Journal of Nonlinear and Convex Analysis
Volume 16, Number 7, 2015, 1425–1437

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON THE CONVERGENCE ANALYSIS OF A NEWTON-LIKE METHOD UNDER WEAK SMOOTHNESS ASSUMPTIONS

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Abstract. The aim of the paper is to discuss the semilocal convergence analysis of a Newton-like method for solving the generalized operator equations containing nondifferentiable term in Banach spaces under ω-type conditions. Our result extends and generalizes the corresponding result discussed by Argyros and Hilout [Improved generalized differentiability conditions for Newton-like methods, J. Complexity, 26 (2010), 316-333]. A numerical example is discussed in support of our main result.

1. Introduction

Let X and Y be Banach spaces and D an open convex subset of X. Throughout the paper, we denote $B_r[x]$ (and $B_r(x)$) the closed ball defined by $B_r[x] = \{y \in X : \|y - x\| \leq r\}$ (and $B_r(x) = \{y \in X : \|y - x\| < r\}$), $B(X, Y)$ the space of bonded linear operators from X to Y, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and Φ denotes the collection of all positive nondecreasing real-valued functions defined on $[0, \infty)$. Consider the nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is an operator defined from X into Y such that F is continuously Fréchet differentiable at each point of D. The problems of differential and integral equations, differential inequalities, optimization problems, variational problems, fixed points and many others can be formulated in terms of finding the solution of nonlinear operator equations (1.1) (see [10, 11, 12, 13, 22]). The Newton method for solving the operator equation (1.1) is given by

$$x_{n+1} = x_n - F'^{-1}_{x_n} F(x_n) \text{ for all } n \in \mathbb{N}_0. \tag{1.2}$$

In [9], Bartle considered the following Newton-like method for solving the operator equation (1.1):

$$x_{n+1} = x_n - F'^{-1}_{z_n} F(x_n) \text{ for all } n \in \mathbb{N}_0, \tag{1.3}$$

where $\{z_n\}$ is a sequence in D. Note that if $z_n = x_0$ for all $n \in \mathbb{N}_0$, (1.3) reduces to modified Newton method given by

$$x_{n+1} = x_n - F'^{-1}_{x_0} F(x_n) \text{ for all } n \in \mathbb{N}_0.$$

2010 *Mathematics Subject Classification.* 49M15, 65K10.
Key words and phrases. Banach space, generalized operator equations, Fréchet derivative, Newton-like method, semilocal convergence.

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In [21], Vijesh and Subrahmaniam considered the combination of (1.2) and (1.3) and discussed the convergence analysis of

$$x_{n+1} = x_n - (\lambda F'_{x_n} + (1-\lambda)F'_{z_n})^{-1}F(x_n) \text{ for all } n \in \mathbb{N}_0, \tag{1.4}$$

where λ is a fixed number in $[0,1]$. Note that for $\lambda = 1$ and $\lambda = 0$, (1.4) reduces to (1.2) and (1.3), respectively. Recently, Argyros and Hilout [1] discussed the convergence analysis of the Newton-like method:

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n), \text{ for all } n \in \mathbb{N}_0, \tag{1.5}$$

where $A(x)$ is a linear operator which approximates F'_x.

The following problem is more general than problem (1.1):

$$F(x) + G(x) = 0, \tag{1.6}$$

where F and G are two operators defined on an open convex subset D of a Banach space X into Banach space Y such that F is continuously Fréchet differentiable at each point of D and G is not necessarily differentiable. The operator equation (1.6) has been considered by many authors, for example, see [2, 3, 4, 5, 6, 7, 8, 15, 16, 17, 18, 19, 20, 21] and references therein. In [1], Argyros and Hilout considered the following iteration process:

$$x_{n+1} = x_n - A(x_n)^{-1}(F(x_n) + G(x_n)), \text{ for all } n \in \mathbb{N}_0, \tag{1.7}$$

for solving the operator equation (1.6). Recently, following the ideas of (1.4), Sahu and Singh [17] discussed the semilocal convergence analysis of

$$x_{n+1} = x_n - (\lambda F'_{x_n} + (1-\lambda)F'_{z_n})^{-1}(F(x_n) + G(x_n)), \text{ for all } n \in \mathbb{N}_0, \tag{1.8}$$

for solving the operator equation (1.6).

Motivated by the iterative algorithms (1.7) and (1.8), the main purpose of this paper is to introduce a Newton-like iterative algorithm to compute the solution of (1.6). We also discuss the semilocal convergence analysis of the sequence generated by proposed algorithm under ω-type conditions. Since our assumptions on the nonlinear operators F and G involving in operator equation (1.6) are fairly general, our main result (Theorem 3.1) covers a wide variety of nonlinear operator equations and it significantly generalizes the corresponding results of Argyros and Hilout [1] and Vijesh and Subrahmanyam [21]. At the end, we give a numerical example to illustrate our main result.

2. Preliminaries

In subsequent sections, we shall make use of the following lemmas.

Lemma 2.1 (Rall [14, Page 50])**.** *Let L be a bounded linear operator on a Banach space X. Then the followings are equivalent:*

(a) *There is a bounded linear operator M on X such that M^{-1} exists, and*

$$\|M - L\| < \frac{1}{\|M^{-1}\|}.$$

(b) *L^{-1} exists.*

Further, if L^{-1} exists, then $\|L^{-1}\| \le \frac{\|M^{-1}\|}{1-\|1-M^{-1}L\|} \le \frac{\|M^{-1}\|}{1-\|M^{-1}\|\|M-L\|}$.

