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CONTRACTIVE SET-VALUED MAPS IN HYPERBOLIC SPACES

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Abstract. We show in the present paper that the set of all noncontractive set-valued maps is σ-porous in the set of all nonexpansive set-valued maps. This result not only extends the previous results in this direction, but also unifies the contractive maps and contractive set-valued maps in some way. Moreover, similar porosity result is established for nonexpansive set-valued maps which are contractive with respect to a given subset.

1. Introduction

There has been considerable interest in nonexpansive/contractive map theory in the last sixty years. A classical theorem of Banach [3] states that every strictly contractive map f defined on a complete metric space X (i.e., $\mathrm{d}(f(x), f(y)) \leq \alpha \mathrm{d}(x, y)$, $\forall x, y \in X$ with some constant $\alpha \in (0,1)$) has the following property:

$$\mathrm{Fix}(f) = \{\bar{x}\} \quad \text{and} \quad f^n(x) \to \bar{x}, \quad \forall x \in X, \tag{1.1}$$

where $\mathrm{Fix}(f)$ denotes the set of all fixed points of f. This result was followed by a series of new works by different authors; see for example [10, 11, 14], etc. However, the fixed point of a nonexpansive map does not exist, in general. It makes sense to study the generic property on nonexpansive maps. This problem was first studied by Blasi and Myjak in [6], where it was showed that, the set of all strictly contractive single-valued maps defined on a convex subset of in a Hilbert space is σ-porous and the set of all nonexpansive single-valued maps defined on a convex subset of a Banach space without fixed point is σ-porous. Rakotch in [19] introduced the notation of contractive map f on a complete metric space X in the sense that there is a decreasing function $\alpha_f : \mathbb{R}^+ \to [0, 1]$ with $\alpha_f(t) < 1$ for $t > 0$ such that

$$\mathrm{d}(f(x), f(y)) \leq \alpha_f(\mathrm{d}(x, y))\mathrm{d}(x, y), \ \forall x, y \in X, \tag{1.2}$$

and showed that every contractive map f defined on X employs the same property (1.1), which extends the classical theorem of Banach in [3] from the strictly contractive map to the contractive map. Based on this notation, Reich and Zaslavski showed in [21] that the set of all nonexpansive single-valued maps which are contractive is generic in a hyperbolic space, which has been strengthened in [22] by showing that the set of all nonexpansive maps fail to be contractive is σ-porous. This result together with [19, Corollary, Page 463] extends the corresponding results in [6] from Banach spaces to hyperbolic spaces. Furthermore, Reich and Zaslavski in [21, 22] studied nonexpansive single-valued maps which are contractive with respect to a

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given subset and established the generic property result and porosity result respectively. It is well-known that a lot of works have been done on set-valued mapping theory and some classical theorems for single-valued maps have been extended to nonexpansive set-valued maps; see, for example, [1, 4, 8, 12, 13, 16–18, 20, 23–25] and the references therein.

The purpose of the present paper is to extend the notions of contractive maps and contractive maps with respect to a set from the single-valued case to the set-valued case, and study the porosity properties for non-contractive set-valued maps and/or non-contractive maps with respect to a set in hyperbolic spaces. By using the notion of admissible family introduced in [15, 18], the main results are stated in Theorems 3.1 and 4.3, which respectively show that the set of all nonexpansive set-valued maps on a star-shaped set fail to be contractive is σ-porous and that the set of all nonexpansive set-valued maps on a star-shaped set fail to be contractive with respect to D_0 is σ-porous. These results extend and/or improve the the corresponding results in [5, 21, 22] from the single-valued case to the set-valued case.

The paper is organized as follows. Section 2 contains notations, terminology which will be used later. In Sections 3 and 4, the porosity properties on the nonexpansive set-valued maps are presented.

