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THE VALUATION OF POWERED OPTIONS

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Abstract. We derive closed-form solution formulae for the pricing of powered options and capped powered options in the Black-Scholes-Merton environment. We employ two methods: One uses the riskless bond as a numéraire and the risk-neutral valuation theory, and the other uses the technique of change of equivalent martingale measures.

1. Introduction

A European option gives the holder the right, but not the obligation, to buy (or sell) a unit share of the underlying asset for a specified (strike) price and at a predetermined (expiration) time. In the Black-Scholes-Merton model, the underlying asset is assumed to follow a geometric Brownian motion, and a plain vanilla European option has a closed-form pricing formula. Exotic options however have extensively been investigated and employed (see, e.g., [1, 9]).

Power and powered options are path-independent exotic derivatives with payoffs dependent on the underlying asset price at expiration time raised in a certain way to some power. More precisely, if S_T is the underlying asset price at the expiration time T of the option, then the payoff of a power option is $\max\{S_T^a - K, 0\}$ and that of a powered option is $(\max\{S_T - K, 0\})^a$, where $a > 0$ is a constant and K is the strike price. Notice that in the case of $a = 1$, both power and powered options are reduced to a vanilla European call option.

The pricing of power options is less complicated than that of powered options. Indeed, Esser [3] obtained a closed-form pricing formula for power options for all $a > 0$. However, as for powered options, she only considered the case where a is a positive integer. The purpose of this paper is to fill in this gap; namely, we are going to derive a closed-form pricing formula for valuation of powered options for an arbitrary real number $a > 0$. Moreover, we also obtain a closed-form pricing formula for capped powered options.

2. Valuation of powered options

Assume the Black-Scholes-Merton model [2, 7]; thus the underlying asset price process $\{S_t\}_{t\geq 0}$ follows the geometric Brownian motion (under the risk-neutral probability measure $\mathbb{P}$):

$$dS_t = S_t(\sigma dW_t + r dt), \tag{2.1}$$

where $\sigma > 0$ is the volatility of the asset S and $r > 0$ is the (constant) risk-free rate of interest, and where $\{W_t\}_{t\geq 0}$ is a Brownian motion on a filtered probability space

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$(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F})$, where $\mathbb{P}$ is the risk-neutral probability measure and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration for the Brownian motion $\{W_t\}_{t\geq 0}$.

It is known that the solution to the stochastic differential equation (2.1) is given by

$$S_t = S_0 \exp\left(\sigma W_t + (r - \frac{1}{2}\sigma^2)t\right), \quad 0 \leq t \leq T. \tag{2.2}$$

Consider a powered option with expiration time T and strike price K. Recall that the payoff of this option is

$$V_T := [(S_T - K)^+]^a = (S_T - K)^a \mathbb{I}_{\{S_T > K\}}, \tag{2.3}$$

where $a > 0$ is a real number, where x^+ is the positive part of a real number $x \in \mathbb{R}$; that is,

$$x^+ = \max\{x, 0\} = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases}$$

and where $\mathbb{I}_D$ is the indicator function of a subset $D \subset \Omega$; namely,

$$I_D(\omega) = \begin{cases} 1, & \text{if } \omega \in D, \\ 0, & \text{if } \omega \notin D. \end{cases}$$

Let $\mathbb{E}$ denote the expectation under the risk-neutral probability measure $\mathbb{P}$. The risk-neutral valuation theory implies that the time-t value of the powered option is given by the formula

$$V_t = e^{-r(T-t)} \mathbb{E}\left[(S_T - K)^a \mathbb{I}_{\{S_T > K\}} | \mathcal{F}_t\right]. \tag{2.4}$$

Throughout the rest of this paper we use $N(\cdot)$ to denote the cumulative distribution function of a standard normal random variable; namely,

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

for $x \in \mathbb{R}$.

