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AN ULM-LIKE METHOD FOR SOLVING NONLINEAR OPERATOR EQUATIONS

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Abstract. In this paper, an Ulm-like method is proposed for solving nonlinear operator equations. This method has an advantage over other known methods since it avoids computing Jacobian matrices and solving Jacobian equations. Under some mild conditions, we prove that this Ulm-like method converges locally to the solution with R-convergence rate 2. Moreover, numerical tests are given in the last section demonstrating the effectiveness of this Ulm-like method.

1. Introduction

Let X and Y be Banach spaces, $D \subseteq X$ be an open subset and let $f : D \subseteq X \to Y$ be a nonlinear operator with the continuous Fréchet derivative denoted by f'. Finding solutions of the nonlinear operator equation

$$f(x) = 0 \tag{1.1}$$

in Banach spaces is a very general subject which is widely considered in both theoretical and applied areas of mathematics. Except special cases, the most commonly used solution methods are iterative when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. The most popular iterative process for solving (1.1) is undoubtedly Newton's method which takes the form $x_{k+1} - x_k = -f'(x_k)^{-1}f(x_k)$. One of the famous results on Newton's method is the well-known Kantorovich theorem (cf.[10]) which guarantees convergence of Newton's sequence to a solution under very mild conditions. Since Newton's method has a sound theoretical basis for many problems and its convergence is rapid, a large number of works in the literature have studied the convergence property of Newton's method (cf.[5, 10, 11]). For recent progress on Newton's method the reader is referred to [2, 4, 20, 21, 22].

However, there are many problems for which Newton's method is not applicable in its original form. A case of interest occurs when the derivative is not continuously invertible, as for instance, dealing with problems involving small divisors, or other important examples [1, 8, 9, 12, 14]. To avoid this problem, Newton-type methods: $x_{k+1} = x_k - H_k f(x_k)$, where H_k is an approximation of $f'(x_k)^{-1}$ are considered. One of these methods was proposed by Moser in [13]. Given $x_0 \in D$ and $B_0 \in \mathcal{L}(Y, X)$, Moser's method is defined as follows

$$\begin{cases} x_{k+1} = x_k - B_k f(x_k) \\ B_{k+1} = 2B_k - B_k f'(x_k) B_k \end{cases} \quad \text{for each } k = 0,\ 1,\ \ldots. \tag{1.2}$$

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The first equation in (1.2) is similar to Newton's method, but replacing the operator $f'(x_k)^{-1}$ by a linear operator B_k. The second equation in (1.2) is Newton's method applied to equation $g_k(B) = 0$ where $g_k : \mathcal{L}(Y, X) \to \mathcal{L}(X, Y)$ is defined by $g_k(B) = B^{-1} - f'(x_k)$. So $\{B_k\}$ gives us an approximation of $\{f'(x_k)^{-1}\}$. Moser's method was developed as a technical tool for investigating the stability of the N-body problem in celestial mechanics. The main difficulty in this, and similar problems involving small divisors, is the solution of a system of nonlinear partial differential equations.

The convergence rate of Moser's method was showed to be $(1 + \sqrt{5})/2$, provided the root of (1.1) is simple [13]. This is unsatisfactory from a numerical point of view because the scheme uses the same amount of information per step as Newton's method, yet, it converges no faster than the secant method. However, quadratic convergence rate can be obtained when the sequence $\{B_k\}$ is generated by

$$B_{k+1} = 2B_k - B_k f'(x_{k+1}) B_k \quad \text{for each } k = 0,\ 1,\ \ldots.$$

This is Ulm's method introduced in [18] and has been further studied in [3, 6, 7, 8, 15, 23]. Notice that, here $f'(x_{k+1})$ appears instead of $f'(x_k)$ in (1.2). This is crucial for obtaining fast convergence. Under the classical assumption that the derivative f' is Lipschitz continuous around the solution, Ulm showed that the method generates successive approximations that converge to a solution of (1.1), asymptotically as fast as Newton's method. Recently, some authors have employed Ulm's method to solve inverse eigenvalue problems and inverse singular value problems [16, 17, 19]. There they found that computing exactly the derivative $f'(x_k)$ at each iteration is costly especially in the case when the system is large.