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map γ from a closed interval $[0, r] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(r) = y$ and $\mathrm{d}(\gamma(t), \gamma(t')) = |t - t'|$ for $t, t' \in [0, r]$. The image $\gamma([0, r])$ of γ is called a geodesic segment joining x and y which when unique is denoted by $[x, y]$. For any $x, y \in X$, we denote the point $z \in [x, y]$ such that $\mathrm{d}(x, z) = t\mathrm{d}(x, y)$ and $\mathrm{d}(z, y) = (1-t)\mathrm{d}(x, y)$ by $z := (1 - t)x \oplus ty$, where $0 \leq t \leq 1$. Similarly, for $x \in X$ and $A \subseteq X$, we denote the set $\{tx \oplus (1-t)a : a \in A\}$ by $tx \oplus (1-t)A$. The space (X, d) is called a geodesic space if each pair of two points of X are joined by a geodesic segment. Let Λ denote the set of all geodesic segments in X. The following definition is taken from [2].

Definition 2.1. Let (X, d) be a geodesic space and $\Gamma \subseteq \Lambda$ a family of geodesic segments. (X, d) is said to be a

(a) Γ-uniquely geodesic space if for each pair of distinct points x and y of X, there is a unique geodesic in Γ which passes through x and y.

(b) uniquely geodesic space if it is a Λ-uniquely geodesic space.

As explained in [2, Remark 6], the class of Γ-uniquely geodesic spaces is strictly larger than the class of uniquely geodesic spaces. The following definition of hyperbolic space is taken from [20], which is a classical example of Γ-uniquely geodesic space.

Definition 2.2. Let (X, d) be a geodesic space and $\Gamma \subseteq \Lambda$ a family of geodesic segments. (X, d) is called a hyperbolic space if X is Γ-uniquely geodesic and the following inequality holds

$$\mathrm{d}(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}\mathrm{d}(y, z), \quad \forall x, y, z \in X. \tag{2.1}$$

Remark 2.3. (a) Classical examples of Hyperbolic spaces are CAT(0) spaces, in particular, all Hadamard manifolds, and the Hilbert ball are hyperbolic spaces. Furthermore, normed linear spaces with $\Gamma := \Gamma_L$, the family of all linear segments are hyperbolic spaces too; see for example [20].

(b) As explained in [20], since the metric d is continuous, (2.1) is equivalent to

$$\mathrm{d}(tx \oplus (1-t)y, tx \oplus (1-t)z) \le (1-t)\mathrm{d}(y,z),\ \forall x,y,z \in X, t \in [0,1]. \tag{2.2}$$

In the remainder of this paper, we assume that $\Gamma \subseteq \Lambda$ and (X, d) is a complete hyperbolic space associated with Γ. Let $A \subseteq X$ be a bounded subset. We use R_A to denote the diameter of A; while the distance function associated to A is defined by

$$\mathrm{d}(x,A) := \inf_{a\in A} \mathrm{d}(x,a), \quad \forall x \in X. \tag{2.3}$$

For any $r > 0$ and $x \in X$, we use $\mathbf{B}_X(x,r)$ and $\mathbf{U}_X(x,r)$ to denote the closed and open ball of X with center x and radius r, respectively.

Let $\mathfrak{B}(X)$ denote the family of nonempty closed bounded subsets of X. Recall that the Hausdorff distance on $\mathfrak{B}(X)$ is defined by

$$\mathrm{h}(A,B) := \max\{\sup_{x\in A} \mathrm{d}(x,B), \sup_{y\in B} \mathrm{d}(y,A)\}, \quad A, B \in \mathfrak{B}(X).$$

It is well-known that $(\mathfrak{B}(X), \mathrm{h})$ is a complete metric space if X is complete, see for example [9]. Let $D \subset X$ be a nonempty, closed and bounded subset and set

$$\mathfrak{B}(D) := \{A \subset D : A \text{ is nonempty and closed}\}. \tag{2.4}$$

Then the space $\mathfrak{B}(D)$ is complete under the metric h. Following [26], a point $a \in D$ is called a star-shaped point of D if $ta + (1-t)x \in D$ for each $x \in D$ and $t \in [0,1]$, and D is called a star-shaped set if it contains a star-shaped point. The set of all star-shaped points of D is denoted by $\mathrm{st}(D)$. Moreover, given $x \in D$ and for any $A \subset D$, we use $\{A_{t;x}\}$ to denote the set family generated by x and A with each $A_{t;x}$ defined by

$$A_{t;x} := tx + (1-t)A,\ \ \forall t \in [0,1].$$

The notion of admissible family in part (b) of the following definition was first introduced by Li and Xu in [15]; see also [18].