2.1. **The integer case.** In this subsection we assume that the exponent a in the payoff (2.3) of the powered option is an integer. Then, using the binomial formula, we can rewrite the payoff V_T as

$$V_T = \sum_{j=0}^{a} \binom{a}{j} S_T^{a-j} (-K)^j \mathbb{I}_{\{S_T > K\}}.$$

Then the pricing formula (2.4) becomes

$$V_t = e^{-r(T-t)} \sum_{j=0}^{a} \binom{a}{j} (-K)^j \mathbb{E}\left[S_T^{a-j} \mathbb{I}_{\{S_T > K\}} | \mathcal{F}_t\right]. \tag{2.5}$$

It turns out that the evaluation of V_t is equivalent to the evaluations of the conditional expectations $\mathbb{E}\left[S_T^{a-j} \mathbb{I}_{\{S_T > K\}} | \mathcal{F}_t\right]$ for $0 \leq j \leq a$. The following result is obtained in [3].

Theorem 2.1. *Assume a is a positive integer. Then the value at time t of the powered option defined by the payoff (2.3) is*

$$V_t = \sum_{j=0}^{a} \binom{a}{j} (-K)^j S_t^{a-j} \exp\left[(a-j-1)\left(r + \frac{1}{2}(a-j)\sigma^2\right)\tau\right] N(d^{(a-j)}), \tag{2.6}$$

where

$$d^{(a-j)} = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S_t}{K} + \left(r + \left(a-j-\frac{1}{2}\right)\sigma^2\right)(T-t)\right]. \tag{2.7}$$

2.2. **The general case.** In this subsection we consider the general case; namely, we assume that the exponent $a > 0$ in the payoff (2.3) of the powered option is not necessarily an integer. We adopt the notation: $[a]$ and $\{a\}$ stand for the integer and decimal parts of a, respectively (e.g., $[2.7] = 2$ and $\{2.7\} = 0.7$); thus $a = [a] + \{a\}$.

We notice that the payoff (2.3) of the powered option can be rewritten as

$$V_T = (S_T - K)^{[a]}(S_T - K)^{\{a\}}\mathbb{I}_{\{S_T > K\}}. \tag{2.8}$$

For the first factor of V_T in (2.8), we use the binomial expansion to get

$$(S_T - K)^{[a]} = \sum_{j=0}^{[a]} \binom{[a]}{j} (-K)^j S_T^{[a]-j}. \tag{2.9}$$

For the middle factor of V_T in (2.8), we employ the following Taylor expansion:

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

for $|x| < 1$ and $\alpha \in [0, 1)$, where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

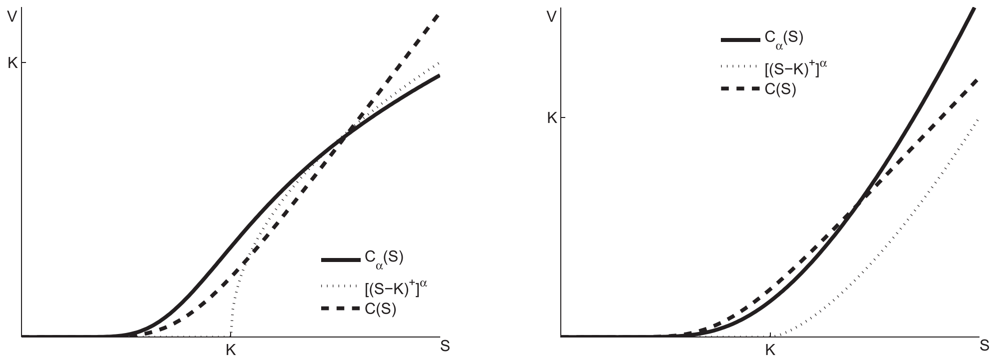
It turns out that, for $S_T > K$,

$$\begin{aligned}(S_T - K)^{\{a\}} &= S_T^{\{a\}}\left(1 - \frac{K}{S_T}\right)^{\{a\}} \\ &= S_T^{\{a\}} \sum_{n=0}^{\infty} \binom{\{a\}}{n}\left(-\frac{K}{S_T}\right)^n \\ &= S_T^{\{a\}} \sum_{n=0}^{\infty} \binom{\{a\}}{n} (-K)^n S_T^{-n}. \end{aligned} \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8), we get that the payoff of the powered option is written as

