_2 = G_1^{-1}$. We prove that G_1 is metrically regular on Z.

For $x := (x_1, x_2) \in Z$, $y := (y_1, y_2) \in Z$ and taking into account that T is metrically regular on X and S^{-1} is metrically regular on Y, we have:

$$\begin{aligned}
D_{d^*}((x_1, x_2), G_2(y_1, y_2)) &= D_{d^*}((x_1, x_2), T^{-1}(y_2) \times S(y_1)) \\
&= D_d(x_1, T^{-1}(y_2)) + D_\rho(x_2, S(y_1)) \\
&\leq k_T \cdot D_\rho(T(x_1), y_2) + k_S \cdot D_d(S^{-1}(x_2), y_1) \\
&\leq \max\{k_T, k_S\}[D_d(S^{-1}(x_2), y_1) + D_\rho(T(x_1), y_2)] \\
&= \max\{k_T, k_S\} \cdot D_{d^*}(S^{-1}(x_2) \times T(x_1), (y_1, y_2)) \\
&= \max\{k_S, k_T\} \cdot D_{d^*}(G_1(x_1, x_2), (y_1, y_2)).
\end{aligned}$$

We denote by $k := \max\{k_T, k_S\} \in (0, 1)$ and we deduce that $G_1 : Z \to P(Z)$ is metrically regular on Z with the norm of regularity $Reg(G) := \max\{k_T, k_S\} \in (0, 1)$. For $y := (y_1, y_2) \in Z$, we have:

$$\sup\{D_{d^*}((x_1, x_2), G_2(y_1, y_2)) : (x_1, x_2) \in Z\} \leq k \cdot D_{d^*}((y_1, y_2), G_1(x_1, x_2)). \tag{2.6}$$

Taking into account that T and S are onto operators we can deduce that for all $t := (t_1, t_2) \in Z$, there exists $(x_1, x_2) \in Z$ such that $(t_1, t_2) \in G_1(x_1, x_2)$. From relation (2.6) we have:

$$\sup\{D_{d^*}((x_1, x_2), G_2(y_1, y_2)) : (x_1, x_2) \in Z\} \leq k \cdot D_{d^*}((y_1, y_2), (t_1, t_2)).$$

For $x := (x_1, x_2) \in Z$, $y := (y_1, y_2) \in Z$, we can write:

$$\sup\{D_{d^*}(x, G_2(y)) : x \in Z\} = e_{d^*}(G_2(y), G_2(t)) \leq k \cdot d^*(y, t). \tag{2.7}$$

Moreover, G_2 has nonempty and closed values. Thus, we can apply Theorem 2.5 for G_2 and we obtain that there exists $z^* \in Z$ such that $z^* \in G_2(z^*)$. Thus, by Lemma 2.4 (b), we deduce that the multivalued coincidence problem (2.1) has at least one solution.

The proof of the second conclusion is similar with the proof of Theorem 2.6. □

Remark 2.8. A similar result take place if we replace, in the proof of the above theorem, the metric d^* with d_* and H_{d^*} with H_{d_*}.

Corollary 2.9. *In Theorem 2.6 and Theorem 2.7 if we consider two subsets $U \subseteq X$ and $V \subseteq Y$ such that the operators $T, S : U \to P(V)$ satisfy all the conditions of these theorems on $U \times V$, then we obtain the same conclusions on $U \times V$.*

Next we give an application of Theorem 2.6.

Theorem 2.10. *Let us consider the multivalued differential inclusion:*

$$x'(t) \in F(t, x(t)), \text{ a.e. } t \in [a, b], \tag{2.8}$$

where $F : [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ is a multivalued operator such that:

(a) *there exists an integrable function $M : [a, b] \to \mathbb{R}_+$ such that for each $u \in \mathbb{R}^n$ we have $F(s, u) \subset M(s)B(0, 1)$, a.e. $s \in [a, b]$;*
(b) *$F(\cdot, u(\cdot)) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is measurable for every $u \in C([a, b)]$;*
(c) *for each $u \in \mathbb{R}^n$, $F(\cdot, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is lower semi-continuous;*
(d) *there exists a continuous function $p : [a, b] \to \mathbb{R}_+$ such that for each $s \in [a, b]$ and each $u, v \in \mathbb{R}^n$ we have that: $H(F(s, u), F(s, v)) \leq p(s) \cdot |u - v|$;*
(e) *$\int_0^t f(\tau, 0)d\tau = O(e^{\int_0^t L(\tau)d\tau})$.*

Then the following conclusions hold:

(i) *there exists at least one solution for the Cauchy problem:*

$$\begin{cases} x'(t) \in F(t, x(t)), \text{ a.e. } t \in [a, b]; \\ x(a) = \alpha, \quad \alpha \in \mathbb{R}^n. \end{cases} \tag{2.9}$$

(ii) *the differential inclusion (2.8) is Ulam-Hyers stable, i.e. for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for each function $y \in C([a, b], \mathbb{R}^n)$ a solution of the inequation*

$$D_{\|\cdot\|_{\mathbb{R}^n}}(y(t), F(t, y(t))) \leq \varepsilon, \quad t \in [a, b],$$

there exists a solution x of differential inclusion (2.8) such that

$$\|x(t) - y(t)\|_{\mathbb{R}^n} \leq c_\varepsilon \cdot \varepsilon, \text{ for each } t \in [a, b].$$

Proof. The problem (2.9) is equivalent to an integral inclusion of Volterra type:

$$x(t) \in \alpha + \int_a^t F(s, x(s))ds, \quad t \in [a, b]. \tag{2.10}$$

Let us consider the multivalued operator $T : C([a, b], \mathbb{R}^n) \to P(C([a, b], \mathbb{R}^n))$ defined by

$$(Tx)(t) := \left\{\alpha + \int_a^t F(s, x(s))ds\right\} e^{-p\int_0^t L(s)ds}$$

and the operator $S : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$ defined by $(Sx)(t) := x(t)e^{-p\int_0^t L(s)ds}$ where $p > 1$ and $C([a, b], \mathbb{R}^n)$ is the space of continuous function on $[a, b]$ with a Bielecki-type norm in $C([a, b], \mathbb{R}^n)$, given by

$$\|x\|_B := \sup_{t\in[a,b]} (\|x(t)\|_{\mathbb{R}^n} \cdot e^{-\tau q(t)}), \text{ where } q(t) := \int_a^t p(s)ds.$$

We observe that (2.9) is equivalent to (2.10) and, by the above notation, equivalent to the inclusion

$$S(x) \in T(x), \quad x \in C([a, b], \mathbb{R}^n). \tag{2.11}$$

We will prove that T is a multivalued contraction on $C([a, b], \mathbb{R}^n)$. Notice first that one may suppose (without affecting the generality of the Lipschitz condition) that the inequality from (d) is strict. Let $x_1, x_2 \in C([a, b], \mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then

$$v_1(t) \in \left\{\alpha + \int_a^t F(s, x_1(s))ds\right\} e^{-p\int_0^t L(s)ds}, t \in [a, b].$$

It follows that

$$v_1(t) \in \left\{\alpha + \int_a^t f_1(s)ds\right\} e^{-p\int_0^t L(s)ds}, \ t \in [a, b],$$

$$\text{for some } f_1(s) \in F(s, x_1(s))ds, \ s \in [a, b].$$

From (d) we have $H(F(s, x_1(s)), F(s, x_2(s))) < p(s) \cdot |x_1(s) - x_2(s)|$. Thus there exists $w \in F(s, x_2(s)$ such that $|f_1(s) - w| \leq p(s) \cdot |x_1(s) - x_2(s)|$, for $s \in [a, b]$.

Let as define $U : [a, b] \to P(\mathbb{R}^n)$, by

$$U(s) := \{w | \ |f_1(s) - w| \leq p(s) \cdot |x_1(s) - x_2(s)|\}.$$

Since the multivalued operator $V(s) := U(s) \cap F(s, x_2(s))$ is measurable, so there exists $f_2(s)$ a measurable selection for V. By (a) we obtain that f_2 is integrable. Hence $f_2(s) \in F(s, x_2(s))$ and $|f_1(s)-f_2(s)| \leq p(s)\cdot|x_1(s)-x_2(s)|$, for each $s \in [a, b]$.