Definition 2.4. Let $x \in D$ and $\chi_D \subseteq \mathfrak{B}(D)$. The set χ_D is called

(a) an x-admissible family if $A_{t;x} \in \chi_D$ whenever $A \in \chi_D$ and $t \in [0,1)$;

(b) an admissible family if it is an x-admissible family for each $x \in D$.

Remark 2.5. Consider the families of subsets of D defined as follows:

$$\chi_D^S := \{\{x\} : x \in D\};$$
$$\chi_D^B := \mathfrak{B}(D);$$
$$\chi_D^K := \{A \in \mathfrak{B}(D) : A \text{ is compact}\};$$
$$\chi_D^C := \{A \in \mathfrak{B}(D) : A \text{ is convex}\}.$$
$$\chi_D^{KC} := \{A \in \mathfrak{B}(D) : A \text{ is compact and convex}\}.$$

Then each of them is a-admissible for any $a \in \mathrm{st}(D)$ and admissible if $\mathrm{st}(D) = D$.

Throughout the remainder of this paper, we further assume that $D \subseteq X$ is a nonempty, closed, bounded Γ-star-shaped subset, and $a \in \mathrm{st}_\Gamma(D)$ is fixed. Set

$$\mathcal{M} := \{F : D \to \chi_D : \mathrm{h}(F(x), F(y)) \leq \mathrm{d}(x, y), \ \ \forall x, y \in X\}. \tag{2.5}$$

Let $\mathcal{M}$ be equipped with the uniform metric ρ defined by

$$\rho(F_1, F_2) := \sup_{x \in D} \mathrm{h}(F_1(x), F_2(x)), \ \ \forall F_1, F_2 \in \mathcal{M}. \tag{2.6}$$

Then $\mathcal{M}$ is complete under the metric ρ. Let $F \in \mathcal{M}$. The following definition is a natural extension of the one in [19] due to Rakotch.

Definition 2.6. F is said to be contractive if there is a function $\alpha_F : [0, R_D] \to [0, 1)$ such that α_F is decreasing and

$$\mathrm{h}(F(x), F(y)) \leq \alpha_F(\mathrm{d}(x, y))\mathrm{d}(x, y), \ \ \forall x, y \in D. \tag{2.7}$$

The notation of a contractive map (single-valued) as well as its modifications and applications were studied by many authors; see, for example [19, 21, 22]. We end this section by the notion of porosity.

Definition 2.7. A subset $Y \subset X$ is said to be porous in X if there are $t \in (0, 1]$ and $r_0 > 0$ such that for every $x \in X$ and $r \in (0, r_0]$ there is a point $y \in X$ such that $\mathbf{B}_X(y, tr) \subseteq \mathbf{B}_X(x, r) \cap (X \setminus Y)$. Y is said to be σ-porous in X if it is a countable union of sets which are porous in X.

Note that in this definition, the statement "for every $x \in X$" can be replaced by "for every $x \in Y$". Clearly, a set which is σ-porous in X is also meager in X, the converse being false, in general. Furthermore, if $X = \mathbb{R}^n$, then each σ-porous set has (Lebesgue) measure zero. See for example [6, 7]. Note further that if Y is σ-porous in X then the complement $X \setminus Y$ is residual in X.

3. Contractive maps

In this section, we will establish the porosity result on the existence of fixed points for nonexpansive set-valued maps in X. Recall that X is a complete hyperbolic space, $D \subseteq X$ is a nonempty, closed, bounded, Γ-star-shaped set, and $a \in \mathrm{st}_\Gamma(D)$. The main theorem of this section is as follows, which seems new for the set-valued case even in Banach spaces.