$$V_T = \sum_{j=0}^{[a]} \sum_{n=0}^{\infty} \binom{[a]}{j}\binom{\{a\}}{n} (-K)^{j+n} S_T^{a-j-n}\mathbb{I}_{\{S_T > K\}}. \tag{2.11}$$

This expression for the payoff V_T enables us to derive a closed-form pricing formula for valuation of powered options.



(a) $a = 0.5$ (b) $a = 1.5$

FIGURE 1. The time $t = 0$ price of powered call option (solid line) with the strike price $K = 1$, exercise time $T = 2$, when $r = 0.1$ and $\sigma = 0.2$; the price of European (vanilla) call option (dashed line) and the intrinsic value of the considered power option (dotted line). Two cases of the power a are considered.

Theorem 2.2. *Let $a > 0$ be a given real number. Then the value at time $t \in [0, T)$ of the powered option defined by payoff (2.3) is given by*

$$\begin{aligned} V_t &= \sum_{j=0}^{[a]} \sum_{n=0}^{\infty} \binom{[a]}{j} \binom{\{a\}}{n} (-K)^{j+n} S_t^{a-j-n} \\ &\qquad \times \exp\left((a-j-n-1)\tau \left(r + \frac{1}{2}\sigma^2 (a-j-n) \right) \right) \\ &\qquad \times N\left(d^{(a-j-n)}(\tau, S_t) \right), \end{aligned} \tag{2.12}$$

where $\tau = T - t$ and, for $x > 0$,

$$d^{(a-j-n)}(\tau, x) = \frac{\log \frac{x}{K} + \left[r + (a-j-n-\frac{1}{2})\sigma^2 \right] \tau}{\sigma \sqrt{\tau}}. \tag{2.13}$$

Proof. Set $\tau = T - t$, the time to maturity. According to the risk-neutral valuation formula (2.4) and by (2.11) we see that the value at time t of the powered option is

$$\begin{aligned} V_t &= e^{-r\tau} \mathbb{E}\left[V_T | \mathcal{F}_t \right] \\ &= e^{-r\tau} \sum_{j=0}^{[a]} \sum_{n=0}^{\infty} \binom{[a]}{j} \binom{\{a\}}{n} (-K)^{j+n} \mathbb{E}\left[S_T^{a-j-n} \mathbb{I}_{\{S_T > K\}} | \mathcal{F}_t \right]. \end{aligned} \tag{2.14}$$

Hence it remains to evaluate the conditional expectations in (2.14) for $0 \le j \le [a]$ and $n \ge 0$.

Since

$$\begin{aligned} S_T &= S_t e^{\sigma(W_T - W_t) + (r - \frac{1}{2}\sigma^2)(T-t)} \\ &= S_t e^{-\sigma\sqrt{\tau} Z + (r - \frac{1}{2}\sigma^2)\tau}, \end{aligned} \tag{2.15}$$

where

$$Z = -\frac{W_T - W_t}{\sqrt{\tau}} \sim N(0,1)$$

which is independent of $\mathcal{F}_t$, we find that $S_T > K$ if and only if

$$Z < \frac{\log \frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} =: d_2(\tau, S_t). \tag{2.16}$$

We then have

$$S_T^{a-j-n} = S_t^{a-j-n} e^{-\sigma(a-j-n)\sqrt{\tau}Z + (a-j-n)(r-\frac{1}{2}\sigma^2)\tau}. \tag{2.17}$$