Consider $v_2(t) = \left\{\alpha + \int_a^t f_2(s)\right\} e^{-p\int_0^t L(s)ds}$, $t \in [a, b]$. Then for each $t \in [a, b]$, we have:

$$\begin{aligned} |v_1(t) - v_2(t)| &\leq \int_a^t |f_1(s) - f_2(s)| e^{-p\int_0^t L(s)ds} ds \\ &\leq e^{-p\int_0^t L(s)ds} \int_a^t p(s)|x_1(s) - x_2(s)|ds \end{aligned}$$

$$
\begin{aligned}
&= e^{-p\int_0^t L(s)ds}\int_a^t p(s)e^{\tau q(s)}|x_1(s)-x_2(s)|e^{-\tau q(s)}ds\\
&\leq e^{-p\int_0^t L(s)ds}\int_a^t p(s)e^{\tau q(s)}\|x_1-x_2\|_B ds\\
&= \frac{e^{-p\int_0^t L(s)ds}}{\tau}\|x_1-x_2\|_B(e^{\tau q(t)}-e^{\tau q(a)})\\
&\leq \frac{e^{-p\int_0^t L(s)ds}}{\tau}\|x_1-x_2\|_B e^{\tau q(t)}.
\end{aligned}
$$

Thus we get $\|v_1 - v_2\|_B \leq \frac{e^{-p\int_0^t L(s)ds}}{\tau}\|x_1-x_2\|_B$. A similar relation can be obtained by interchanging the role of x_1 and x_2. By choosing now $\tau > e^{-p\int_0^t L(s)ds}$ we get that $H_{\|\cdot\|_B}(T(x_1),T(x_2)) \leq \frac{e^{-p\int_0^t L(s)ds}}{\tau}\|x_1-x_2\|_B$, which prove that T is a multivalued contraction with constant $\alpha := \frac{e^{-p\int_0^t L(s)ds}}{\tau}$.

We will prove that S is metrically regular on $C([a,b],\mathbb{R}^n)$.
We have $(S^{-1}y)(t) = y(t)\cdot e^{p\int_0^t L(s)ds}$. By calculation we get:

$$
\begin{aligned}
|x(t)-(S^{-1}y)(t)| &= |x(t)-y(t)\cdot e^{p\int_0^t L(s)ds}|\\
&= |e^{p\int_0^t L(s)ds}|\cdot e^{\tau q(s)}\cdot|x(t)\cdot e^{-p\int_0^t L(s)ds}-y(t)|\cdot e^{-\tau q(s)}\\
&= e^{p\int_0^t L(s)ds}\cdot e^{\tau q(s)}\cdot\|(Sx)-y\|_B
\end{aligned}
$$

and further $\|x-(S^{-1}y)\|_B = e^{p\int_0^t L(s)ds}\cdot\|(Sx)-y\|_B$. So, S is metrically regular with constant $e^{p\int_0^t L(s)ds} > 0$.

So, all the conditions from Theorem 2.6 are satisfy, then there exists $\bar{x} \in C([a,b],\mathbb{R}^n)$ suchh that $S(\bar{x}) \in T(\bar{x})$. There exists at least one solution for the Cauchy problem (2.9).

For the second conclusion, we observe that, from definition of S that the operator S^{-1} has compact values and from Theorem 8.8.2 Bang-Bang Principle) from [2], we deduce that the operator T has compact values. So, from Theorem 2.6 we that the differential inclusion (2.9) is Ulam-Hyers stable.

□

Next, we give an example for Corollary 2.9 with Theorem 2.6.

