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LOCAL CUBIC CONVERGENCE OF A FAMILY OF DEFORMED SUPER-HALLEY METHOD

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Abstract. The convergence problem of a family of deformed Super-Halley iterations with parmeters for solving nonlinear operator equations in Banach spaces is studied. Under the assumption that the derivative of the operator satisfies the Lipschitz condition, the local cubical convergence of the family of deformed Super-Halley iterations is established.

1. Introduction

Let X and Y be two real or complex Banach spaces. Let $\Omega \subseteq X$ be an open convex subset. Let $F : \Omega \subseteq X \to Y$ be a second Fréchet differentiable nonlinear operator. Consider the nonlinear equation

$$F(x) = 0. \tag{1.1}$$

Newton's method and its variations are the most efficient methods known for solving the nonlinear equation (1.1). One of the main results on Newton's method is the well-known Kantorovich's theorem (see [14]), which has the advantage that Newton's sequence converges to a solution under very mild conditions. Another important result on Newton's method is the Smale's point estimate theory in [17] (see also [1]). Other results on Newton's method such as the estimates of the radii of convergence balls were given by Traub and Wozniakowski [18] and Wang [19] independently. A big step in this direction was made by Wang in [20, 21], where some generalized Lipschitz conditions are introduced and so Kantorovich's theorem and Smale's theory were unified and extended. Newton's method and its variations are also explored extensively in ([9, 22, 24, 28] and references therein).

Several kinds of cubic generalizations for Newton's method are introduced. The most important two are the Euler method and the Halley method (see for example, [2,3,25]). Another more general family of the cubic extensions is the family of Euler-Halley type methods in Banach spaces, which includes the Euler method and the Halley method as its special cases and has been studied extensively in ([4,7,8,11,23] and references therein). More precisely, the family of Euler-Halley type methods is given as follow:

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Algorithm EH. Let $\alpha \in [0,1]$, and let $x_0 \in X$ be given. Have $x_0, x_1, \ldots, x_n$. Define x_{n+1} by

$$x_{n+1} = x_n - [I + \frac{1}{2} L_F(x_n)[I - \alpha L_F(x_n)]^{-1}] F'(x_n)^{-1} F(x_n),$$

where

$$L_F(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x).$$

In the case when $\alpha = 0$ and $\alpha = \frac{1}{2}$, Algorithm EH is reduced to the Euler method and the Halley method, respectively. Furthermore, if $\alpha = 1$, then Algorithm EH is reduced to the supper-Halley method or the convex acceleration of Newton's method (cf. [11]).

One shortcoming of Algorithm EH is that we have to compute the second-derivative of F at each step, while the computation cost of second-derivative is very expensive. To overcome this difficulty, in [5,6], the second-derivative operator is replaced by a finite difference between first derivatives, while the cubic convergence rate is reserved under some mild conditions (see [6,10,12,13] for more details). More precisely,

$$F''(x_n)(z_n - x_n) \approx F'(z_n) - F'(x_n),$$

where $z_n = x_n + \lambda(y_n - x_n), \lambda \in [0,1]$ and $y_n = x_n - F'(x_n)^{-1} F(x_n)$. Our main interests are focused on a modification of the super-Halley method introduced in [5,6], which is given as follows:

Algorithm 1.1. Let $\lambda \in (0,1]$, and let $x_0 \in X$ be given. Have $x_0, x_1, \ldots, x_n$. Define x_{n+1} by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ H(x_n, y_n) &= \frac{1}{\lambda} F'(x_n)^{-1} [F'(x_n + \lambda(y_n - x_n)) - F'(x_n)], \\ Q(x_n, y_n) &= -\frac{1}{2} H(x_n, y_n)[I + H(x_n, y_n)]^{-1}, \\ x_{n+1} &= y_n + Q(x_n, y_n)(y_n - x_n), \quad n = 1, 2, \ldots. \end{aligned}$$

Semilocal convergence analysis of Algorithm 1.1 is provided in [5,6,26,27]. The main purpose of the present paper is to study the local convergence analysis of Algorithm 1.1. Under the assumption that the derivative of the operator satisfies the Lipschitz condition, the local cubical convergence of Algorithm 1.1 is established and the estimate of convergence radius is also presented.