It follows from (2.17) and the independence of Z with $\mathcal{F}_t$ that

$$\begin{aligned} \mathbb{E}\left[S_T^{a-j-n}\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t\right] &= S_t^{a-j-n}e^{(a-j-n)(r-\frac{1}{2}\sigma^2)\tau} \\ &\quad \times\mathbb{E}\left[e^{-\sigma(a-j-n)\sqrt{\tau}Z}\mathbb{I}_{\{Z<d_2(\tau,S_t)\}}|\mathcal{F}_t\right] \\ &= f(\tau, S_t), \end{aligned} \tag{2.18}$$

where $f(\tau, x)$ is given by

$$f(\tau,x) = x^{a-j-n}e^{(a-j-n)(r-\frac{1}{2}\sigma^2)\tau}\mathbb{E}\left[e^{-\sigma(a-j-n)\sqrt{\tau}Z}\mathbb{I}_{\{Z<d_2(\tau,x)\}}\right].$$

Since $Z \sim N(0,1)$, we obtain

$$\begin{aligned} f(\tau,x) &= x^{a-j-n}e^{(a-j-n)(r-\frac{1}{2}\sigma^2)\tau}\int_{-\infty}^{d_2(\tau,x)}\frac{1}{\sqrt{2\pi}}e^{-\sigma(a-j-n)\sqrt{\tau}z-\frac{1}{2}z^2}dz \\ &= x^{a-j-n}e^{(a-j-n)\tau[r+\frac{1}{2}\sigma^2(a-j-n-1)]}\int_{-\infty}^{d_2(\tau,x)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z+\sigma(a-j-n)\sqrt{\tau})^2}dz. \end{aligned}$$

Applying the substitution $v = z + \sigma(a-j-n)\sqrt{\tau}$ and setting

$$\begin{aligned} d^{(a-j-n)}(\tau,x) &:= d_2(\tau,x) + \sigma(a-j-n)\sqrt{\tau} \\ &= \frac{\log\frac{x}{K} + \left[r + (a-j-n-\frac{1}{2})\sigma^2\right]\tau}{\sigma\sqrt{\tau}}, \end{aligned} \tag{2.19}$$

we get

$$\begin{aligned} f(\tau,x) &= x^{a-j-n}e^{(a-j-n)\tau[r+\frac{1}{2}\sigma^2(a-j-n-1)]}\int_{-\infty}^{d^{(a-j-n)}(\tau,x)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}dv \\ &= x^{a-j-n}e^{(a-j-n)\tau[r+\frac{1}{2}\sigma^2(a-j-n-1)]}N\left(d^{(a-j-n)}(\tau,x)\right). \end{aligned} \tag{2.20}$$

Substituting (2.20) and (2.18) into (2.14) and observing

$$(a-j-n)\tau[r+\frac{1}{2}\sigma^2(a-j-n-1)] - r\tau = \tau(a-j-n-1)[r+\frac{1}{2}\sigma^2(a-j-n)],$$

we immediately obtain the pricing formula (2.12). □

It is easily checked that if $\{a\} = 0$ (i.e., a is a positive integer), then the formula (2.12) is reduced to the formula (2.6).

It is also not hard to compute the delta of the powered option from the formula (2.6).

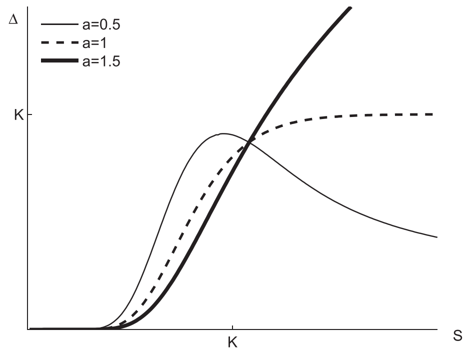


FIGURE 2. The Δ function as a function of the stock price S. We consider $t=0, K=1, T=2, r=0.1$ and $\sigma=0.2$. Three cases of the power a are considered (for $a=1$ we obtain the Δ function for the European vanilla option).

