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THE SPLIT FEASIBILITY PROBLEM AND THE SHRINKING PROJECTION METHOD IN BANACH SPACES

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Abstract. In this paper, we consider the split feasibility problem in Banach spaces. Using the shrinking projection method, we prove two strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. It seems that such theorems are first in Banach spaces.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [6] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [2], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem; see, for instance, [2, 5, 7, 10, 19].

On the other hand, in 2003, Nakajo and Takahashi [11] proved the following strong convergence theorem by using the hybrid method in mathematical programming. Let C be a nonempty, closed and convex subset of H. For a mapping $T : C \to C$, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

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where $P_{C_n \cap Q_n}$ is the metric projection from H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset [0,1]$ is chosen so that $0 \leq \limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Takahashi, Takeuchi and Kubota [18] also obtained the following result by using the shrinking projection method:

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x \in H$. For $C_1 = C$ and $x_1 \in C$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x,, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$.

In this paper, motivated by the split feasibility problem in Hilbert spaces, we consider the problem in Banach spaces. Using the shrinking projection method, we prove two strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. It seems that such theorems are first in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot\, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [16] that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v \rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [14].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}$$

for all ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.5) is attained uniformly for $y \in U$. It is known that if the norm of E is Fréchet differentiable, then J is norm to norm continuous. For more details, see [14] and [15]. We know the following result.

Lemma 2.1 ([14]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call a mapping P_C the metric projection of E onto C.

Lemma 2.2 ([14]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \ge 0, \quad \forall y \in C$.

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2; \tag{2.6}$$

$$\phi(x,y)=\phi(x,z)+\phi(z,y)+2\langle x-z, Jz-Jy\rangle; \tag{2.7}$$

$$2\langle x-y, Jz-Jw\rangle=\phi(x,w)+\phi(y,z)-\phi(x,z)-\phi(y,w). \tag{2.8}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x,y)=0 \quad \text{if and only if} \quad x=y. \tag{2.9}$$

The following lemma was proved by Kamimura and Takahashi [8].

Lemma 2.3 ([8]). *Let E be a uniformly convex Banach space and let $r>0$. Then there exists a strictly increasing, continuous, and convex function $g:[0,2r]\to[0,\infty)$ such that $g(0)=0$ and*

$$g(\|x-y\|)\le\phi(x,y)$$

for all $x,y\in B_r$, where $B_r=\{z\in E:\|z\|\le r\}$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Then we know that for any $x\in E$, there exists a unique element $z\in C$ such that $\phi(z,x)\le\phi(y,x)$ for all $y\in C$. Putting $z=Q_Cx$, we call a mapping Q_C the generalized projection of E onto C; see [1] and [8].

Lemma 2.4 ([1], [8]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1\in E$ and $z\in C$. Then, the following conditions are equivalent:*

(1) $z=Q_Cx_1$;
(2) $\langle z-y, Jx_1-Jz\rangle\ge 0, \quad \forall y\in C$.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x\in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\}\subset E$ such that $\{x_n\}$ converges strongly to x and $x_n\in C_n$ for all $n\in\mathbb{N}$. Similarly, $y\in\text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\}\subset E$ such that $\{y_i\}$ converges weakly to y and $y_i\in C_{n_i}$ for all $i\in\mathbb{N}$. If C_0 satisfies

$$C_0=\operatorname*{s-Li}_n C_n=\operatorname*{w-Ls}_n C_n, \tag{2.10}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [9] and we write $C_0=\text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty}C_n$ in the sense of Mosco. For more details, see [9]. The following lemma was proved by Tsukada [20].

Lemma 2.5 ([20]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0=M\text{-lim}_{n\to\infty}C_n$ exists and nonempty, then for each $x\in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3. Main results

In this section, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [18], we first prove a strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces. Before proving the theorem, we need the following result and lemma.

Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and let P_C be the metric projection of E onto C. Using Lemma 2.3, we can prove that P_C is continuous. In fact, let $x_n \to x_0$. Since P_C is the metric projection of E onto C, we have from Lemma 2.2 that

$$\langle P_C x_n - y, J(x_n - P_C x_n)\rangle \geq 0, \quad \forall y \in C.$$

Then we have $\langle P_C x_n - x_n + x_n - y, J(x_n - P_C x_n)\rangle \geq 0$ and hence

$$\begin{aligned}\|x_n - y\|\|x_n - P_C x_n\| &\geq \langle x_n - y, J(x_n - P_C x_n)\rangle \\ &\geq \langle x_n - P_C x_n, J(x_n - P_C x_n)\rangle \\ &= \|x_n - P_C x_n\|^2.\end{aligned}$$

This means that $\{x_n - P_C x_n\}$ is bounded. Furthermore, since P_C is the metric projection of E onto C, we have that $\langle P_C x_n - P_C x_0, J(x_n - P_C x_n)\rangle \geq 0$ and

$$\langle P_C x_0 - P_C x_n, J(x_0 - P_C x_0)\rangle \geq 0.$$

Then we have

$$\langle P_C x_n - P_C x_0, J(x_n - P_C x_n) - J(x_0 - P_C x_0)\rangle \geq 0.$$

Using (2.8) and Lemma 2.3, we have that

$$\begin{aligned}2\langle x_n - x_0, &J(x_n - P_C x_n) - J(x_0 - P_C x_0)\rangle \\ &\geq 2\langle x_n - P_C x_n - (x_0 - P_C x_0), J(x_n - P_C x_n) - J(x_0 - P_C x_0)\rangle \\ &= \phi(x_n - P_C x_n, x_0 - P_C x_0) + \phi(x_0 - P_C x_0, x_n - P_C x_n) \\ &\geq g(\|x_n - P_C x_n - (x_0 - P_C x_0)\|) + g(\|x_0 - P_C x_0 - (x_n - P_C x_n)\|) \\ &= 2g(\|x_n - P_C x_n - (x_0 - P_C x_0)\|),\end{aligned}$$

where g is a strictly increasing, continuous, and convex function in Lemma 2.3. Therefore, if $x_n \to x_0$, then $P_C x_n \to P_C x_0$. Therefore, P_C is continuous.

Lemma 3.1. *Let E and F be strictly convex, reflexive and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and F, respectively. Let P_C and P_D be the metric projections of E onto C and F onto D, respectively and let Q_C and Q_D be the generalized projections of E onto C and F onto D, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $r > 0$ and $z \in E$. Then the following are equivalent:*

(i) $z = P_C\big(z - rJ_E^{-1}A^*J_F(Az - P_D Az)\big)$;
(ii) $z = Q_C J_E^{-1}\big(J_E z - rA^*(J_F Az - J_F Q_D Az)\big)$;
(iii) $z \in C \cap A^{-1}D$.

Proof. The proof of (i) $\Longleftrightarrow$ (iii) is in [17].

(ii) $\Rightarrow$ (iii). Since $C \cap A^{-1}D \neq \emptyset$, there exists $z_0 \in C \cap A^{-1}D$, i.e., $z_0 \in C$ and $Az_0 \in D$. Assuming $z = Q_C J_E^{-1}\big(J_E z - rA^*(J_F Az - J_F Q_D Az)\big)$, we have from the properties of Q_C that

$$\langle z - y, J_E J_E^{-1}\big(J_E z - rA^*(J_F Az - J_F Q_D Az)\big) - J_E z\rangle \geq 0, \quad \forall y \in C.$$

This implies that

$$\langle z - y, J_E z - rA^*(J_F Az - J_F Q_D Az) - J_E z\rangle \geq 0.$$

Thus we have that

$$\langle z - y, -rA^*(J_F Az - J_F Q_D Az)\rangle \geq 0$$

and hence

$$\langle z - y, A^*(J_F Az - J_F Q_D Az)\rangle \leq 0.$$

Since A^* is the adjoint operator, we have that

$$\langle Az - Ay, J_F Az - J_F Q_D Az\rangle \leq 0.$$

From $z_0 \in C$ we have that

$$\langle Az - Az_0, J_F Az - J_F Q_D Az\rangle \leq 0. \tag{3.1}$$

On the other hand, since Q_D is the generalized projection of F onto D, we have that

$$\langle Q_D Az - v, J_F Az - J_F Q_D Az\rangle \geq 0, \quad \forall v \in D.$$