Theorem 3.1. *There exists a set $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{M} \setminus \mathcal{N}$ is σ-porous in $\mathcal{M}$ and each $F \in \mathcal{N}$ is contractive.*

Proof. Let $n \in \mathbb{N}$ and define $\mathcal{M}_n \subset \mathcal{M}$ by

$$\mathcal{M}_n := \{F \in \mathcal{M} : \exists k_n \in (0, 1) \text{ s.t. } \forall x, y \in D \text{ with } \mathrm{d}(x, y) \geq \frac{R_D}{2n}, \mathrm{h}(F(x), F(y)) \leq k_n \mathrm{d}(x, y)\}.$$

It suffices to show that

(a) each $F \in \cap_{n \in \mathbb{N}} \mathcal{M}_n$ is contractive;

(b) $\mathcal{M}\setminus\mathcal{M}_n$ is porous in $\mathcal{M}$.

To show **(a)**, let $F\in\cap_{n\in\mathbb{N}}\mathcal{M}_n$. By (**??**), for each $n\in\mathbb{N}$, there is $k_n\in(0,1)$ such that

$$\mathrm{h}(F(x),F(y))\le k_n\mathrm{d}(x,y),\quad \forall x,y\in D \text{ with } \mathrm{d}(x,y)\ge\frac{R_D}{2n}. \tag{3.1}$$

Without loss of generality, we may assume that $k_n\le k_{n+1}$. Define $\alpha_F:[0,R_D]\to[0,1)$ by

$$\alpha_F(d):=k_n,\quad \frac{R_D}{2(n-1)}>d\ge\frac{R_D}{2n}. \tag{3.2}$$

Then α_F is decreasing on $[0,R_D]$ and $\alpha_F(d)<1$ for all $d>0$. So **(a)** is proved.

Below we show **(b)**. To do this, set

$$\alpha:=\frac{1}{8}\min\{R_D,1\}(2n)^{-1}(R_D+1)^{-1}\quad\text{and}\quad r_0:=1. \tag{3.3}$$

Let $F\in\mathcal{M}$, $r\in(0,r_0]$ and

$$\lambda:=\frac{r}{2(R_D+1)}. \tag{3.4}$$

Define $G:D\to\chi_D$ by

$$G(x):=\lambda a\oplus(1-\lambda)F(x),\ \forall x\in D. \tag{3.5}$$

By (2.2), one has that

$$\mathrm{h}(G(x),G(y))\le(1-\lambda)\mathrm{h}(F(x),F(y))\le(1-\lambda)\mathrm{d}(x,y),\quad\forall x,y\in D \tag{3.6}$$

and

$$\mathrm{h}(G(x),F(x))\le\lambda\mathrm{h}(a,F(x))\le\lambda R_D,\quad\forall x\in D. \tag{3.7}$$

Then $G\in\mathcal{M}$ and

$$\rho(F,G)\le\lambda R_D. \tag{3.8}$$

It suffices to show that

$$\mathbf{B}_{\mathcal{M}}(G,\alpha r)\subset\mathbf{B}_{\mathcal{M}}(F,r)\cap\mathcal{M}_n. \tag{3.9}$$

To show (3.9), let $H\in\mathbf{B}_{\mathcal{M}}(G,\alpha r)$ and $x,y\in D$ with $\mathrm{d}(x,y)\ge\frac{R_D}{2n}$. It follows from (3.6) that

$$\mathrm{d}(x,y)-\mathrm{h}(G(x),G(y))\ge\lambda\mathrm{d}(x,y)\ge\lambda\frac{R_D}{2n}. \tag{3.10}$$

Note that

$$\begin{aligned}\mathrm{h}(H(x),H(y)) &\le \mathrm{h}(H(x),G(x))+\mathrm{h}(G(x),G(y))+\mathrm{h}(G(y),H(y))\\ &\le \mathrm{h}(G(x),G(y))+2\alpha r.\end{aligned} \tag{3.11}$$

This together with (3.3),(3.4) and (3.10) implies that

$$\begin{aligned}\mathrm{d}(x,y)-\mathrm{h}(H(x),H(y)) &\ge \mathrm{d}(x,y)-\mathrm{h}(G(x),G(y))-2\alpha r\\ &\ge \lambda\frac{R_D}{2n}-2\alpha r\\ &\ge \frac{rR_D}{8n(R_D+1)}.\end{aligned} \tag{3.12}$$