The purpose of the present paper is, motivated by Ulm's method, to propose a Ulm-like method for solving the nonlinear operator equation $f(x) = 0$. Given $x_0 \in D$ and $B_0 \in \mathcal{L}(Y, X)$, the Ulm-like method is defined by

$$\begin{cases} x_{k+1} = x_k - B_k f(x_k) \\ B_{k+1} = 2B_k - B_k A_{k+1} B_k \end{cases} \qquad \text{for each } k = 0,\ 1,\ ,\ldots, \tag{1.3}$$

where A_{k+1} is an approximation of the derivative $f'(x_{k+1})$. This method exhibits several attractive features. First, it is inverse free: we do not need to solve a linear equation at each iteration. Second, it is derivative free: we do not need to compute the Frécher derivative at each iteration. Third, in addition to solve the nonlinear equation (1.1), the method produces successive approximations $\{B_k\}$ to the value of $f'(x^*)^{-1}$, being x^* a solution of (1.1). This property is very helpful especially when one investigates the sensitivity of the solution to small perturbations. Furthermore, under certain assumptions, the radius of the convergence ball for the Ulm-like method is estimated, and the quadratic convergence property is proved. Numerical experiment is given in the last section illustrating the convergence performance of the Ulm-like method.

2. Convergence analysis

Let X and Y be Banach spaces. Let $\mathbf{B}(x,\ R)$ stands for the open ball in X with center x and radius $R > 0$. Let D be an open subset of X and let $f : D \subseteq X \to Y$ be a nonlinear operator with the continuous Fréchet derivative denoted by f'. Let $x^* \in D$ be a solution of the nonlinear equation $f(x) = 0$. Throughout the whole paper we shall always assume that the inverse $f'(x^*)^{-1}$ exists and that f' satisfies Lipschitz condition on $\mathbf{B}(x^*,\ R)$ with the Lipschitz constant L:

$$\|f'(x) - f'(y)\| \leq L\|x - y\| \quad \text{for each } x,\ y \in \mathbf{B}(x^*, R). \tag{2.1}$$

Let $\{x_k\}$ be generated by the Ulm-like method. Let A_k be an approximation to $f'(x_k)$ such that

$$\|A_k - f'(x_k)\| \leq \eta_k \|f(x_k)\| \quad \text{for each } k = 0,\ 1,\ \ldots. \tag{2.2}$$

Here $\{\eta_k\}$ is a nonnegative-valued sequence satisfying $\sup\limits_{k\geq 0} \eta_k \leq \eta$ where η is a nonnegative constant. Let

$$0 < R_L < \min\left\{1,\ R,\ \frac{1}{\left(\frac{L}{2}\eta + L + \eta\|f'(x^*)\|\right)\|f'(x^*)^{-1}\|}\right\}. \tag{2.3}$$

Then we have the following lemma which is crucial for the proof of the main theorem.

Lemma 2.1. *If* $x_k \in \mathbf{B}(x^*, R_L)$. *Then the following assertions hold.*

(i) $\|A_k - f'(x_k)\| \leq \eta \left(\frac{L}{2} + \|f'(x^*)\|\right) \|x_k - x^*\|$.

(ii) A_k *is invertible and moreover*

$$\|A_k^{-1}\| \leq \frac{\|f'(x^*)^{-1}\|}{1 - \left(\frac{L}{2}\eta + L + \eta\|f'(x^*)\|\right)\|f'(x^*)^{-1}\| R_L}.$$

Proof. Suppose that $x_k \in \mathbf{B}(x^*, R_L)$. For the proof of assertion (i), let us write $x_k^\theta = x^* + \theta(x_k - x^*)$ where $0 \leq \theta \leq 1$. Then, noting that $f(x^*) = 0$, we can write

$$f(x_k) = f(x_k) - f(x^*) = \int_0^1 [f'(x_k^\theta) - f'(x^*)]d\theta(x_k - x^*) + f'(x^*)(x_k - x^*).$$