From $Az_0 \in D$ we have that

$$\langle Q_D Az - Az_0, J_F Az - J_F Q_D Az\rangle \geq 0. \tag{3.2}$$

Using (3.1) and (3.2), we have that

$$\langle Az - Q_D Az, J_F Az - J_F Q_D Az\rangle \leq 0$$

and hence

$$\phi(Az, Q_D Az) + \phi(Q_D Az, Az) \leq 0.$$

This implies that $Az = Q_D Az$. Using this and

$$z = Q_C J_E^{-1}\big(J_E z - rA^*(J_F Az - J_F Q_D Az)\big),$$

we have that $z = Q_C z$. Therefore $z \in C \cap A^{-1}D$.

(iii) $\Rightarrow$ (ii). Since $z \in C \cap A^{-1}D$, we have that $Az \in D$ and $z \in C$. It follows that $Az = Q_D Az$ and $z = Q_C z$. Thus we have

$$Q_C J_E^{-1}\big(J_E z - rA^*(J_F Az - J_F Q_D Az)\big) = Q_C z = z.$$

The proof is complete. □

Theorem 3.2. *Let H be a Hilbert space and let F be a uniformly convex Banach space whose norm is Fréchet differentiable. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint* *operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$, $C_1 = H$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C\Big(x_n - rA^*J_F(Ax_n - P_DAx_n)\Big), \\ C_{n+1} = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}}u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|A\|^2 \leq 2$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D}u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined. Let $x_1 \in H$ and $z_n = P_C\Big(x_n - rA^*J_F(Ax_n - P_DAx_n)\Big)$ with $0 < r \leq \frac{2}{\|A\|^2}$. We have that for $z \in C \cap A^{-1}D$,

$$\begin{aligned} \|z_n - z\|^2 &= \|P_C\Big(x_n - rA^*J_F(Ax_n - P_DAx_n)\Big) - P_Cz\|^2 \\ &\leq \|x_n - rA^*J_F(Ax_n - P_DAx_n) - z\|^2 \\ &= \|x_n - z - rA^*J_F(Ax_n - P_DAx_n)\|^2 \\ &= \|x_n - z\|^2 - 2\langle x_n - z, rA^*J_F(Ax_n - P_DAx_n)\rangle \\ &\quad + \|rA^*J_F(Ax_n - P_DAx_n)\|^2 \\ &\leq \|x_n - z\|^2 - 2r\langle Ax_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\ &\quad + r^2\|A\|^2\|J_F(Ax_n - P_DAx_n)\|^2 \\ &= \|x_n - z\|^2 - 2r\langle Ax_n - P_DAx_n + P_DAx_n - Az, J_F(Ax_n - P_DAx_n)\rangle \quad (3.3) \\ &\quad + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 \\ &\leq \|x_n - z\|^2 - 2r\|Ax_n - P_DAx_n\|^2 \\ &\quad - 2r\langle P_DAx_n - Az, J_F(Ax_n - P_DAx_n)\rangle + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 \\ &\leq \|x_n - z\|^2 - 2r\|Ax_n - P_DAx_n\|^2 + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 \\ &\leq \|x_n - z\|^2 - 2r\|Ax_n - P_DAx_n\|^2 + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 \\ &= \|x_n - z\|^2 + r(r\|A\|^2 - 2)\|Ax_n - P_DAx_n\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned}$$

Therefore, $C \cap A^{-1}D \subset C_n$ for all $n \in \mathbb{N}$. Moreover, since

$$\begin{aligned} \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} &= \{z \in H : \|z_n - z\|^2 \leq \|x_n - z\|^2\} \\ &= \{z \in H : \|z_n\|^2 - \|x_n\|^2 \leq 2\,\langle z_n - x_n, z\rangle\}, \end{aligned}$$

it is closed and convex. Applying these facts inductively, we obtain that C_n are nonempty, closed, and convex for all $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then since $C_0 \supset C \cap A^{-1}D \neq \emptyset$, C_0 is also nonempty. Let $w_n = P_{C_n}u$ for every $n \in \mathbb{N}$. Then, by Lemma 2.5, we have $w_n \to z_0 = P_{C_0}u$. Since a metric projection on H is nonexpansive, it follows that

$$\|x_n - z_0\| \leq \|x_n - w_n\| + \|w_n - z_0\|$$

$$
\begin{aligned}
&= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - z_0\| \\
&\le \|u_n - u\| + \|w_n - z_0\|
\end{aligned}
$$

and hence $x_n \to z_0$.