Thus

$$\mathrm{h}(H(x), H(y)) \le \mathrm{d}(x, y) - \frac{rR_D}{8n(R_D+1)} \le (1 - \frac{r}{8n(R_D+1)})\mathrm{d}(x, y). \tag{3.13}$$

Noting that (3.13) holds for all $x, y \in D$ such that $\mathrm{d}(x, y) \ge \frac{R_D}{2n}$, we conclude that $H \in \mathcal{M}_n$ and so

$$\mathbf{B}_{\mathcal{M}}(G, \alpha r) \subset \mathcal{M}_n. \tag{3.14}$$

On the other hand, by (3.8), one has

$$\mathrm{h}(H(x), F(x)) \le \mathrm{h}(H(x), G(x)) + \mathrm{h}(G(x), F(x)) \le \alpha r + \lambda R_D \le r, \tag{3.15}$$

which means that $\mathbf{B}_{\mathcal{M}}(G, \alpha r) \subset \mathbf{B}_{\mathcal{M}}(F, r)$. This together with (3.14) implies that (3.9) holds. Set $\mathcal{N} := \cap_{n\in\mathbb{N}}\mathcal{M}_n$ and the proof is complete. □

Applying Theorem 3.1 to the special admissible family χ_D^S, we get the following corollary, which extends and improves the corresponding results in [22] and [21]. In particular, Corollary 3.2 was proved in [22, Theorem 2.2] in the Banach space setting.

Corollary 3.2. *Let $\mathcal{M}_S := \{f : D \to \chi_D^S : f$ is nonexpansive$\}$. Then there exists a set $\mathcal{N}_S \subset \mathcal{M}_S$ such that $\mathcal{M}_S \backslash \mathcal{N}_S$ is σ-porous in $\mathcal{M}_S$ and each $f \in \mathcal{N}_S$ is contractive.*

4. Attractive sets

As assumed in the previous sections, E is a complete hyperbolic space, $D \subseteq E$ is a nonempty, closed, bounded, Γ-star-shaped set. In this section, let D_0 be a convex subset of D. Set

$$\mathcal{M}_{D_0} := \{F \in \mathcal{M} : x \in F(x), \forall x \in D_0\}. \tag{4.1}$$

Clearly, $\mathcal{M}_{D_0}$ is a closed subset of $\mathcal{M}$. The next definition is an extension of the single-valued case.

Definition 4.1. Let $F \in \mathcal{M}_{D_0}$. F is said to be contractive with respect to D_0 if there is a function $\alpha_F : [0, R_D] \to [0, 1)$ with α_F being decreasing such that

$$\inf_{f\in F(x)} \mathrm{d}(f, D_0) \le \alpha(\mathrm{d}(x, D_0))\mathrm{d}(x, D_0), \quad \forall x \in D. \tag{4.2}$$

Remark 4.2. As in the single-valued case, for any $F \in \mathcal{M}_{D_0}$, F is contractive implies that F is contractive with respect to D_0. In fact, let $F \in \mathcal{M}_{D_0}$ be contractive and $y \in D_0$ be arbitrary. Then $y \in F(y)$ and

$$\inf_{f\in F(x)} \mathrm{d}(y, f) \le \mathrm{h}(F(x), F(y)) \le \alpha_F(\mathrm{d}(x, y))\mathrm{d}(x, y). \tag{4.3}$$

It follows that

$$\inf_{f\in F(x), y\in D_0} \mathrm{d}(y, f) \le \inf_{y\in D_0} \alpha_F(\mathrm{d}(x, y))\mathrm{d}(x, y). \tag{4.4}$$

Noting that α_F is decreasing, (4.2) follows from (4.4) immediately.

Let $\mathcal{N}_{D_0}$ denote the set of all $F \in \mathcal{M}_{D_0}$ such that F is contractive with respect to D_0. Furthermore, let

$$\mathcal{M}_{D_0}^S := \{f \in \mathcal{M}_S : f(x) = x, \forall x \in D_0\}. \tag{4.5}$$

The main result in this section is as follows, which seems new for the set-valued case even in Banach spaces.