Combining this with (2.1), we obtain that

$$\begin{aligned}
\|f(x_k)\| &\leq \int_0^1 \|f'(x_k^\theta) - f'(x^*)\|d\theta\|x_k - x^*\| + \|f'(x^*)\| \cdot \|x_k - x^*\| \\
&\leq \int_0^1 L\theta\|x_k - x^*\|^2 d\theta + \|f'(x^*)\| \cdot \|x_k - x^*\| \\
&= \frac{L}{2}\|x_k - x^*\|^2 + \|f'(x^*)\| \cdot \|x_k - x^*\|.
\end{aligned}$$

Thus, by (2.2) and the fact that $\|x_k - x^*\| < R_L < 1$, one has

$$\|A_k - f'(x_k)\| \leq \eta_k\|f(x_k)\| \leq \eta\left(\frac{L}{2} + \|f'(x^*)\|\right)\|x_k - x^*\|.$$

That is to say assertion (i) holds and hence, by (2.1), we have

$$\|A_k - f'(x^*)\| \leq \|A_k - f'(x_k)\| + \|f'(x_k) - f'(x^*)\|$$

$$\leq \left(\frac{L}{2}\eta + L + \eta\|f'(x^*)\|\right)\|x_k - x^*\|.$$

It follows from (2.3) and the assumption that

$$\|f'(x^*)^{-1}\| \cdot \|A_k - f'(x^*)\| \leq \left(\frac{L}{2}\eta + L + \eta\|f'(x^*)\|\right)\|f'(x^*)^{-1}\|R_L < 1.$$

Consequently, using Banach's lemma, we can derive that A_k is invertible and moreover

$$\|A_k^{-1}\| \leq \frac{\|f'(x^*)^{-1}\|}{1 - \left(\frac{L}{2}\eta + L + \eta\|f'(x^*)\|\right)\|f'(x^*)^{-1}\|R_L}.$$

This proves assertion (ii) and so the whole lemma. □

Note that in the Ulm-like method, sequence $\{B_k\}$ is generated by the algorithm except for B_0. Below, we prove that if B_0 approximates A_0^{-1}, then the sequence $\{x_k\}$ generated by the Ulm-like method converges locally to x^* with R-convergence rate 2. For this end, let B_0 satisfy that

$$\|I - B_0A_0\| \leq \mu, \tag{2.4}$$

where μ is a positive constant.

Theorem 2.1. *Suppose that the Jacobian matrix $f'(x^*)$ is invertible and that f' satisfies the Lipschitz condition* (2.1) *on $\mathbf{B}(x^*, R_L)$. Suppose also that* (2.2) *holds. Then there exist positive numbers δ and μ such that for any $x_0 \in \mathbf{B}(x^*, \delta)$ and B_0 satisfying* (2.4)*, the sequence $\{x_k\}$ generated by the Ulm-like method with initial point x_0 converges to x^*. Moreover, the following estimates hold for each $k = 0, 1, \ldots$.*

$$\|x_k - x^*\| \leq \tau\left(\frac{\delta}{\tau}\right)^{2^k} \tag{2.5}$$

and

$$\|I - B_kA_k\| \leq \frac{1}{3}\left(\frac{\delta}{\tau}\right)^{2^k}. \tag{2.6}$$

Here τ is a positive constant.