Since $z_0 \in C_0 \subset C_{n+1}$, we have $\|z_n - z_0\| \le \|x_n - z_0\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $z_n \to z_0$. Since P_C, A, A^*, J_F and P_D are continuous, the mapping $P_C(I - rA^*J_F(A - P_DA))$ is continuous. Then we have that

$$
\begin{aligned}
&\|z_n - P_C(I - rA^*J_F(A - P_DA))z_0\| \\
&\quad = \|P_C(I - rA^*J_F(A - P_DA))x_n - P_C(I - rA^*J_F(A - P_DA))z_0\| \to 0.
\end{aligned}
$$

Hence we have that

$$
\begin{aligned}
&\|z_0 - P_C(I - rA^*J_F(A - P_DA))z_0\| \\
&\quad \le \|z_0 - z_n\| + \|z_n - P_C(I - rA^*J_F(A - P_DA))z_0\| \\
&\quad \to 0.
\end{aligned}
$$

This implies $z_0 \in C \cap A^{-1}D$ by Lemma 3.1. Since $P_{C_0}u = z_0 \in C \cap A^{-1}D$ and $C \cap A^{-1}D \subset C_0$, we have $z_0 = P_{C\cap A^{-1}D}u$, which completes the proof. □

We do not know whether a Hilbert space H in Theorem 3.2 is replaced by a Banach space E or not and whether the metric projections in Theorem 3.2 are replaced by the generalized projections or not. Furthermore, we do not know whether such a theorem (Theorem 3.2) holds or not for the hybrid method of Nakajo and Takahashi (Theorem 1.1).

Next, using the shrinking projection method, we prove another strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces.

Theorem 3.3. *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and F, respectively. Let P_D be the metric projection of F onto D. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $x_1 \in E$ and let $C_1 = C$. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
z_n = x_n - rJ_E^{-1}A^*J_F(Ax_n - P_DAx_n), \\
C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \ge 0\}, \\
x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N},
\end{cases}
$$

where $0 < r < \frac{1}{\|A\|^2}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D}x_1$.

Proof. It is obvious that C_n are closed and convex for all $n \in \mathbb{N}$. We show that $C \cap A^{-1}D \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $C \cap A^{-1}D \subset C = C_1$. Suppose that $C \cap A^{-1}D \subset C_k$ for some $k \in \mathbb{N}$. To show $C \cap A^{-1}D \subset C_{k+1}$, let us show that $\langle z_k - z, J_E(x_k - z_k)\rangle \ge 0$ for all $z \in A^{-1}D$. In fact, we have that for all $z \in A^{-1}D$,

$$
\begin{aligned}
\langle z_k - z,\, J_E(x_k - z_k)\rangle &= \langle z_k - x_k + x_k - z, J_E(x_k - z_k)\rangle \\
&= \langle -rJ_E^{-1}A^*J_F(Ax_k - P_DAx_k) \\
&\qquad + x_k - z, J_E(rJ_E^{-1}A^*J_F(Ax_k - P_DAx_k))\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle -rJ_E^{-1}A^*J_F(Ax_k - P_DAx_k) + x_k - z, rA^*J_F(Ax_k - P_DAx_k)\rangle \\
&= -r^2\|A^*J_F(Ax_k - P_DAx_k)\|^2 + \langle x_k - z, rA^*J_F(Ax_k - P_DAx_k)\rangle \\
&= -r^2\|A^*J_F(Ax_k - P_DAx_k)\|^2 + r\langle Ax_k - Az, J_F(Ax_k - P_DAx_k)\rangle \qquad (3.4)\\
&= -r^2\|A^*J_F(Ax_k - P_DAx_k)\|^2 \\
&\qquad + r\langle Ax_k - P_DAx_k + P_DAx_k - Az, J_F(Ax_k - P_DAx_k)\rangle \\
&= -r^2\|A^*J_F(Ax_k - P_DAx_k)\|^2 \\
&\qquad + r\|Ax_k - P_DAx_k\|^2 + r\langle P_DAx_k - Az, J_F(Ax_k - P_DAx_k)\rangle \\
&\geq -r^2\|A\|^2\|Ax_k - P_DAx_k\|^2 + r\|Ax_k - P_DAx_k\|^2 \\
&= r(1 - r\|A\|^2)\|Ax_k - P_DAx_k\|^2 \\
&\geq 0.
\end{aligned}
$$