Theorem 4.3. *Suppose that there is $p \in \mathcal{M}_{D_0}^S$ such that $p(D) = D_0$. Then the set $\mathcal{M}_{D_0} \setminus \mathcal{N}_{D_0}$ is σ-porous in $\mathcal{M}_{D_0}$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary and let $\mathcal{M}_{D_0}^n$ denote the set of all $F \in \mathcal{M}_{D_0}$ which have the following property.

$$\exists k_n \in (0,1) \text{ s.t. } \forall x \in D \text{ with } \mathrm{d}(x, D_0) \geq \frac{\min\{R_D, 1\}}{n}, \inf_{f \in F(x)} \mathrm{d}(f, D_0) \leq k_n \mathrm{d}(x, D_0). \tag{4.6}$$

To complete the proof, we only need to show that
 (a) $\cap_{n\in\mathbb{N}} \mathcal{M}_{D_0}^n \subseteq \mathcal{N}_{D_0}$;
 (b) the set $\mathcal{M}_{D_0} \setminus \mathcal{M}_{D_0}^n$ is porous in $\mathcal{M}_{D_0}$.

To show **(a)**, let $F \in \cap_{n\in\mathbb{N}} \mathcal{M}_{D_0}^n$. By (4.6), for each $n \in \mathbb{N}$, there is $k_n \in (0,1)$ such that

$$\inf_{f\in F(x)} \mathrm{d}(f, D_0) \leq k_n \mathrm{d}(x, D_0), \quad \forall x \in D \text{ with } \mathrm{d}(x, D_0) \geq \frac{\min\{R_D, 1\}}{n}. \tag{4.7}$$

Without loss of generality, we may assume that $k_n \leq k_{n+1}$. Define $\alpha_F : [0, R_D] \to [0, 1)$ by

$$\alpha_F(d) := k_n, \quad \frac{\min\{R_D, 1\}}{n-1} > d \geq \frac{\min\{R_D, 1\}}{n}. \tag{4.8}$$

Then α_F is decreasing on $[0, R_D]$ and $\alpha_F(d) < 1$ for all $d > 0$. So **(a)** is proved.

Below we show **(b)**. For this purpose, set

$$\alpha := \frac{1}{8} \min\{R_D, 1\}(2n)^{-1}(R_D + 1)^{-1} \text{ and } r_0 := 1. \tag{4.9}$$

Let $F \in \mathcal{M}_{D_0}$, $r \in (0, r_0]$,

$$\lambda := \frac{r}{2(R_D + 1)} \tag{4.10}$$

and define $G : D \to \chi_D$ by

$$G(x) := \lambda p(x) \oplus (1 - \lambda)F(x), \ \forall x \in D. \tag{4.11}$$

Then $x \in G(x)$ for all $x \in D_0$. By (2.2), one has

$$\mathrm{h}(G(x), F(x)) \leq \lambda \mathrm{d}(p(x), F(x)) \leq \lambda R_D, \quad \forall x \in D \tag{4.12}$$

and

$$\mathrm{h}(G(x), G(y)) \leq \mathrm{d}(x, y), \quad \forall x, y \in D. \tag{4.13}$$

So $G \in \mathcal{M}_{D_0}$. It suffices to show that

$$\mathbf{B}_{\mathcal{M}_{D_0}}(G, \alpha r) \subset \mathbf{B}_{\mathcal{M}_{D_0}}(F, r) \cap \mathcal{M}_{D_0}^n. \tag{4.14}$$