Proof. We write for simplicity,

$$\rho = \frac{\|f'(x^*)^{-1}\|}{1 - \left(\frac{L}{2}\eta + L + \eta\|f'(x^*)\|\right)\|f'(x^*)^{-1}\|R_L}.$$

Set

$$\tau = \frac{1}{6\rho(L\eta + 2\eta\|f'(x^*)\| + 2L)}. \tag{2.7}$$

Take δ and μ such that

$$0 < \delta < \min\{R_L,\ \tau\} \quad \text{and} \quad 0 < \mu \leq 2\rho\delta(L\eta + 2\eta\|f'(x^*)\| + 2L). \tag{2.8}$$

We shall show that τ, δ, and μ are as desired. Let $x_0 \in B(x^*, \delta)$ and B_0 satisfy (2.4). It suffices to verify that (2.5)-(2.6) hold for each $k = 0, 1, \ldots$. We proceed by mathematical induction. Clearly, (2.5) is trivial for $k = 0$ by the assumption. By (2.4), (2.7), and (2.8), we have

$$\|I - B_0 A_0\| \le \mu \le 2\rho\delta(L\eta + 2\eta\|f'(x^*)\| + 2L) = \frac{\delta}{3\tau}.$$

That is, estimate (2.6) holds for $k = 0$. Now assume that (2.5)-(2.6) hold for $k = m$. Then, one has

$$\|x_m - x^*\| \le \tau \left(\frac{\delta}{\tau}\right)^{2^m} \tag{2.9}$$

and

$$\|I - B_m A_m\| \le \frac{1}{3}\left(\frac{\delta}{\tau}\right)^{2^m}. \tag{2.10}$$

It follows from (2.8) that

$$\|x_m - x^*\| \le \tau \left(\frac{\delta}{\tau}\right)^{2^m} < \delta < R_L.$$

Thus, Lemma 2.1 is applicable to concluding that

$$\|A_m - f'(x_m)\| \le \eta\left(\frac{L}{2} + \|f'(x^*)\|\right)\|x_m - x^*\| \le \eta\left(\frac{L}{2} + \|f'(x^*)\|\right)\tau\left(\frac{\delta}{\tau}\right)^{2^m} \tag{2.11}$$

and that

$$\|A_m^{-1}\| \le \frac{\|f'(x^*)^{-1}\|}{1 - \left(\frac{L}{2}\eta + L + \eta\|f'(x^*)\|\right)\|f'(x^*)^{-1}\|R_L} = \rho, \tag{2.12}$$

where the equality holds because of the definition of ρ. Since $\delta < \tau$, we derive from (2.10) and (2.12) that

$$\begin{aligned} \|B_m\| &\le \|B_m A_m\| \cdot \|A_m^{-1}\| \le (1 + \|I - B_m A_m\|) \cdot \|A_m^{-1}\| \\ &\le \rho\left[1 + \frac{1}{3}\left(\frac{\delta}{\tau}\right)^{2^m}\right] \\ &\le \sqrt{2}\rho. \end{aligned} \tag{2.13}$$

Note by (1.3) that

$$\begin{aligned} x_{m+1} - x^* &= x_m - x^* - B_m(f(x_m) - f(x^*)) \\ &= x_m - x^* - \int_0^1 B_m f'(x_m^\theta)(x_m - x^*)d\theta \\ &= \int_0^1 [I - B_m f'(x_m) + B_m(f'(x_m) - f'(x_m^\theta))](x_m - x^*)d\theta, \end{aligned}$$

where $x_m^\theta = x^* + \theta(x_m - x^*)$ for each $0 \le \theta \le 1$. Since $\|x_m - x^*\| \le R_L$ and $\|x_m^\theta - x^*\| = \theta\|x_m - x^*\| \le \|x_m - x^*\| \le R_L$, it follows from the Lipschitz condition that

$$
\begin{aligned}
\|x_{m+1}-x^*\| &\leq \int_0^1 (\|I-B_m f'(x_m)\| \\
&\quad +L(1-\theta)\|B_m\|\cdot\|x_m-x^*\|)\cdot\|x_m-x^*\|d\theta \\
&= \|I-B_m f'(x_m)\|\cdot\|x_m-x^*\| + \frac{L}{2}\|B_m\|\cdot\|x_m-x^*\|^2.
\end{aligned} \tag{2.14}
$$