Lemma 2.2. *Let $H, H_1, l_0, l_1, l_2, \eta$ be some nonnegative real numbers, $\lambda \in [0,1]$ be a fixed number and $\omega, \omega_0, \omega_1, \omega_2, \omega_3 \in \Phi$. Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_n \le \delta$ for all $n \in \mathbb{N}_0$ and for some $\delta \ge 0$. Denote*

$$\psi_0 = \frac{H_1\omega_1(\eta) + \lambda\omega_2(0) + (1-\lambda)[\omega_1(\delta) + \omega_2(\delta)] + \omega_3(\eta) + (2-\lambda)l_1 + l_2}{1 - l_0 - \lambda\omega_0(\eta) - (1-\lambda)\omega_0(\delta)}$$

and

$$\psi(r) = \frac{H\omega(\eta) + \lambda\omega_2(r) + (1-\lambda)[\omega(\delta + r) + \omega_2(\delta)] + \omega_3(r) + l_2}{1 - l_0 - \lambda\omega_0(r) - (1-\lambda)\omega_0(\delta)}.$$

Assume that the scalar equation

$$\left(1 + \frac{\psi_0}{1-\psi(r)}\right)\eta = r \tag{2.1}$$

has a minimum positive zero r^* such that

$$\lambda\omega_0(r^*) + (1-\lambda)\omega_0(\delta) < 1 - l_0, \psi_0 < 1 \text{ and } \psi(r^*) < 1. \tag{2.2}$$

Then, the sequence $\{t_n\}$ defined by

$$\begin{aligned} t_0 &= 0, t_1 = \eta, t_2 = t_1 + \psi_0\eta \text{ and} \\ t_{n+1} &= t_n + \frac{(t_n - t_{n-1})}{1 - l_0 - \lambda\omega_0(t_n) - (1-\lambda)\omega_0(\delta_n)}(H\omega(t_n - t_{n-1}) + \lambda\omega_2(t_{n-1}) \\ &\quad +(1-\lambda)(\omega(t_{n-1} + \delta_{n-1}) + \omega_2(\delta_{n-1})) + \omega_3(t_n - t_{n-1}) + l_2) \text{ for } n \ge 2 \end{aligned} \tag{2.3}$$

is well defined, nondecreasing, bounded above and hence convergent and converges to its least upper bound t^*. Moreover, the following estimate holds:

$$t_n - t_{n-1} \le \psi_0\eta(\psi(r^*))^{n-2}, n \ge 2. \tag{2.4}$$

Proof. Since r^* is the minimum positive root of (2.1), we have $t_1 = \eta \le r^*$ and $t_1 - t_0 \le \eta$. Using induction on n, we show that t_n is well defined, $t_{n-1} \le t_n \le r^*$ and (2.4) hold for all $n \ge 2$. Using (2.1) and (2.3), we note that $t_2 \le r^*$ and $t_2 - t_1 \le \psi_0\eta$. Thus, our assertion holds for $n = 2$. Let $k \ge 2$ be an integer. Assume that our assertion hold for $n = 2, 3, \ldots, k$. As $t_k < r^*$, it follows from (2.2) and (2.3) that t_{k+1} is well defined. Using (2.3), we have

$$\begin{aligned} t_{k+1} &= t_k + \frac{(t_k - t_{k-1})}{1 - l_0 - \lambda\omega_0(t_k) - (1-\lambda)\omega_0(\delta_k)}(H\omega(t_k - t_{k-1}) + \lambda\omega_2(t_{k-1}) \\ &\quad + (1-\lambda)(\omega(t_{k-1} + \delta_{k-1}) + \omega_2(\delta_{k-1})) + \omega_3(t_k - t_{k-1}) + l_2) \\ &\le t_k \\ &\quad + \frac{H\omega(\eta) + \lambda\omega_2(r^*) + (1-\lambda)(\omega(r^* + \delta) + \omega_2(\delta)) + \omega_3(r^*) + l_2}{1 - l_0 - \lambda\omega_0(r^*) - (1-\lambda)\omega_0(\delta)}(t_k - t_{k-1}) \\ &= t_k + \psi(r^*)(t_k - t_{k-1}) \\ &\le t_{k-1} + \psi(r^*)(t_{k-1} - t_{k-2}) + \psi(r^*)(t_k - t_{k-1}) \\ &\le t_2 + \psi(r^*)(t_2 - t_1) + \psi(r^*)(t_3 - t_2) \\ &\quad + \cdots + \psi(r^*)(t_{k-1} - t_{k-2}) + \psi(r^*)(t_k - t_{k-1}) \end{aligned} \tag{2.5}$$

$$\leq \Big(1 + \psi_0 + \psi_0\psi(r^*) + \psi_0(\psi(r^*))^2 + \cdots + \psi_0(\psi(r^*))^{k-1}\Big)\eta$$
$$\leq \left(1 + \frac{\psi_0}{1-\psi(r^*)}\right)\eta$$
$$= r^*.$$

Hence, we have $t_{k+1} \leq r^*$. By definition of $\omega, \omega_0, \omega_1, \omega_2, \omega_3$ and (2.3), it follows that $t_k \leq t_{k+1}$. Using (2.5), we have

$$\begin{aligned} t_{k+1} - t_k &\leq \psi(r^*)(t_k - t_{k-1}) \\ &\leq \psi_0\eta(\psi(r^*))^{k-1}. \end{aligned}$$

Thus, (2.4) holds for $n = k+1$. Hence, the induction hypothesis is complete. Therefore, the sequence $\{t_n\}$ is nondecreasing, bounded above and as such it converges to its unique least upper bound t^* for some $t^* \in [\eta, r^*]$. □

3. Convergence analysis

Let F and G be two operators defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Fréchet differentiable at each point of D. Let $A(x) \in B(X,Y)$ be an operator which approximates F'_x. Let $\{z_n\}$ be a sequence in D. To solve the operator equation (1.6), we define the Newton-like algorithm as follows: Starting with $x_0 \in D$ and after $x_n \in D$ is defined, we define the next iteration x_{n+1} as follows:

$$x_{n+1} = x_n - (\lambda A(x_n) + (1-\lambda)A(z_n))^{-1}(F(x_n) + G(x_n)), \text{ for all } n \in \mathbb{N}_0. \tag{3.1}$$

Note that when $\lambda = 1$, (3.1) reduces to (1.7) and when $A(x) = F'_x$, (3.1) reduces to (1.8).