Example 2.11. Let us consider the following multivalued operators: $T : [0,1] \to P([0,1])$, $T(x) = [0, \frac{x}{2}]$ and $S : [0,1] \to P([0,1])$, $S(x) = [0,x]$.
We consider the following coincidence problem:

$$T(x) \cap S(x) \neq \emptyset. \tag{2.12}$$

The multivalued operator T is a contraction with constant $\frac{1}{2} < 1$, because:

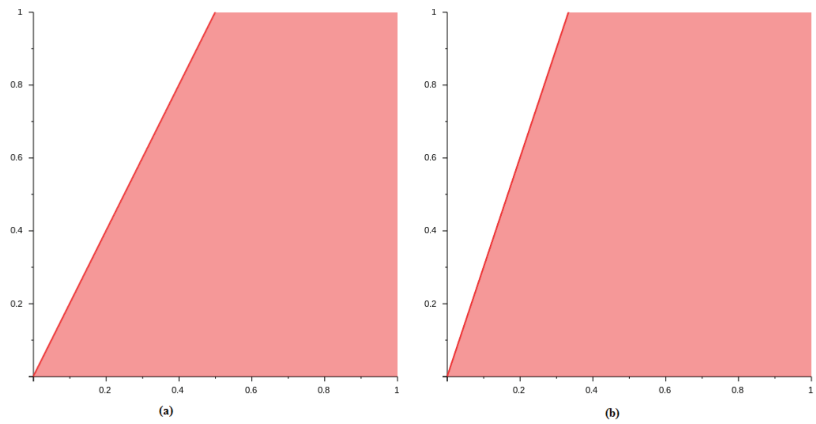
$$H(T(x),T(y)) = H\left(\left[0,\frac{x}{2}\right],\left[0,\frac{y}{2}\right]\right) = \max\left\{0,\frac{1}{2}|x-y|\right\} = \frac{1}{2}|x-y|.$$

We will show that S is metrically regular on $U \times V \subseteq \mathbb{R}^2$.
So, $S^{-1} : [0,1] \to P([0,1])$, $S^{-1}(y) = [y,1]$.

$$D(S^{-1}(y), x) = \inf_{v \in S^{-1}(y)} |v - x| = \inf_{v \in [y,1]} |v - x| = y - x.$$

$$D(y, S(x)) = \inf_{u \in S(x)} |y - u| = \inf_{u \in [0,x]} |y - u| = y.$$

So, S is metrically regular if $y - x \leq k \cdot y \Rightarrow (1-k)y \leq x$, with $k < 1$.
For $k = \frac{1}{2}$ we deduce that S is metrically regular on subset $\{(x,y) \subseteq \mathbb{R}^2 | x \in [0,1], y \leq 2x\}$ (Figure (a)), and for $k = \frac{2}{3}$ we deduce that S is metrically regular on subset $\{(x,y) \subseteq \mathbb{R}^2 | x \in [0,1], y \leq 3x\}$ (Figure (b)). So, all the conditions of



Theorem 2.6 are satisfy on a subset of $\mathbb{R}^2$. We apply Corollary 2.9 and we obtain that there exists at least one solution for coincidence problem (2.12).

3. The vector valued case

Let us consider the above approach using now a Hausdorff-Pompeiu type vector metric.

If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i, \text{ for each } i \in \{1, 2, ..., m\}$$

and

$$x < y \text{ if and only if } x_i < y_i, \text{ for each } i \in \{1, 2, ..., m\}.$$

Notice that, through this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$.
Let us introduce now some vector-valued metrics of Perov's type. Let (X, d) and (Y, ρ) be two metric spaces. Let $Z := X \times Y$ and define on $Z \times Z$ the vector metric

$$d^V(u,v) := \begin{pmatrix} d(u_1, v_1) \\ \rho(u_2, v_2) \end{pmatrix}, \text{ for each } u = (u_1, u_2), v = (v_1, v_2) \in Z.$$

In the same framework as above, let us consider a Hausdorff-Pompeiu type vector functional given by $H^* : (P(X) \times P(Y)) \times (P(X) \times P(Y)) \to \mathbb{R}^2_+$ given by

$$H^*(A \times B, U \times V) := \begin{pmatrix} H_d(A,U) \\ H_\rho(B,V) \end{pmatrix}.$$

From the definition, it follows that H^* is a vector metric on $P_{cl}(X) \times P_{cl}(Y)$. Similarly, we can define $D^* : (X \times Y) \times (P(X) \times P(Y)) \to \mathbb{R}^2_+$ given by

$$D^*((x,y), U \times V) := \begin{pmatrix} D_d(x,U) \\ D_\rho(y,V) \end{pmatrix}.$$

The following lemma follows immediately by the definition of H^* and the properties of the Hausdorff-Pompeiu functional.