Then, $C \cap A^{-1}D \subset C_{k+1}$. We have by mathematical induction that $C \cap A^{-1}D \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $C_0 \supset C \cap A^{-1}D \neq \emptyset$, C_0 is nonempty. Since $C_0 = \text{M-lim}_{n\to\infty} C_n$ and $x_n = P_{C_n}x_1$ for every $n \in \mathbb{N}$, by Lemma 2.5 we have

$$x_n \to z_0 = P_{C_0}x_1.$$

Since $z_0 \in C_0 \subset C_{n+1}$ and $z_n = P_{C_{n+1}}x_n$, we have $\|x_n - z_n\| \leq \|x_n - z_0\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $x_n - z_n \to 0$.

On the other hand, we know that

$$\|x_n - z_n\| = \|J_E(x_n - z_n)\| = \|rA^*J_F(Ax_n - P_DAx_n)\|.$$

Since $\|x_n - z_n\| \to 0$ and $0 < r\|A\|^2 < 1$, we have that $\|A^*J_F(Ax_n - P_DAx_n)\| \to 0$. Then we get from (3.4) that

$$\lim_{n\to\infty} \|Ax_n - P_DAx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Note that $w \in C$. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. It follows from $\lim_{n\to\infty} \|Ax_n - P_DAx_n\| = 0$ that $P_DAx_{n_i} \rightharpoonup Aw$ and $\|J_F(Ax_n - P_DAx_n)\| = \|Ax_n - P_DAx_n\| \to 0$. Since P_D is the metric projection of F onto D, we have that $\langle P_DAx_n - P_DAw, J_F(Ax_n - P_DAx_n)\rangle \geq 0$ and

$$\langle P_DAw - P_DAx_n, J_F(Aw - P_DAw)\rangle \geq 0$$

and hence

$$\langle P_DAx_n - P_DAw, J_F(Ax_n - P_DAx_n) - J_F(Aw - P_DAw)\rangle \geq 0.$$

Since $P_DAx_{n_i} \rightharpoonup Aw$ and $\|J_F(Ax_n - P_DAx_n)\| \to 0$, we have that

$$-\|Aw - P_DAw\|^2 = \langle Aw - P_DAw, -J_F(Aw - P_DAw)\rangle \geq 0$$

and hence $Aw = P_DAw$. This implies that $w \in C \cap A^{-1}D$.

Since $C \cap A^{-1}D$ is nonempty, closed and convex, there exists $z_1 \in C \cap A^{-1}D$ such that $z_1 = P_{C\cap A^{-1}D}x_1$. From $x_{n+1} = P_{C_{n+1}}x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in C_{n+1}$. Since $z_1 \in C \cap A^{-1}D \subset C_{n+1}$, we have that

$$\|x_1 - x_{n+1}\| \le \|x_1 - z_1\|. \tag{3.5}$$

From $z_1 = P_{C\cap A^{-1}D}x_1$, $w \in C \cap A^{-1}D$ and (3.5), we have that

$$\begin{aligned}\|x_1 - z_1\| \le \|x_1 - w\| &\le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\le \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - z_1\|.\end{aligned}$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_1\|.$$

From the Kadec-Klee property of E, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_1.$$

Therefore, we have $x_n \to w = z_1$. This completes the proof. □

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