Let $W \in \mathbf{B}_{\mathcal{M}_{D_0}}(G, \alpha r)$ and $x \in D$ with $\mathrm{d}(x, D_0) \geq \frac{\min\{R_D, 1\}}{n}$. For each $\epsilon > 0$, there is $z \in D_0$ such that

$$\mathrm{d}(x, z) \leq \mathrm{d}(x, D_0) + \epsilon. \tag{4.15}$$

Noting that $z \in F(z)$, (2.2) and (4.15), one has that

$$\begin{aligned}
\inf_{g \in G(x)} \mathrm{d}(g, D_0) &= \inf_{f \in F(x)} \mathrm{d}(\lambda p(x) \oplus (1-\lambda) f, D_0) \\
&\leq \inf_{f \in F(x)} \mathrm{d}(\lambda p(x) \oplus (1-\lambda) f, \lambda p(x) \oplus (1-\lambda) z) \\
&\leq \inf_{f \in F(x)} (1-\lambda) \mathrm{d}(f, z) \\
&= (1-\lambda) \mathrm{d}(z, F(x)) \\
&\leq (1-\lambda) \mathrm{h}(F(z), F(x)) \\
&\leq (1-\lambda) \mathrm{d}(x, z) \\
&\leq (1-\lambda) \mathrm{d}(x, D_0) + (1-\lambda)\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\inf_{g \in G(x)} \mathrm{d}(g, D_0) \leq (1-\lambda) \mathrm{d}(x, D_0). \tag{4.16}$$

Note that for $x_1, x_2 \in D$, $|\mathrm{d}(x_1, D_0) - \mathrm{d}(x_2, D_0)| \leq \mathrm{d}(x_1, x_2)$. It follows from (4.16) that

$$\inf_{w \in W(x)} \mathrm{d}(w, D_0) \leq \inf_{w \in W(x)} \mathrm{d}(w, g) + \mathrm{d}(g, D_0) \leq \alpha r + \mathrm{d}(g, D_0) \tag{4.17}$$

and hence

$$\inf_{w \in W(x)} \mathrm{d}(w, D_0) \leq \inf_{g \in G(x)} \mathrm{d}(g, D_0) + \alpha r \leq (1-\lambda) \mathrm{d}(x, D_0) + \alpha r. \tag{4.18}$$

Thus

$$\begin{aligned}
\inf_{w \in W(x)} \mathrm{d}(w, D_0) &\leq (1-\lambda) \mathrm{d}(x, D_0) + \alpha r \\
&= \mathrm{d}(x, D_0) \left(1 - \lambda + \frac{\alpha r}{\mathrm{d}(x, D_0)}\right) \\
&\leq \mathrm{d}(x, D_0) \left(1 - \frac{r}{2(R_D+1)} + \frac{\alpha n r}{\min\{R_D, 1\}}\right) \\
&\leq \mathrm{d}(x, D_0) \left(1 - \frac{r}{2(R_D+1)} + \frac{r}{16(R_D+1)}\right) \\
&= \mathrm{d}(x, D_0) \left(1 - \frac{7r}{16(R_D+1)}\right),
\end{aligned}$$

which implies that $W \in \mathcal{M}_{D_0}^n$ and hence

$$\mathbf{B}_{\mathcal{M}_{D_0}}(G, \alpha r) \subset \mathcal{M}_{D_0}^n. \tag{4.19}$$

On the other hand, by (4.9), (4.10) and (4.12), one gets

$$\rho(W, F) \leq \rho(W, G) + \rho(G, F) \leq \alpha r + \lambda R_D \leq r, \tag{4.20}$$

which means that (4.14) holds and the proof is complete. □

Applying Theorem 4.1 to the special a-admissible family $\mathfrak{A}_D^S$. We have the following corollary, which extends the corresponding results in [21, 22]. In particular, under the assumption that D is convex, Corollary 4.4 was proved in [22] in the Banach space setting.

Corollary 4.4. *Suppose that there is* $p \in \mathcal{M}_{D_0}^S$ *such that* $p(D) = D_0$ *and let* $\mathcal{N}_{D_0}^S$ *denote the set of all* $f \in \mathcal{M}_{D_0}^S$ *such that* f *is contractive with respect to* D_0. *Then the set* $\mathcal{M}_{D_0}^S \setminus \mathcal{N}_{D_0}^S$ *is* σ*-porous in* $\mathcal{M}_{D_0}^S$.

5. Concluding remark

As metioned in the introduction section, every contractive single-valued map has a unique fixed point. However, we do not know whether every contractive set-valued map has a fixed point or not. Actually, it seems little to be known about the existence of fixed point for the set-valued case.

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