The following theorem is the main result of this paper which guarantees the convergence of the proposed algorithm (3.1) to a solution of the problem (1.6).

Theorem 3.1. *Let F and G be two operators defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Fréchet differentiable at each point of D. Let $A(x) \in B(X,Y)$ be an operator which approximates F'_x for $x \in D$. Let $\{\delta_n\}$ be a sequence of nonnegative real numbers such that $\delta_n \leq \delta$ for all $n \in \mathbb{N}_0$ and for some $\delta \geq 0$. For $x_0 \in D$, let $\{z_n\}_{n=0}^{\infty}$ be a sequence in D satisfying $\|z_n - x_0\| \leq \delta_n$ and $A(x_0)^{-1}, (\lambda A(x_0) + (1-\lambda)A(z_0))^{-1} \in B(Y,X)$ for some fixed $\lambda \in [0,1]$. For $x, y \in D$ and for some nonnegative numbers η, l_0, l_1, l_2, assume that the operators F, F' and $A(x)$ satisfy the following conditions:*

- ($\mathcal{C}1$) $\|(\lambda A(x_0) + (1-\lambda)A(z_0))^{-1}(F(x_0) + G(x_0))\| \leq \eta$;
- ($\mathcal{C}2$) $\|A(x_0)^{-1}(F'_x - F'_y)\| \leq \omega(\|x-y\|)$;
- ($\mathcal{C}3$) $\|A(x_0)^{-1}(A(x) - A(x_0))\| \leq \omega_0(\|x - x_0\|) + l_0$;
- ($\mathcal{C}4$) $\|A(x_0)^{-1}(F'_x - F'_{x_0})\| \leq \omega_1(\|x - x_0\|) + l_1$;
- ($\mathcal{C}5$) $\|A(x_0)^{-1}(F'_x - A(x))\| \leq \omega_2(\|x - x_0\|) + l_2$;
- ($\mathcal{C}6$) $\|A(x_0)^{-1}(G(x) - G(y))\| \leq \omega_3(\|x-y\|)\|x-y\|$,

where $\omega, \omega_0, \omega_1, \omega_2, \omega_3 \in \Phi$. Assume that

$$\omega(ts) \leq h(t)\omega(s), \omega_1(ts) \leq h_1(t)\omega_1(s) \text{ for all } t \in [0,1] \text{ and for all } s \in [0,\infty) \tag{3.2}$$

for some positive nondecreasing continuous functions h and h_1 defined in $[0,1]$. Denote $H = \int_0^1 h(t)dt$ and $H_1 = \int_0^1 h_1(t)dt$. Suppose that the scalar equation defined by (2.1) has a minimum positive zero r^* such that (2.2) is satisfied and $B_{r^*}[x_0] \subseteq D_0$. Then, we have the following:

(a) *The sequence $\{x_n\}$ generated by (3.1) is well defined, remains in $B_{r^*}[x_0]$ and converges to a solution $x^* \in B_{r^*}[x_0]$ of (1.6). Moreover, the following error estimates hold:*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \text{ for all } n \in \mathbb{N}_0, \tag{3.3}$$

$$\|x_{n+1} - x_0\| \leq t_{n+1} \text{ for all } n \in \mathbb{N}_0 \tag{3.4}$$

and

$$\|x_{n+1} - x^*\| \leq t^* - t_{n+1} \text{ for all } n \in \mathbb{N}_0, \tag{3.5}$$

where $\{t_n\}$ is a sequence generated by (2.3) and t^* is the limit of sequence $\{t_n\}$.

(b) *Further, if r^{**} is a positive number such that*

$$\theta = \frac{H\omega(r^{**}) + \lambda\omega_2(r^*) + (1-\lambda)(\omega(r^*+\delta) + \omega(\delta)) + \omega_3(r^{**}) + l_2}{1 - l_0 - \lambda\omega_0(r^*) - (1-\lambda)\omega_0(\delta)} < 1, \tag{3.6}$$

then the solution x^* of (1.6) is unique in $B_{r^{**}}[x_0] \cap D$.

Proof. (a) It follows from Lemma 2.2 that the sequence $\{t_n\}$ defined by (2.3) is convergent and converges to its unique least upper bound t^* for some $t^* \in [\eta, r^*]$. We now proceed with the following steps:

Step 1. $\{x_n\}$ is well defined and it remains in $B_{r^*}[x_0]$ and (3.3) and (3.4) hold for all $n \geq 0$.

Note $\|x_1 - x_0\| = \|((\lambda A(x_0) + (1-\lambda)A(z_0))^{-1}(F(x_0) + G(x_0))\| \leq \eta = t_1 - t_0 \leq r^*$.