Corollary 2.3. *The delta of the powered option defined by payoff (2.3) is given by*

$$\Delta_t = \sum_{j=0}^{[a]}\sum_{n=0}^{\infty}\binom{[a]}{j}\binom{\{a\}}{n}(-K)^{j+n}\exp\left((a-j-n-1)\tau\left(r+\frac{1}{2}\sigma^2(a-j-n)\right)\right)$$
$$\times S_t^{a-j-n-1}\Bigg[(a-j-n)N\left(d^{(j,n)}(\tau,S_t)\right)$$
$$+\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}\left(d^{(a-j-n)}(\tau,S_t)\right)^2\right)\Bigg]. \tag{2.21}$$

Proof. Since $\Delta_t = \frac{\partial V_t}{\partial S_t}$, we can easily find from (2.6) and (2.19) that (2.21) holds. □

2.3. **Equivalent martingale measures.** We provide with another approach to evaluating the pricing formula (2.12). In order to evaluate the conditional expectations in (2.14), we employ the technique of change of equivalent martingale measures [5, 4, 3, 8]; thus we are able to remove the factor S_T^{a-j-n} in the conditional expectation in (2.14) which reduces the computation of this conditional expectation to the computation of the event that the option ends in the money (under another equivalent martingale measure). We proceed as follows.

For $b \in \mathbb{R}$, let

$$Z^{(b)} = \frac{S_T^b}{\mathbb{E}[S_T^b]}. \tag{2.22}$$

Then $Z^{(b)}$ is a positive-valued random variable, $\mathcal{F}_T$-measurable, and $\mathbb{E}[Z^{(b)}]=1$; hence we can define an equivalent measure $\mathbb{P}^{(b)}$ by

$$\mathbb{P}^{(b)}(A) = \mathbb{E}[\mathbb{I}_A Z^{(b)}] = \int_A Z^{(b)}(\omega)d\mathbb{P}(\omega), \quad A \in \mathcal{F}. \tag{2.23}$$

Let $\mathbb{E}^{(b)}$ be the corresponding expectation.

Since $S_T^b = S_0^b e^X$, where

$$X = b\sigma W_T + b\left(r - \frac{1}{2}\sigma^2\right)T,$$

we compute

$$\begin{aligned}\mathbb{E}[S_T^b] &= S_0^b \mathbb{E}[e^X] \\ &= S_0^b \exp\left(\mathbb{E}[X] + \frac{1}{2}\mathrm{Var}(X)\right) \\ &= S_0^b \exp\left(b\left(r - \frac{1}{2}\sigma^2\right)T + \frac{1}{2}(b\sigma)^2 T\right).\end{aligned}$$

Hence, by (2.22),

$$Z^{(b)} = \exp\left(b\sigma W_T - \frac{1}{2}(b\sigma)^2 T\right). \tag{2.24}$$

Setting

$$W_t^{(b)} = W_t - b\sigma t.$$

Then by Girsanov's theorem (cf. [6, 8]), we obtain that $\{W_t^{(b)}\}$ is a standard Brownian motion under the measure $\mathbb{P}^{(b)}$.

Furthermore, under the measure $\mathbb{P}^{(b)}$, S_t follows the geometric Brownian motion

$$dS_t = S_t((r + b\sigma^2)dt + \sigma dW_t^{(b)}).$$

It follows that

$$S_t = S_0 \exp\left(\sigma W_t^{(b)} + \left(r + \left(b - \frac{1}{2}\right)\sigma^2\right)t\right)$$

and

$$S_T = S_t \exp\left(\sigma(W_T^{(b)} - W_t^{(b)}) + \left(r + \left(b - \frac{1}{2}\right)\sigma^2\right)(T - t)\right).$$

Now set

$$Y = -\frac{W_T^{(b)} - W_t^{(b)}}{\sqrt{\tau}} \sim N(0,1), \quad \tau = T - t.$$