Lemma 3.1. *Let (X,d) and (Y,ρ) be two metric spaces. Let $A, C \subset X$ and $B, D \subset Y$ and $q > 1$. Then, for any $z \in A \times B$ there exists $w \in C \times D$ such that*

$$d^V(z,w) \leq qH^*(A \times B, C \times D).$$

A classical result in matrix analysis is the following theorem (see, e.g., [1]).

Theorem 3.2. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *A is convergent towards zero;*
(ii) *$A^n \to 0$ as $n \to \infty$;*
(iii) *The spectral radius $\rho(A)$ of A is strictly less than 1, i.e., the eigenvalues of A are in the open unit disc;*
(iv) *The matrix $I - A$ is nonsingular and*

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots ; \tag{3.1}$$

(v) *The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;*
(vi) *$A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$;*
(vii) *The matrices qA and Aq converge to 0, for each $q \in (1, Q)$, where $Q := \frac{1}{\rho(A)}$.*

Notice that we have the following result.

Lemma 3.3 (see [13]). *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ be a matrix convergent to zero. Then, there exists $Q > 1$ such that for any $q \in (1, Q)$ the matrix qA is convergent to 0.*

We present now a vector version of Nadler's fixed point theorem. See also [18] for a similar result.

Theorem 3.4 ([10]). *Let (X,d) and (Y,ρ) be two complete metric spaces. Let $F := F_1 \times F_2$ be a multivalued operator such that $F_1 : X \times Y \to P_{cl}(X)$ and $F_2 : X \times Y \to P_{cl}(Y)$ are two multivalued operators with the property that there exists a matrix $A \in M_{22}(\mathbb{R}_+)$ which converges to zero such that*

$$H^*(F(u), F(v)) \leq Ad^V(u,v), \text{ for all } u = (u_1,u_2), v = (v_1,v_2) \in X \times Y.$$

Then:

(i) *$Fix(F) \neq \emptyset$, i.e., there exists $z^* := (z_1^*, z_2^*) \in X \times Y$ such that $z_1^* \in F_1(z^*)$ and $z_2^* \in F_2(z^*)$;*

(ii) *for each* $(z,w) \in Graph(F)$ *there exists a sequence* $(z_n)_{n\in\mathbb{N}}$ *(with* $z_0 = z$, $z_1 = w$ *and* $z_{n+1} \in F(z_n)$*, for each* $n \in \mathbb{N}^*$*) such that* $(z_n)_{n\in\mathbb{N}}$ *is convergent to a fixed point* z^* *of* F *and* $d^V(z,z^*) \le (I-A)^{-1}d^V(z,w)$.

(iii) *if* $(z_n)_{n\in\mathbb{N}} \to z^* \in Fix(F)$ *as* $n \to +\infty$*, then*

$$d^V(z_n,z^*) \le A^n(I-A)^{-1}d^V(z_0,z_1),$$

for each $n \in \mathbb{N}^*$.

We will recall now the concept of generalized Ulam-Hyers stablility, in a vector form, for coincidence point problems.