Hence, $x_1 \in B_{r^*}[x_0]$ and (3.3) and (3.4) hold for $n = 0$. Using induction on n, we show that x_n is well defined for each $n \geq 2$ and remains in $B_{r^*}[x_0]$ and (3.3)-(3.4) hold for all $n \geq 1$. Put $L_n = \lambda A(x_n) + (1-\lambda)A(z_n)$. From (3.1), we have

$$\begin{aligned}
& \|A(x_0)^{-1}(A(x_0) - L_1)\| \\
= \; & \|A(x_0)^{-1}(\lambda A(x_0) + (1-\lambda)A(x_0) - \lambda A(x_1) - (1-\lambda)A(z_1))\| \\
\leq \; & \lambda\|A(x_0)^{-1}(A(x_1) - A(x_0))\| + (1-\lambda)\|A(x_0)^{-1}(A(z_1) - A(x_0))\| \\
\leq \; & \lambda\omega_0(\|x_1 - x_0\|) + (1-\lambda)\omega_0(\|z_1 - x_0\|) + l_0 \\
\leq \; & \lambda\omega_0(t_1) + (1-\lambda)\omega_0(\delta_1) + l_0 \\
\leq \; & \lambda\omega_0(r^*) + (1-\lambda)\omega_0(\delta) + l_0 < 1.
\end{aligned}$$

Hence, by Banach lemma, L_1^{-1} exists and

$$\|L_1^{-1}A(x_0)\| \leq \frac{1}{1 - l_0 - \lambda\omega_0(t_1) - (1-\lambda)\omega_0(\delta_1)} \leq \frac{1}{1 - l_0 - \lambda\omega_0(r^*) - (1-\lambda)\omega_0(\delta)}.$$

Therefore, x_2 is well defined. Now, we have

$$\|x_2 - x_1\| = \|{L_1}^{-1}(F(x_1) + G(x_1))\|$$

$$
\begin{aligned}
&= \|{L_1}^{-1}(F(x_1) + G(x_1) - F(x_0) - G(x_0) - L_0(x_1 - x_0))\| \\
&= \Big\|{L_1}^{-1}\Big(\int_0^1 (F'_{x_0+t(x_1-x_0)} - \lambda A(x_0) - (1-\lambda)A(z_0))(x_1 - x_0)dt \\
&\quad + G(x_1) - G(x_0)\Big)\Big\| \\
&\leq \|{L_1}^{-1}A(x_0)\|\Big(\lambda\int_0^1 (\|A(x_0)^{-1}(F'_{x_0+t(x_1-x_0)} - A(x_0))\| \\
&\quad + (1-\lambda)\|A(x_0)^{-1}(F'_{x_0+t(x_1-x_0)} - A(z_0))\|)\|x_1 - x_0\|dt \\
&\quad + \|A(x_0)^{-1}(G(x_1) - G(x_0))\|\Big) \\
&= \|{L_1}^{-1}A(x_0)\|\Big(\int_0^1 (\lambda\|A(x_0)^{-1}(F'_{x_0+t(x_1-x_0)} - F'_{x_0} + F'_{x_0} - A(x_0))\| \\
&\quad + (1-\lambda)\|A(x_0)^{-1}(F'_{x_0+t(x_1-x_0)} - F'_{x_0} + F'_{x_0} - F'_{z_0} + F'_{z_0} - A(z_0))\|) \\
&\quad \times \|x_1 - x_0\|dt + \|A(x_0)^{-1}(G(x_1) - G(x_0))\|\Big).
\end{aligned}
$$

Using $(\mathcal{C}2) - (\mathcal{C}6)$ and (3.2), we get

$$
\begin{aligned}
& \|x_2 - x_1\| \\
\leq\ & \|{L_1}^{-1}A(x_0)\|(H_1\omega_1(\|x_1 - x_0\|) + \lambda\omega_2(0) + (1-\lambda)(\omega_1(\|z_0 - x_0\|) \\
& +\omega_2(\|z_0 - x_0\|)) + \omega_3(\|x_1 - x_0\|) + (2-\lambda)l_1 + l_2)\|x_1 - x_0\| \\
\leq\ & \left[\frac{H_1\omega_1(\eta) + \lambda\omega_2(0) + (1-\lambda)(\omega_1(\delta) + \omega_2(\delta)) + \omega_3(\eta) + (2-\lambda)l_1 + l_2}{1 - l_0 - \lambda\omega_0(\eta) - (1-\lambda)\omega_0(\delta)}\right]\eta \\
=\ & \psi_0\eta = t_2 - t_1,
\end{aligned}
$$

which shows (3.3) holds for $n = 1$. Note

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq t_2 - t_1 + t_1 - t_0 = t_2 \leq r^*,$$

which shows (3.4) holds for $n = 1$ and $x_2 \in B_{r^*}[x_0]$. Thus, our assertion holds for $n = 1$. Let $k \geq 1$ be an integer. Assume that our assertion holds for some positive integer $n = k$. From (3.1), we have

$$
\begin{aligned}
& \|A(x_0)^{-1}(A(x_0) - L_k)\| \\
=\ & \|A(x_0)^{-1}(\lambda A(x_0) + (1-\lambda)A(x_0) - \lambda A(x_k) - (1-\lambda)A(z_k))\| \\
\leq\ & \lambda\|A(x_0)^{-1}(A(x_k) - A(x_0))\| + (1-\lambda)\|A(x_0)^{-1}(A(z_k) - A(x_0))\| \\
\leq\ & \lambda\omega_0(\|x_k - x_0\|) + (1-\lambda)\omega_0(\|z_k - x_0\|) + l_0 \\
\leq\ & \lambda\omega_0(t_k) + (1-\lambda)\omega_0(\delta_k) + l_0 \\
\leq\ & \lambda\omega_0(r^*) + (1-\lambda)\omega_0(\delta) + l_0 < 1.
\end{aligned}
$$

Hence, by Banach lemma, L_k^{-1} exists and

$$\|L_k^{-1}A(x_0)\| \leq \frac{1}{1 - l_0 - \lambda\omega_0(t_k) - (1-\lambda)\omega_0(\delta_k)} \leq \frac{1}{1 - l_0 - \lambda\omega_0(r^*) - (1-\lambda)\omega_0(\delta)}.$$