Then we write

$$S_T = S_t \exp\left(-\sigma\sqrt{\tau}Y + \left(r + \left(b - \frac{1}{2}\right)\sigma^2\right)\tau\right). \tag{2.25}$$

Therefore, $S_T > K$ if and only if

$$Y < \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{S_t}{K}\right) + \left(r + \left(b - \frac{1}{2}\right)\sigma^2\right)\tau\right] = d^{(b)}(\tau, S_t). \tag{2.26}$$

Consequently (noting that Y is independent of $\mathcal{F}_t$),

$$\mathbb{P}^{(b)}\{S_T > K|\mathcal{F}_t\} = \mathbb{P}^{(b)}\{Y < d^{(b)}(\tau, S_t)\} = N(d^{(b)}(\tau, S_t)).$$

Define $Z_t^{(b)}$ by

$$Z_t^{(b)} = \mathbb{E}[Z^{(b)}|\mathcal{F}_t] = \exp\left(a\sigma W_t - \frac{1}{2}(b\sigma)^2 t\right)$$

$$= \frac{S_t^b}{S_0^b}\exp\left(-\frac{1}{2}(b\sigma)^2 t - \left(r - \frac{1}{2}\sigma^2\right)bt\right).$$

Next we compute

$$\begin{aligned}
\mathbb{E}[S_T^b \mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t] &= Z_t^{(b)}\mathbb{E}^{(b)}\left[\frac{1}{Z^{(b)}}S_T^b\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t\right] \\
&= Z_t^{(b)}\mathbb{E}^{(b)}\left[\frac{\mathbb{E}[S_T^b]}{S_T^b}S_T^b\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t\right] \\
&= Z_t^{(b)}\mathbb{E}[S_T^b]\mathbb{P}^{(b)}\{S_T>K|\mathcal{F}_t\} \\
&= \frac{S_t^b}{S_0^b}\exp\left(-\frac{1}{2}(b\sigma)^2 t - \left(r - \frac{1}{2}\sigma^2\right)at\right) \\
&\quad \times S_0^b\exp\left(b\left(r-\frac{1}{2}\sigma^2\right)T + \frac{1}{2}(b\sigma)^2 T\right)N(d^{(b)}(\tau,S_t)).
\end{aligned}$$

It follows that, for $b \in \mathbb{R}$,

$$\mathbb{E}[S_T^b\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t] = S_t^b\exp\left(\frac{1}{2}(b\sigma)^2\tau + \left(r-\frac{1}{2}\sigma^2\right)b\tau\right)N(d^{(b)}(\tau,S_t)). \tag{2.27}$$

Upon setting $b = a - j - n$ in (2.27), we immediately get

$$\begin{aligned}
\mathbb{E}\left[S_T^{a-j-n}\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t\right] &= S_t^{a-j-n}\exp\left(\frac{1}{2}\sigma^2(a-j-n)^2\tau\right. \\
&\quad \left. + \left(r-\frac{1}{2}\sigma^2\right)(a-j-n)\tau\right)N(d^{(a-j-n)}(\tau,S_t)) \\
&= f(\tau,S_t),
\end{aligned} \tag{2.28}$$

where $f(\tau, x)$ is given by (2.20).

Finally, substituting (2.28) into (2.14), we again obtain the pricing formula (2.12).