In addition, by (2.10), (2.11), and (2.13), we have

$$
\begin{aligned}
\|I-B_m f'(x_m)\| &\leq \|I-B_m A_m\| + \|B_m\|\cdot\|A_m - f'(x_m)\| \\
&\leq \frac{1}{3}\left(\frac{\delta}{\tau}\right)^{2^m} + \sqrt{2}\rho\eta\left(\frac{L}{2}+\|f'(x^*)\|\right)\tau\left(\frac{\delta}{\tau}\right)^{2^m}.
\end{aligned}
$$

This together with (2.13)-(2.14) as well as (2.9) gives that

$$
\begin{aligned}
\|x_{m+1}-x^*\| &\leq \left[\frac{1}{3}\left(\frac{\delta}{\tau}\right)^{2^m} + \sqrt{2}\rho\eta\left(\frac{L}{2}+\|f'(x^*)\|\right)\tau\left(\frac{\delta}{\tau}\right)^{2^m}\right]\tau\left(\frac{\delta}{\tau}\right)^{2^m} \\
&\quad + \frac{\sqrt{2}}{2}L\rho\tau^2\left(\frac{\delta}{\tau}\right)^{2^{m+1}} \\
&= \left[\frac{1}{3}+\sqrt{2}\rho\tau\left(\frac{L}{2}\eta+\eta\|f'(x^*)\|+\frac{L}{2}\right)\right]\tau\left(\frac{\delta}{\tau}\right)^{2^{m+1}}.
\end{aligned}
$$

Thus, thanks to the definitions of δ and τ, we can derive

$$
\|x_{m+1}-x^*\| \leq \tau\left(\frac{\delta}{\tau}\right)^{2^{m+1}} < \delta < R_L.
$$

Consequently, (2.5) holds for $k=m+1$ and hence, by Lemma 2.1(i),

$$
\|A_{m+1}-f'(x_{m+1})\| \leq \eta\left(\frac{L}{2}+\|f'(x^*)\|\right)\tau\left(\frac{\delta}{\tau}\right)^{2^{m+1}}. \tag{2.15}
$$

Below, we verify (2.6) holds for $k=m+1$. We shall note by the Lipschitz condition that

$$
\begin{aligned}
\|A_{m+1}-A_m\| &\leq \|A_{m+1}-f'(x_{m+1})\| + \|f'(x_{m+1})-f'(x_m)\| + \|A_m - f'(x_m)\| \\
&\leq \|A_{m+1}-f'(x_{m+1})\| + \|A_m - f'(x_m)\| + L\|x_{m+1}-x_m\| \\
&\leq \|A_{m+1}-f'(x_{m+1})\| + \|A_m - f'(x_m)\| \\
&\quad +L(\|x_{m+1}-x^*\|+\|x_m-x^*\|).
\end{aligned}
$$

Then, using (2.11), (2.15) and (2.5)(with $k=m,\ m+1$), we get that

$$
\begin{aligned}
\|A_{m+1}-A_m\| &\leq \left(\frac{\eta}{2}L+\eta\|f'(x^*)\|+L\right)\left[\left(\frac{\delta}{\tau}\right)^{2^m}+1\right]\tau\left(\frac{\delta}{\tau}\right)^{2^m} \\
&\leq (\eta L + 2\eta\|f'(x^*)\|+2L)\tau\left(\frac{\delta}{\tau}\right)^{2^m},
\end{aligned}
$$

where the last inequality holds because that $\delta<\tau$. Moreover, by (1.3),

$$
I-B_{m+1}A_{m+1} = I-(2B_m - B_m A_{m+1} B_m)A_{m+1} = (I-B_m A_{m+1})^2.
$$

Thus, it follows from (2.6)(with $k = m$), (2.13) as well as the definition of τ that

$$\begin{aligned} \|I - B_{m+1}A_{m+1}\| &\leq (\|I - B_mA_m\| + \|B_m\| \cdot \|A_{m+1} - A_m\|)^2 \\ &\leq 2\|I - B_mA_m\|^2 + 2\|B_m\|^2 \cdot \|A_{m+1} - A_m\|^2 \\ &\leq \frac{2}{9}\left(\frac{\delta}{\tau}\right)^{2^{m+1}} + 4\rho^2(\eta L + 2\eta\|f'(x^*)\| + 2L)^2 \cdot \tau^2 \left(\frac{\delta}{\tau}\right)^{2^{m+1}} \\ &= \frac{1}{3}\left(\frac{\delta}{\tau}\right)^{2^{m+1}}. \end{aligned}$$