Therefore, x_{k+1} is well defined. Now, we have

$$
\begin{aligned}
&\|x_{k+1}-x_k\| \\
&= \|L_k{}^{-1}(F(x_k)+G(x_k))\| \\
&= \|L_k{}^{-1}(F(x_k)+G(x_k)-F(x_{k-1})-G(x_{k-1})-L_{k-1}(x_k-x_{k-1}))\| \\
&\le \|L_k{}^{-1}A(x_0)\|\bigg(\int_0^1 \|A(x_0)^{-1}(F'_{x_{k-1}+t(x_k-x_{k-1})}-\lambda A(x_{k-1})-(1-\lambda)A(z_{k-1}))\| \\
&\quad \times\|x_k-x_{k-1}\|dt+\|A(x_0)^{-1}(G(x_k)-G(x_{k-1}))\|\bigg) \\
&\le \|L_k{}^{-1}A(x_0)\|\bigg[\int_0^1\bigg(\lambda\|A(x_0)^{-1}(F'_{x_{k-1}+t(x_k-x_{k-1})}-F'_{x_{k-1}}+F'_{x_{k-1}}-A(x_{k-1}))\| \\
&\quad +(1-\lambda)\|A(x_0)^{-1}(F'_{x_{k-1}+t(x_k-x_{k-1})}-F'_{x_{k-1}}+F'_{x_{k-1}}-F'_{z_{k-1}}+F'_{z_{k-1}}-A(z_{k-1}))\|dt\bigg) \\
&\quad \times\|x_k-x_{k-1}\|+\|A(x_0)^{-1}(G(x_k)-G(x_{k-1}))\|\bigg].
\end{aligned}
$$

Again, by using $(\mathcal{C}2)-(\mathcal{C}6)$ and (3.2), we get

$$
\begin{aligned}
& \|x_{k+1}-x_k\| \\
\le\ & \|L_k{}^{-1}A(x_0)\|\Big[H\omega(\|x_k-x_{k-1}\|)+\lambda\omega_2(\|x_{k-1}-x_0\|) \\
& +(1-\lambda)(\omega(\|z_{k-1}-x_{k-1}\|) \\
& +\omega_2(\|z_{k-1}-x_0\|))+\omega_3(\|x_k-x_{k-1}\|)+l_2\Big]\|x_k-x_{k-1}\| \\
\le\ & \frac{(t_k-t_{k-1})}{1-l_0-\lambda\omega_0(t_k)-(1-\lambda)\omega_0(\delta_k)}\Big[H\omega(t_k-t_{k-1})+\lambda\omega_2(t_{k-1}) \\
& +(1-\lambda)(\omega(\delta_{k-1}+t_{k-1})+\omega_2(\delta_{k-1}))+\omega_3(t_k-t_{k-1})+l_2\Big] \\
=\ & t_{k+1}-t_k,
\end{aligned}
$$

which shows that (3.3) holds for $n=k+1$. Note

$$\|x_{k+1}-x_0\|\le\|x_{k+1}-x_k\|+\cdots+\|x_1-x_0\|\le t_{k+1}\le r^*,$$

which shows (3.4) holds for $n=k$ and $x_{k+1}\in B_{r^*}[x_0]$. Thus, our assertion holds for all $n\ge 0$.

Step 2. $x_n\to x^*$, where $x^*\in B_{r^*}[x_0]$ is a solution of (1.6) and (3.5) holds for all $n\in\mathbb{N}_0$.

Using (3.3), we have

$$\|x_{n+m}-x_n\| \ \le\ \|x_{n+m}-x_{n+m-1}\|+\|x_{n+m-1}-x_{n+m-2}\|+\cdots+\|x_{n+1}-x_n\|$$

$$= \sum_{k=n}^{n+m-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+m-1} (t_{k+1} - t_k) = t_{n+m} - t_n. \tag{3.7}$$

It follows that $\{x_n\}$ is Cauchy and hence converges to some $x^* \in B_{r^*}[x_0]$. From (3.1), we get

$$\begin{aligned}
& \|A(x_0)^{-1}(F(x_n) + G(x_n))\| \\
\leq\ & \|A(x_0)^{-1}(L_n)\|\|x_{n+1} - x_n\| \\
\leq\ & \|A(x_0)^{-1}(\lambda A(x_n) + (1-\lambda)A(z_n) - \lambda A(x_0) + \lambda A(x_0) \\
& -(1-\lambda)A(x_0) + (1-\lambda)A(x_0))\|\|x_{n+1} - x_n\| \\
=\ & \|A(x_0)^{-1}(\lambda(A(x_n) - A(x_0)) + (1-\lambda)(A(z_n) - A(x_0)) \\
& +A(x_0)\|\|x_{n+1} - x_n\| \\
\leq\ & (\lambda\omega_0(\|x_n - x_0\|) + (1-\lambda)\omega_0(\|z_n - x_0\|) + l_0 + 1)\|\|x_{n+1} - x_n\| \\
\leq\ & (\lambda\omega_0(r^*) + (1-\lambda)\omega_0(\delta) + l_0 + 1)(t_{n+1} - t_n) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

From the continuity of F and G, we have $F(x^*)+G(x^*) = 0$. Thus, x^* is a solution of (1.6). Taking limit as $m \to \infty$ in (3.7), we get (3.5).