Definition 3.5. Let (X,d) and (Y,ρ) be two metric spaces and $S,T : X \to P(Y)$ be two multivalued operators. The multivalued coincidence problem (2.1) is called generalized Ulam-Hyers v-stable if and only if there exists $\psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for every $\varepsilon := (\varepsilon_1,\varepsilon_2)$ (with $\varepsilon_1,\varepsilon_2 > 0$) and for each ε-solution $w^* := (u^*,v^*) \in X \times Y$ of the multivalued coincidence problem (2.1) (i.e., a solution of the following approximative coincidence problem

$$D_\rho(S(u),v) \le \varepsilon_1 \text{ and } D_\rho(T(u),v) \le \varepsilon_2),$$

there exists a solution $z^* := (x^*,y^*)$ of (2.1) such that

$$d^V(z^*,w^*) \le \psi(\varepsilon). \tag{3.2}$$

If there exists a matrix $C \in \mathcal{M}_{22}(\mathbb{R}_+)$ such that $\psi(t) = Ct$ for each $t \in \mathbb{R}^2_+$, then the multivalued coincidence problem (2.1) is said to be Ulam-Hyers v-stable.

We can prove now another existence and Ulam-Hyers stability result for the multivalued coincidence problem.

Theorem 3.6. *Let* (X,d) *and* (Y,ρ) *be two complete metric spaces. Let* $T,S : X \to P(Y)$ *be two multivalued operators, such that* S *is an onto operator and:*

(i) $T : X \to P(Y)$ *is Lipschitz with constant* $k_T > 0$;

(ii) $S : X \to P(Y)$ *is metrically regular on* X *with constant* $k_S > 0$ *and* $S^{-1}(y)$ *is closed for each* $y \in Y$;

(iii) $k_T \cdot k_S < 1$.

Then there exists at least one solution of multivalued coincidence problem (2.1).

If, in addition, S^{-1} *and* T *have compact values then the problem* (2.1) *is Ulam-Hyers stable.*

Proof. Let $Z := X \times Y$ and $G_1 : Z \to P(Z)$ defined by $G_1(u,v) = S^{-1}(v) \times T(u)$. For $u := (u_1,u_2) \in Z$, $v := (v_1,v_2) \in Z$, $x \in G_1(u)$ and taking into account that S is metrically regular on X and T is Lipschitz, we have:

$$\begin{aligned}
D^*(x,G_1(v)) &= D^*((x_1,x_2),G_1(v_1,v_2)) = D^*((x_1,x_2),(S^{-1}(v_2) \times T(v_1))) \\
&= \begin{pmatrix} D_d(x_1,S^{-1}(v_2)) \\ D_\rho(x_2,T(v_1)) \end{pmatrix} \le \begin{pmatrix} k_S \cdot D_\rho(v_2,S(x_1)) \\ H_\rho(T(u_1),T(v_1)) \end{pmatrix} \\
&\le \begin{pmatrix} k_S \cdot \rho(u_2,v_2) \\ k_T \cdot d(u_1,v_1) \end{pmatrix} = \begin{pmatrix} 0 & k_S \\ k_T & 0 \end{pmatrix} \cdot \begin{pmatrix} d(u_1,v_1) \\ \rho(u_2,v_2) \end{pmatrix} \\
&= \begin{pmatrix} 0 & k_S \\ k_T & 0 \end{pmatrix} \cdot d^V(u,v).
\end{aligned}$$

If we denote by $A := \begin{pmatrix} 0 & k_S \\ k_T & 0 \end{pmatrix}$, we obtain:

$$\begin{aligned} \sup_{x\in G_1(u)} D^*(x, G_1(v)) &= \sup_{(x_1,x_2)\in G_1(u_1,u_2)} D^*((x_1, x_2), G_1(v_1, v_2)) \\ &= e_{d^V}(G_1(u_1, u_2), G_1(v_1, v_2)) \\ &\le A \cdot \begin{pmatrix} d(u_1, v_1) \\ \rho(u_2, v_2) \end{pmatrix} = A \cdot d^V(u, v). \end{aligned} \tag{3.3}$$

Similarly, for $y := (y_1, y_2) \in G_1(v)$, we have $D^*(y, G_1(u)) \le A \cdot d^V(u, v)$. So,

$$\begin{aligned} \sup_{y\in G_1(v)} D^*(y, G_1(u)) &= \sup_{(y_1,y_2)\in G_1(v_1,v_2)} D^*((y_1, y_2), G_1(u_1, u_2)) \\ &= e_{d^V}(G_1(v_1, v_2), G_1(u_1, u_2)) \\ &\le A \cdot \begin{pmatrix} d(u_1, v_1) \\ \rho(u_2, v_2) \end{pmatrix} = A \cdot d^V(u, v). \end{aligned} \tag{3.4}$$