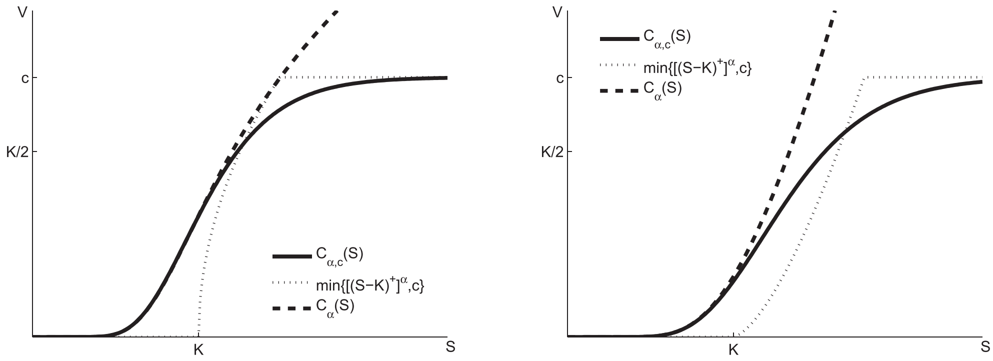
3. Capped powered options

The payoff of a powered option is capped to limit the risk for the writer of the option. Putting another way, we have that a capped powered option has an upper bound $\hat{c} > 0$ for the payoff to limit the possible loss of the writer. Namely, the payoff of a capped powered option has the form

$$\hat{V}_T = \min\{[(S_T-K)^+]^a, \hat{c}\} = \begin{cases} [(S_T-K)^+]^a, & \text{if } [(S_T-K)^+]^a \le \hat{c}, \\ \hat{c}, & \text{if } [(S_T-K)^+]^a > \hat{c}. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $a > 0$ be a given real positive number. The value at time $t \in [0, T)$, $\hat{V}_t$, of the capped powered option defined by the payoff* (3.1) *is*

$$\begin{aligned}
\hat{V}_t &= \sum_{j=0}^{[a]}\sum_{n=0}^{\infty}\binom{[a]}{j}\binom{\{a\}}{n}(-K)^{j+n}S_t^{a-j-n} \\
&\quad \times\exp\left((a-j-n-1)\tau\left(r+\frac{1}{2}\sigma^2(a-j-n)\right)\right) \\
&\quad \times\left[N\left(d^{(a-j-n)}(\tau,S_t)\right) - N\left(\hat{d}^{(a-j-n)}(\tau,S_t)\right)\right] + \hat{c}N\left(\hat{d}_2(\tau,S_t)\right),
\end{aligned} \tag{3.2}$$



(a) $a = 0.5$ (b) $a = 1.5$

FIGURE 3. The time $t = 0$ price of capped powered call option (solid line) with $\hat{c} = 0.7$, the strike price $K = 1$, exercise time $T = 2$, when $r = 0.1$ and $\sigma = 0.2$; its intrinsic payoff function (dotted); and the price of related not-capped powered option (dashed line).

where $\tau = T - t$, *and for* $b \in \mathbb{R}$ *and* $x > 0$,

$$d^{(b)}(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left[\log\left(\frac{x}{K}\right) + \left(r + \left(b - \frac{1}{2}\right)\sigma^2\right)\tau \right],$$

$$\hat{d}^{(b)}(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left[\log\left(\frac{x}{\hat{K}}\right) + \left(r + \left(b - \frac{1}{2}\right)\sigma^2\right)\tau \right],$$

and

$$\hat{d}_2(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left[\log\left(\frac{x}{\hat{K}}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau \right] = \hat{d}^{(0)}(\tau, x).$$

Proof. By the risk-neutral valuation formula, the time-t value of the capped powered option is given by

$$\begin{aligned} \hat{V}_t &= e^{-r(T-t)}\mathbb{E}[\hat{V}_T|\mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}\left[\min\{[(S_T - K)^+]^a, \hat{c}\}|\mathcal{F}_t\right] \end{aligned}$$