2. Preliminaries

Throughout the whole paper, we always assume that $L, \gamma > 0$ and $\lambda \in (0,1]$. Below, we will give some lemmas which will be useful in the next section.

Lemma 2.1. *Let h be a function defined by*

$$h(t) := -t + \frac{\gamma}{2} t^2 + \frac{L}{6} t^3 \quad \text{for each } t \in \mathbb{R}. \tag{2.1}$$

Then, there exists $r_0 \in (0, +\infty)$ such that $h'(r_0) = 0$, $h'(t) < 0$ for all $t \in (0, r_0)$ and $h'(t) > 0$ for all $t > r_0$.

Proof. Since $h'(t) = -1 + \gamma t + \frac{L}{2}t^2$ and

$$h''(t) = \gamma + Lt > 0 \quad \text{for each } t > 0, \tag{2.2}$$

we get that $h'(t)$ is increasing on $(0, +\infty)$. Note that $h'(0) = -1$ and $h'(+\infty) > 0$. Hence, there exists $r_0 \in (0, +\infty)$ such that $h'(r_0) = 0$, $h'(t) < 0$ for all $t \in (0, r_0)$ and $h'(t) > 0$ for all $t > r_0$. □

Let $\lambda \in (0, 1]$. Define

$$U(t) := 1 + h'(t)^{-2}h(t)[h''(t) + \frac{L\lambda}{2}(h'(t)^{-1}h(t))] \quad \text{for each } t \in [0, r_0). \tag{2.3}$$

Lemma 2.2. *Let h, U be given by (2.1) and (2.3), respectively. Then, U is monotone decreasing on $(0, r_0)$ and there exists $r_1 \in (0, r_0)$ such that $U(r_1) = 0$.*

Proof. Note by (2.3) that

$$\begin{aligned} U'(t) &= \tfrac{h'(t)^2 - 2h(t)h''(t)}{h'(t)^3}[h''(t) + \tfrac{L\lambda}{2}(h'(t)^{-1}h(t))] \\ &+ h'(t)^{-2}h(t)[h'''(t) + \tfrac{L\lambda(h'(t)^2 - h(t)h''(t))}{2h'(t)^2}]. \end{aligned} \tag{2.4}$$

Let $t \in (0, r_0)$. Then, $h(t) < 0, h'(t) < 0$ and $h''(t) > 0$. This implies that

$$\frac{h'(t)^2 - 2h(t)h''(t)}{h'(t)^3}[h''(t) + \frac{L\lambda}{2}(h'(t)^{-1}h(t))] < 0,$$

$\frac{L\lambda(h'(t)^2 - h(t)h''(t))}{2h'(t)^2} > 0$ and $h'(t)^{-2}h(t) < 0$. Note further that $h'''(t) = L > 0$. Thus, it follows from (2.4) that $U'(t) < 0$ which implies that $U(\cdot)$ is monotone decreasing on $(0, r_0)$. As $U(0) = 1$ and $U(t) \to -\infty$ if $t \to r_0$, we obtain that there exists $r_1 \in (0, r_0)$ such that $U(r_1) = 0$. □

Define $S(t) := \frac{h'(t)^{-1}h(t)}{t}$ for each $t \in (0, r_0)$ and

$$\begin{aligned} P_\lambda(t) :&= \{-\tfrac{Lh'(t)^{-1}}{6} + \tfrac{(\gamma + 2h''(t))h'(t)^{-2}h''(t)}{12} - \tfrac{L\lambda S(t)^3 h'(t)^{-1}}{4U(t)} \\ &+ \tfrac{h'(t)^{-2}h''(t)S(t)(\frac{\gamma + 2h''(t)}{6} + S(t)[h''(t) + \frac{L\lambda}{2}(h'(t)^{-1}h(t))])}{2U(t)}\}t^3 \quad \text{for each } t \in (0, r_1). \end{aligned} \tag{2.5}$$