From relations (3.4) and (**??**) we obtain:

$$H^*(G_1(u), G_1(v)) = \max\{e_{d^V}(G_1(u), G_1(v)), e_{d^V}(G_1(v), G_1(u))\} \le A \cdot d^V(u, v),$$

for all $(u, v) \in Z \times Z$. Since A is convergent to zero, we can apply to G_1 the multivalued version of Perov's fixed point theorem (see Theorem 3.4) and we obtain that there exists $z^* \in Z$ such that $z^* = G_1(z^*)$. Thus by Lemma 2.4 (a), the multivalued coincidence problem (2.1) has at least one solution.

For the second conclusion, let $\varepsilon_1, \varepsilon_2 > 0$ and let $w^* := (u^*, v^*) \in Z$ be a solution of the approximative coincidence problem, i.e.,

$$D_\rho(S(u^*), v^*) \le \varepsilon_1 \text{ and } D_\rho(T(u^*), v^*) \le \varepsilon_2, \tag{3.5}$$

By (3.5) and the compactness of the values of T, there exists $t_2^* \in T(u^*)$ such that

$$\rho(t_2^*, v^*) = D_\rho(T(u^*), v^*) \le \varepsilon_2.$$

Since $S^{-1}(v^*)$ is compact, there exists $t_1^* \in S^{-1}(v^*)$ such that $d(u^*, t_1^*) = D_d(u^*, S^{-1}(v^*))$. Taking into account that S is metrically regular on X, we get:

$$d(u^*, t_1^*) = D_d(u^*, S^{-1}(v^*)) \le D_\rho(S(u^*), v^*) \le \varepsilon_1.$$

By the first part of our proof, we know there exists $z^* := (x^*, y^*) \in Z$ a solution of the multivalued coincidence problem (2.1). Then, from the second part of (ii) in Theorem 3.4 we get that

$$d^V(w^*, z^*) \le (I - A)^{-1} d^V(w^*, t), \text{ for any } t \in G_1(w^*).$$

Thus, for $t := t^* = (t_1^*, t_2^*) \in S^{-1}(v^*) \times T(u^*) = G_1(w^*)$, we have that

$$d^V(w^*, z^*) \le (I - A)^{-1} \begin{pmatrix} d(u^*, t_1^*) \\ \rho(v^*, t_2^*) \end{pmatrix} \le (I - A)^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = (I - A)^{-1}\varepsilon,$$

proving that the multivalued coincidence problem (2.1) is generalized Ulam-Hyers stable with a function $\psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $\psi(t) = (I - A)^{-1}t$. □

Remark 3.7. Notice that the assumptions on S and T in Theorem 3.6 are much more relaxed than those in Theorem 2.6, which shows the advantages of working with the vector metric technique.

We will present now an application of Theorem 3.6.

Theorem 3.8. *Consider the differential equation*

$$x' = f(t, x) \tag{3.6}$$

with initial condition

$$x(0) = \xi. \tag{3.7}$$

Suppose that the function f is defined in the half-plane $t \geq 0$, $-\infty < x < +\infty$ and satisfies following conditions:

- *i) $f(t, x)$ is a continuous function of x for almost all $t \geq 0$;*
- *ii) $f(t, x)$ is a measurable function of t for all $x \in \mathbb{R}$;*
- *iii) Lipschitz inequality, i.e.*

$$|f(t, x) - f(t, y)| \leq L(t)|x - y|,$$

where L is locally integrable function on the interval $(0, \infty)$;
- *iv) $\int_0^t f(\tau, 0)d\tau = O(e^{\int_0^t L(\tau)d\tau})$.*