(b) We show that x^* is the unique solution of (1.6) in $B_{r^{**}}[x_0] \cap D$. Suppose that y^* is another solution of (1.6) in $B_{r^{**}}[x_0] \cap D$. For $n \in \mathbb{N}$, we obtain in turn

$$\begin{aligned}
& \|y^* - x_{n+1}\| \\
&= \|y^* - x_n + L_n{}^{-1}(F(x_n) + G(x_n))\| \\
&= \|L_n{}^{-1}(F(y^*) - F(x_n) + G(y^*) - G(x_n) - L_n(y^* - x_n))\| \\
&= \|L_n{}^{-1}A(x_0)\|\bigg(\int_0^1 \|A(x_0)^{-1}(F'_{x_n+t(y^*-x_n)} - L_n)\|\|y^* - x_n\|dt \\
&\quad + \|A(x_0)^{-1}(G(x_n) - G(y^*))\|\bigg) \\
&= \|L_n{}^{-1}A(x_0)\|\bigg[\int_0^1 (\lambda\|A(x_0)^{-1}(F'_{x_n+t(y^*-x_n)} - A(x_n))\| \\
&\quad + (1-\lambda)\|A(x_0)^{-1}(F'_{x_n+t(y^*-x_n)} - A(z_n))\|)\|y^* - x_n\|dt \\
&\quad + \|A(x_0)^{-1}(G(y^*) - G(x_n))\|\bigg] \\
&= \|L_n{}^{-1}A(x_0)\|\bigg[\int_0^1 (\lambda\|A(x_0)^{-1}(F'_{x_n+t(y^*-x_n)} - F'_{x_n} + F'_{x_n} - A(x_n))\| \\
&\quad + (1-\lambda)\|A(x_0)^{-1}(F'_{x_n+t(y^*-x_n)} - F'_{x_n} + F'_{x_n} - F'_{z_n} + F'_{z_n} - A(z_n))\|)\|y^* - x_n\|dt \\
&\quad + \|A(x_0)^{-1}(G(y^*) - G(x_n))\|\bigg].
\end{aligned}$$

Using $(\mathcal{C}2)-(\mathcal{C}6)$ and (3.2), we get

$$\begin{aligned}
&\|y^*-x_{n+1}\|\\
\leq\ &\left[\frac{H\omega(\|y^*-x_n\|)+\lambda\omega_2(\|x_n-x_0\|)+(1-\lambda)(\omega(\|z_n-x_n\|)+\omega_2(\|z_n-x_0\|))}{1-l_0-\lambda\omega_0(\|x_n-x_0\|)-(1-\lambda)\omega_0(\|z_n-x_0\|)}\right.\\
&\left.+\frac{\omega_3(\|y^*-x_n\|)+l_2}{1-l_0-\lambda\omega_0(\|x_n-x_0\|)-(1-\lambda)\omega_0(\|z_n-x_0\|)}\right]\|y^*-x_n\|\\
\leq\ &\left[\frac{H\omega(\|y^*-x_n\|)+\lambda\omega_2(t_n)+(1-\lambda)(\omega(\delta_n+t_n)+\omega_2(\delta_n))}{1-l_0-\lambda\omega_0(t_n)-(1-\lambda)\omega_0(\delta_n)}\right.\\
&\left.+\frac{\omega_3(\|y^*-x_n\|)+l_2}{1-l_0-\lambda\omega_0(t_n)-(1-\lambda)\omega_0(\delta_n)}\right]\|y^*-x_n\|,
\end{aligned}$$

which gives

$$\begin{aligned}
\|y^*-x_{n+1}\| \leq\ &\left[\frac{H\omega(\|y^*-x_n\|)+\lambda\omega_2(r^*)+(1-\lambda)(\omega(\delta+r^*)+\omega_2(\delta))}{1-l_0-\lambda\omega_0(r^*)-(1-\lambda)\omega_0(\delta)}\right.\\
&\left.+\frac{\omega_3(\|y^*-x_n\|)+l_2}{1-l_0-\lambda\omega_0(r^*)-(1-\lambda)\omega_0(\delta)}\right]\|y^*-x_n\|. \qquad (3.8)
\end{aligned}$$

We now prove using induction that

$$\|y^*-x_{n+1}\|\leq\theta\|y^*-x_n\| \qquad (3.9)$$

holds for all $n\in\mathbb{N}_0$. Using (3.6) and (3.8), we see that (3.9) holds for $n=0$. Let k be a positive integer and suppose that (3.9) holds for $n=k-1$. Again using (3.6) and (3.8), we have

$$\begin{aligned}
&\|y^*-x_{k+1}\|\\
\leq\ &\frac{H\omega(\|y^*-x_k\|)+\lambda\omega_2(r^*)+(1-\lambda)(\omega(\delta+r^*)+\omega_2(\delta))+\omega_3(\|y^*-x_k\|)+l_2}{1-l_0-\lambda\omega_0(r^*)-(1-\lambda)\omega_0(\delta)}\\
&\times\|y^*-x_k\|\\
\leq\ &\frac{H\omega(\|y^*-x_0\|)+\lambda\omega_2(r^*)+(1-\lambda)(\omega(\delta+r^*)+\omega_2(\delta))+\omega_3(\|y^*-x_0\|)+l_2}{1-l_0-\lambda\omega_0(r^*)-(1-\lambda)\omega_0(\delta)}\\
&\times\|y^*-x_k\|\\
\leq\ &\frac{H\omega(r^{**})+\lambda\omega_2(r^*)+(1-\lambda)(\omega(r^*+\delta)+\omega_2(\delta))+\omega_3(r^{**})+l_2}{1-l_0-\lambda\omega_0(r^*)-(1-\lambda)\omega_0(\delta)}\|y^*-x_k\|\\
=\ &\theta\|y^*-x_k\|.
\end{aligned}$$

Thus, (3.9) holds for $n=k$. Hence, (3.9) holds for all $n\in\mathbb{N}_0$. Since $\theta<1$. We conclude from (3.9) that $\lim_{n\to\infty}x_n=y^*$. But we have shown that $\lim_{n\to\infty}x_n=x^*$. Hence, we deduce $x^*=y^*$. □

If $\delta=0$, we get $z_n=x_0$ for all $n\in\mathbb{N}_0$. The following corollary follows from Theorem 3.1.