Notice that we can express the payoff $\hat{V}_T$ in another way as follows:

$$\begin{aligned} \hat{V}_T &= \min\left\{(S_T - K)^a \mathbb{I}_{\{S_T > K\}}, \hat{c}\right\} \\ &= (S_T - K)^a \mathbb{I}_{\{\hat{c}^{1/a} + K > S_T > K\}} + \hat{c}\mathbb{I}_{\{S_T \geq \hat{c}^{1/a} + K\}} \\ &= (S_T - K)^a \mathbb{I}_{\{S_T > K\}} - (S_T - K)^a \mathbb{I}_{\{S_T \geq \hat{K}\}} + \hat{c}\mathbb{I}_{\{S_T \geq \hat{K}\}}, \end{aligned}$$

where $\hat{K} = \hat{c}^{1/a} + K$. It then follows that

$$\begin{aligned} \hat{V}_t &= e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^a \mathbb{I}_{\{S_T > K\}}|\mathcal{F}_t\right] \\ &\quad - e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^a \mathbb{I}_{\{S_T \geq \hat{K}\}}|\mathcal{F}_t\right] \\ &\quad + \hat{c}\mathbb{E}\left[\mathbb{I}_{\{S_T \geq \hat{K}\}}|\mathcal{F}_t\right] \\ &\equiv I_1 - I_2 + I_3. \end{aligned} \tag{3.3}$$

We have that the first term I_1 in (3.3) is precisely the time-t value of the powered option which is determined in Theorem 2.2; namely,

$$\begin{aligned} I_1 &= \sum_{j=0}^{[a]}\sum_{n=0}^{\infty}\binom{[a]}{j}\binom{\{a\}}{n}(-K)^{j+n}S_t^{a-j-n} \\ &\qquad \times \exp\left((a-j-n-1)\tau\left(r+\frac{1}{2}\sigma^2(a-j-n)\right)\right) \\ &\qquad \times N\left(d^{(a-j-n)}(\tau,S_t)\right). \end{aligned} \tag{3.4}$$

The second term I_2 is similar to the first term I_1 except that the K in the indicator function of I_1 is replaced with $\hat{K}$ in the indicator function of I_2. So, repeating the proof of Theorem 2.2 we get

$$\begin{aligned} I_2 &= \sum_{j=0}^{[a]}\sum_{n=0}^{\infty}\binom{[a]}{j}\binom{\{a\}}{n}(-K)^{j+n}S_t^{a-j-n} \\ &\qquad \times \exp\left((a-j-n-1)\tau\left(r+\frac{1}{2}\sigma^2(a-j-n)\right)\right) \\ &\qquad \times N\left(\hat{d}^{(a-j-n)}(\tau,S_t)\right), \end{aligned} \tag{3.5}$$

where, for $x > 0$,

$$\hat{d}^{(a-j-n)}(\tau,x) = \frac{\log\frac{x}{\hat{K}} + \left[r+(a-j-n-\frac{1}{2})\sigma^2\right]\tau}{\sigma\sqrt{\tau}}. \tag{3.6}$$

To compute I_3, we use (2.15) and replace the K in (2.16) with $\hat{K}$ to find that $S_T \geq \hat{K}$ if and only if

$$Z \leq \frac{\log\frac{S_t}{\hat{K}} + (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = \hat{d}_2(\tau,S_t). \tag{3.7}$$

Since $Z \sim N(0,1)$ and is independent of $\mathcal{F}_t$, it turns out that

$$\begin{aligned} I_3 &= \hat{c}\mathbb{E}\left[\mathbb{I}_{\{S_T\geq\hat{K}\}}|\mathcal{F}_t\right] \\ &= \hat{c}\mathbb{P}\{Z \leq \hat{d}_2(\tau,S_t)\} \\ &= \hat{c}N\left(\hat{d}_2(\tau,S_t)\right). \end{aligned} \tag{3.8}$$

Finally, substituting (3.4), (3.5) and (3.8) into (3.3), we obtain (3.2). □

4. Conclusion

We have obtained a closed-form solution formula for the price of a powered option in the general case where the exponent a is an arbitrary positive real number. In the derivation of this formula, we have employed two approaches. In the first approach, we have utilized the risk-neutral valuation formula, and in the second approach, we have applied the trick of change of equivalent martingale measures, which is a technique used widely in option pricing theory [5, 3, 8]. We have then computed the delta of a powered option. Moreover, we have also derived a closed-form solution formula for the pricing of a capped powered option.

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