Then the differential equation (3.6) *has for every $\xi \in \mathbb{R}$ a unique solution and the equation* (3.6) *is Ulam-Hyers stable.*

Proof. Let us consider the set

$$A = \{x \in C[0, \infty) : x(t) = O(e^{\int_0^t L(\tau)d\tau})\}.$$

We define the operators $S, T : A \to B$ by

$$(Tx)(t) = \left\{ \int_0^t f(\tau, x(\tau))d\tau + \xi \right\} e^{-p \int_0^t L(\tau)d\tau},$$

$$(Sx)(t) = x(t)e^{-p \int_0^t L(\tau)d\tau},$$

where B is a Banach space of bounded continuous functions on $[0, \infty)$ with the norm $\|x\| = \sup\limits_{(0,\infty)} |x(t)|$ and $p > 1$. By simple calculation we have

$$|(Tx)(t) - (Ty)(t)| \leq \frac{e^{-p \int_0^t L(\tau)d\tau}}{p} \|x - y\|$$

and further $\|Tx - Ty\| \leq \dfrac{e^{-p \int_0^t L(\tau)d\tau}}{p} \|x - y\|$. So T is a Lipschitz operator with constant $\dfrac{e^{-p \int_0^t L(\tau)d\tau}}{p} > 0$.

We will prove that S is metrically regular on A. We have $(S^{-1}y)(t) = y(t) \cdot e^{p \int_0^t L(\tau)d\tau}$. By calculation we get:

$$\begin{aligned}
|x(t) - (S^{-1}y)(t)| &= |x(t) - y(t) \cdot e^{p \int_0^t L(\tau)d\tau}| \\
&= |e^{p \int_0^t L(\tau)d\tau}| \cdot |x(t) \cdot e^{-p \int_0^t L(\tau)d\tau} - y(t)| \\
&= e^{p \int_0^t L(\tau)d\tau} \cdot |(Sx)(t) - y(t)|
\end{aligned}$$

and further $\|x-(S^{-1}y)\| = e^{p\int_0^t L(\tau)d\tau} \cdot \|(Sx)-y\|$. So, S is metrically regular with constant $e^{p\int_0^t L(\tau)d\tau} > 0$.

The condition (iii) by Theorem 3.6 is satisfy because $\dfrac{e^{-p\int_0^t L(\tau)d\tau}}{p} \cdot e^{p\int_0^t L(\tau)d\tau} = \dfrac{1}{p} < 1$.

So, all the conditions from Theorem 3.6 are satisfy, then there exists $\bar{x} \in A$ such that $S(\bar{x}) = T(\bar{x})$. From this we have

$$\bar{x}(t) = \int_0^t f(\tau, \bar{x}(\tau))d\tau + \xi.$$

Then the differential equation (3.6) has, for every $\xi \in, \mathbb{R}$ at least one solution with the initial condition $\bar{x}(0) = \xi$.

We willl prove that the solution for the differential equation (3.6) is unique. Suppose that there exists $\bar{x}, \bar{y} \in A$ solutions for the differential equation (3.6). In this case, we have:

$$\bar{x}(t) = \int_0^t f(\tau, \bar{x}(\tau))d\tau + \xi.$$

$$\bar{y}(t) = \int_0^t f(\tau, \bar{y}(\tau))d\tau + \xi.$$

Taking into account that f is Lipschitz, we have:

$$|\bar{x}(t) - \bar{y}(t)| \leq \int_0^t |f(\tau, \bar{x}(\tau)) - f(\tau, \bar{y}(\tau))|d\tau \leq \int_0^t L(\tau)|\bar{x}(\tau) - \bar{y}(\tau)|d\tau$$

Using Gronwall Lemma, we obtain that $|\bar{x}(t) - \bar{y}(t)| \leq 0 \Longrightarrow \bar{x}(t) = \bar{y}(t)$, so the differential equation (3.6) has, for every $\xi \in, \mathbb{R}$ a unique solution with the initial condition $\bar{x}(0) = \xi$.

Because S^{-1} and T have compact values, we deduce that the equation (3.6) is Ulam-Hyers stable. □

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