Corollary 3.2. *Let F and G be two operators defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Fréchet differentiable at each point of D. Further, let $A(x)\in B(X,Y)$ be an operator which approximates F'_x, for $x\in D$. For $x_0\in D_0$, assume that $A(x_0)^{-1}$ exists.*

For $\omega, \omega_0, \omega_1, \omega_2, \omega_3 \in \Phi$, assume that $(\mathcal{C}2) - (\mathcal{C}6)$ and (3.2) are satisfied and the following condition holds:

$(\mathcal{C}'1)$ $\|A(x_0)^{-1}(F(x_0) + G(x_0))\| \leq \eta$, *for some* $\eta \geq 0$;

Let $H = \int_0^1 h(t)dt$ *and* $H_1 = \int_0^1 h_1(t)dt$. *Denote*

$$\psi_0 = \frac{H_1\omega_1(\eta) + \lambda\omega_2(0) + (1-\lambda)(\omega_1(0) + \omega_2(0)) + \omega_3(\eta) + (2-\lambda)l_1 + l_2}{1 - l_0 - \lambda\omega_0(\eta) - (1-\lambda)\omega_0(0)}$$

and

$$\psi(r) = \frac{H\omega(\eta) + \lambda\omega_2(r) + (1-\lambda)(\omega(r) + \omega_2(0)) + \omega_3(r) + l_2}{1 - l_0 - \lambda\omega_0(r) - (1-\lambda)\omega_0(0)}.$$

Assume that the scalar equation (2.1) has a minimum positive root r^* with

$$\lambda\omega_0(r^*) + (1-\lambda)\omega_0(0) < 1 - l_0 \text{ and } \psi(r^*) < 1, \psi_0 < 1.$$

Assume that $B_{r_0}[x_0] \subseteq D_0$. Then, we have the following:

(a) *The sequence $\{x_n\}$ generated by*

$$x_{n+1} = x_n - (\lambda A(x_n) + (1-\lambda)A(x_0))^{-1}(F(x_n) + G(x_n)) \text{ for all } n \in \mathbb{N}_0$$

is well defined, remains in $B_{r_0}[x_0]$ and converges to a solution $x^* \in B_{r_0}[x_0]$ of the operator equation (1.6). Moreover, the error estimates (3.3)-(3.4) hold, where t^* is the limit of the sequence $\{t_n\}$ generated by

$$\begin{aligned} t_0 &= 0, t_1 = \eta, t_2 = t_1 + \psi_0\eta \text{ and for } n \geq 2, \\ t_{n+1} &= t_n + \frac{(t_n - t_{n-1})}{1 - l_0 - \lambda\omega_0(t_n) - (1-\lambda)\omega_0(0)}(H\omega(t_n - t_{n-1}) + \lambda\omega_2(t_{n-1}) \\ &\quad + (1-\lambda)(\omega(t_{n-1}) + \omega_2(0)) + \omega_3(t_n - t_{n-1}) + l_2). \end{aligned}$$

(b) *If r^{**} is a positive number such that*

$$\frac{H\omega(r^{**}) + \lambda\omega_2(r^*) + (1-\lambda)(\omega(r^*) + \omega(0)) + \omega_3(r^{**}) + l_2}{1 - l_0 - \lambda\omega_0(r^*) - (1-\lambda)\omega_0(0)} < 1,$$

then the solution x^* of (1.6) is unique in $B_{r^{**}}[x_0] \cap D_0$.

We now have the following semilocal convergence analysis of (1.7) which follows from Corollary 3.2 for $\lambda = 1$.

Corollary 3.3. *Let F and G be two operators defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Fréchet differentiable at each point of D. Further, let $A(x) \in B(X, Y)$ be an operator which approximates F'_x, for $x \in D$. For $x_0 \in D_0$, assume that $A(x_0)^{-1}$ exists. For $\omega, \omega_0, \omega_1, \omega_2, \omega_3 \in \Phi$, assume that $(\mathcal{C}2) - (\mathcal{C}6)$ with (3.2). Further, assume that $(\mathcal{C}'1)$ holds. Let $H = \int_0^1 h(t)dt$ and $H_1 = \int_0^1 h_1(t)dt$. Denote*

$$\psi_0 = \frac{H_1\omega_1(\eta) + \omega_2(0) + \omega_3(\eta) + l_1 + l_2}{1 - l_0 - \omega_0(\eta)}\eta, \psi(r) = \frac{H\omega(\eta) + \omega_2(r) + \omega_3(r) + l_2}{1 - l_0 - \omega_0(r)}$$

and assume that the scalar equation (2.1) has a minimum positive root r^* with

$$\psi(r^*) < 1, \psi_0 < 1 \text{ and } \omega_0(r^*) < 1 - l_0.$$

Assume that $B_{r_0}[x_0] \subseteq D_0$. Then, we have the following:

(a) *The sequence* $\{x_n\}$ *generated by (1.7) is well defined, remains in* $B_{r_0}[x_0]$ *and converges to a solution* $x^* \in B_{r_0}[x_0]$ *of the operator equation (1.6). Moreover, the error estimates (3.3)-(3.4) hold, where* t^* *is the limit of the sequence* $\{t_n\}$ *generated by*

$$\begin{cases} t_0 = 0, t_1 = \eta, t_2 = t_1 + \psi_0\eta, \\ t_{n+1} = t_n + \left(\frac{H\omega(t_n - t_{n-1}) + \omega_2(t_{n-1}) + \omega_3(t_n - t_{n-1}) + l_2}{1 - l_0 - \omega_0(t_n)}\right)(t_n - t_{n-1}), \quad n \geq 2. \end{cases}$$

(b) *If* r^{**} *is a positive number such that*

$$\frac{H\omega(r^{**}) + \omega_2(r^*) + \omega_3(r^{**}) + l_2}{1 - l_0 - \omega_0(r^*)} < 1,$$

then the solution x^* *of (1.6) is unique in* $B_{r^{**}}[x_0] \cap D_0$.