This conforms that (2.6) holds for $k = m + 1$ and the proof is complete. □

3. A numerical example

Consider the two-point boundary value problem

$$\begin{cases} x'' + x^2 = 0, \\ x(0) = x(1) = 0. \end{cases} \tag{3.1}$$

We divide the interval $[0, 1]$ into $m + 1$ subintervals and we get $h = \frac{1}{m+1}$. Let $d_0, d_1, \ldots, d_{m+1}$ be the points of subdivision with $0 = d_0 < d_1 < \cdots < d_{m+1} = 1$. An approximation for the second derivative may be chosen as

$$\begin{cases} x_i'' = \dfrac{x_{i-1} - 2x_i + x_{i+1}}{h^2}, & x_i = x(d_i) \text{ for each } i = 1,\ 2,\ \ldots,\ m. \\ x_0 = x_1 = 0, \end{cases}$$

Let the operator $\phi : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$\phi(x) = (x_1^2, x_2^2, \ldots, x_m^2)^T \quad \text{for each } x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m.$$

To get an approximation to the solution of (3.1), we need to solve the following nonlinear equation:

$$f(x) := Mx + h^2\phi(x) = 0 \quad \text{for each } x \in \mathbb{R}^m, \tag{3.2}$$

where

$$M = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}_{m\times m}.$$

Obviously, $x^* = \mathbf{0}$ is a solution of (3.2) and

$$f'(x) = M + 2h^2 \text{diag}(x_1,\ x_2,\ \ldots,\ x_m).$$

Hence $f'(x^*) = M$. Furthermore, it is easy to verify that

$$\|f'(x) - f'(y)\| \leq 2h^2\|x - y\| \quad \text{for each } x, y \in \mathbb{R}^m$$

where $\|\cdot\|$ denotes the ∞-norm. Thus thanks to the results from Section 2, there exists a radius δ such that for each $x_0 \in \mathbf{B}(x^*, \delta)$, the sequence $\{x_k\}$ generated by the Ulm-like method converges to $x^* = 0$ with convergence order 2. For different choices of η_k, the convergence performance of the algorithm are illustrated in the following tables. Here we take $m = 4$ and $x_0 = (0.2, 0.2, 0.2, 0.2)^T$ in Table 1, while in Table 2, $m = 19$ and $x_0 = (0.01, 0.01, \ldots, 0.01)^T$.

TABLE 1. Values of $\alpha_k := \|x_k - x^*\|_\infty$ for different η_k ($m = 4$)

k	Ulm's method	Ulm-like method	
		$\eta_k \equiv \frac{1}{20}$	$\eta_k \equiv \frac{1}{10}$
0	$4.0e-1$	$4.0e-1$	$4.0e-1$
1	$8.51e-3$	$8.51e-3$	$8.51e-3$
2	$3.76e-4$	$6.49e-4$	$7.17e-4$
3	$1.12e-6$	$4.20e-6$	$5.49e-6$
4	$3.74e-11$	$2.36e-10$	$3.10e-10$
5	0.00	0.00	0.00

TABLE 2. Values of $\alpha_k := \|x_k - x^*\|_\infty$ for different η_k ($m = 19$)

k	Ulm's method	Ulm-like method	
		$\eta_k \equiv \frac{1}{30}$	$\eta_k \equiv \frac{1}{20}$
0	$4.36e-2$	$4.36e-2$	$4.36e-2$
1	$4.09e-5$	$4.09e-5$	$4.09e-5$
2	$8.32e-8$	$6.73e-7$	$1.03e-6$
3	$5.29e-13$	$1.83e-10$	$6.47e-10$
4	0.00	0.00	0.00

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