We now present an example to show the effectiveness and convergence of the sequence generated by the proposed iterative scheme.

Example 3.4. *Consider the operator equation*

$$30x + x^2 + x^3 + |x| = 0. \tag{3.10}$$

Let $X = Y = \mathbb{R}, D = (-1.1, 1.1)$ *and define functions* $F, G : D \to Y$ *by*

$$F(x) = 30x + x^2 + x^3, G(x) = |x|.$$

Let $x_0 = \frac{1}{20}$ *and* $\lambda = \frac{1}{2}$. *Let* $\{z_n\}$ *be a sequence in* D *defined by*

$$z_n = \frac{1}{n + 20}, n \in \mathbb{N}_0.$$

Observe that

$$\|z_n - x_0\| = \left|\frac{1}{n + 20} - \frac{1}{20}\right| \leq \frac{n}{20(n + 20)} = \delta_n < \frac{1}{20} = \delta.$$

Define $A(x)$ *by*

$$A(x) = 30 + 2x, x \in D.$$

Let $a = 0.29, b = 0.11, c = 0.1, d = 0.1$. *Since*

$$A(x_0) = A(z_0) = 30 + \frac{1}{20} = \frac{301}{10} \neq 0,$$

it follows that $A(x_0)^{-1}$ *exists and* $\|A(x_0)^{-1}\| = \frac{10}{301}$. *Note that* $(\lambda A(x_0) + (1 - \lambda)A(z_0))^{-1} = \frac{10}{301}$. *Now, we have*

$$\|(\lambda A(x_0) + (1 - \lambda)A(z_0))^{-1}(F(x_0) + G(x_0))\| \approx 0.05158 < 0.06 = \eta.$$

Let $x, y \in D$,*Then we have*

$$\|A(x_0)^{-1}(F'_x - F'_y)\| = \|A(x_0)^{-1}\|(2 + 3|x + y|)\|x - y\| < a\|x - y\| = \omega(\|x - y\|),$$

where $\omega(t) = at, t \geq 0$. *Let* $\omega_1(t) = \omega(t), l_1 = 0$. *Clearly,* $\omega, \omega_1 \in \Phi$. *Next, we have*

$$\begin{aligned} \|A(x_0)^{-1}(F'_x - A(x))\| &\leq \|A(x_0)^{-1}\|\|F'_x - A(x)\| \\ &\leq \frac{30\|x\|}{301}\|x - x_0\| + \frac{30\|x\|\|x_0\|}{301} \\ &< b\|x - x_0\| + 0.006 \end{aligned}$$

$$= \omega_2(\|x - x_0\|) + l_2,$$

where $\omega_2(t) = bt, l_2 = 0.006$. *Finally, we have*

$$\begin{aligned} \|A(x_0)^{-1}(A(x) - A(x_0))\| &= 2\|A(x_0)^{-1}\|\|x - x_0\| \\ &< c\|x - x_0\| \\ &= \omega_0(\|x - x_0\|) + l_0, \end{aligned}$$

where $\omega_0(t) = ct, l_0 = 0$. *Now, we have*

$$\begin{aligned} \|A(x_0)^{-1}(G(x) - G(y))\| &= \frac{10}{301}\|x - y\| \approx 0.03322\|x - y\| < d\|x - y\| \\ &= \omega_3(\|x - y\|)\|x - y\|, \end{aligned}$$

where $\omega_3(t) = d, \forall t \geq 0$. *Note the scalar equation (2.1) reduces to the quadratic polynomial*

$$Ar^2 + Br + C = 0, \tag{3.11}$$

where $A = a+b+c, B = \delta a+(\delta-\eta)b+(\delta-\eta-\psi_0\eta)c+2d+2l_2-2$ *and* $C = -\eta(\eta+\delta)a-\delta\eta b-\delta\eta(1+\psi_0)c-2d\eta-2l_2\eta+2\psi_0\eta+2\eta$. *Since* $B^2-4AC \approx 1.536416619182858 > 0$, *the equation (3.11) has real and distinct roots, which are* $\frac{-B\pm\sqrt{B^2-4AC}}{2A}$. *For* $r^* = \frac{-B-\sqrt{B^2-4AC}}{2A} \approx 0.087920205078916$, *one can see that* r^* *is the minimum positive zero of (3.11),* $\omega_0(r^*) + \omega_0(\delta) \approx 0.018792020507892 < 2 - l_0, \psi(r^*) \approx 0.344194808039604 < 1$ *and* $B_{r^*}[x_0] = (x_0-r^*, x_0+r^*) \subseteq (-1.1, 1.1) = D$. *Further, let* r^{**} *be any number with* $0 < r^{**} < \frac{2-(\delta+r^*)(a+b+c)-2d-2l_2}{a} \approx 4.741516887794973$. *Then,* $\theta < 1$. *Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, the sequence generated by (3.1) is well defined, remains in* $(x_0 - r^*, x_0 + r^*)$ *and converges to a solution* $x^* \in (x_0 - r^*, x_0 + r^*)$ *of (3.10) with the error estimates (3.3)-(3.5). Further, the solution* x^* *of (1.6) is unique in* $B_{r^{**}}[x_0] \cap D$.

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Manuscript received September 1, 2014
revised October 9, 2014

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