Lemma 2.3. *Let P_λ be given by (2.5). Then, the following two assertions hold:*

(i) *the two functions $t \to \frac{P_\lambda(t)}{t^3}$ and $t \to \frac{P_\lambda(t)}{t}$ are monotone increasing on $(0, r_1)$;*

(ii) *P_λ has a unique fixed point $r_\lambda \in (0, r_1)$, that is, $P_\lambda(r_\lambda) = r_\lambda$.*

Proof. (i). Note by definition that the functions $h'', -h'^{-1}, h'^{-2}$ and $h'^{-1}h$ are monotone increasing on $(0, r_1)$. Note further that

$$S(t) = \frac{h'(t)^{-1}h(t)}{t} = \frac{1 - \frac{1}{2}\gamma t - \frac{L}{6}t^2}{1 - \gamma t - \frac{L}{2}t^2} = 1 + \frac{\frac{1}{2}\gamma t + \frac{L}{3}t^2}{1 - \gamma t - \frac{L}{2}t^2}.$$

It's easy to verify by definition that S is monotone increasing on $(0, r_1)$. As U is monotone decreasing on $(0, r_1)$, it's easy to show that the two functions $t \to \frac{P_\lambda(t)}{t^3}$ and $t \to \frac{P_\lambda(t)}{t}$ are monotone increasing on $(0, r_1)$.

(ii). Note that $\lim_{t \to 0^+} \frac{P_\lambda(t)}{t} = 0$ and $\lim_{t \to r_1^-} \frac{P_\lambda(t)}{t} = +\infty$. As $\frac{P_\lambda(t)}{t}$ is continuous on $(0, r_1)$, we get from (i) that there exists a unique point $r_\lambda \in (0, r_1)$ such that $\frac{P_\lambda(r_\lambda)}{r_\lambda} = 1$. □

3. Convergence of the algorithm

Let X be a Banach space. Let $x \in X$, and let $r > 0$. We use $\mathbf{B}(x,r)$ and $\overline{\mathbf{B}(x,r)}$ to denote, respectively, the open metric ball and the closed metric ball at x with radius r, that is,

$$\mathbf{B}(x,r) = \{y \in X : \|x-y\| < r\} \quad \text{and} \quad \overline{\mathbf{B}(x,r)} = \{y \in X : \|x-y\| \le r\}.$$

Recall that we assume that $L, \gamma > 0$ and $\lambda \in (0,1]$.

Lemma 3.1. *Let $x^* \in X$. Suppose that*

$$\|F'(x^*)^{-1}F''(x^*)\| \le \gamma \tag{3.1}$$

and

$$\|F'(x^*)^{-1}[F''(x') - F''(x)]\| \le L\|x'-x\| \quad \text{for each } x, x' \in \mathbf{B}(x^*, r_0). \tag{3.2}$$

Then for all $x \in \mathbf{B}(x^*, r_0)$,

$$\|F'(x^*)^{-1}F''(x)\| \le h''(t), \tag{3.3}$$

$F'(x)^{-1}$ exists and

$$\|F'(x)^{-1}F'(x^*)\| \le -h'(t)^{-1}, \tag{3.4}$$

where $t = \|x - x^*\|$.

Proof. It follows from (3.1) and (3.2) that

$$\begin{aligned} \|F'(x^*)^{-1}F''(x)\| &\le \|F'(x^*)^{-1}F''(x^*)\| + \|F'(x^*)^{-1}[F''(x) - F''(x^*)]\| \\ &\le \gamma + L\|x - x^*\| = h''(t). \end{aligned}$$

Hence, (3.3) is seen to hold. Since

$$F'(x) = F'(x^*) + F''(x^*)(x-x^*) + \int_0^1 (F''(x^* + \tau(x-x^*)) - F''(x^*))d\tau(x-x^*),$$

we have

$$\begin{aligned} \|F'(x^*)^{-1}F'(x) - I\| \le \|F'(x^*)^{-1}F''(x^*)\|\|x-x^*\| \\ + \int_0^1 \|F'(x^*)^{-1}(F''(x^* + \tau(x-x^*)) - F''(x^*))\|d\tau\|x-x^*\|. \end{aligned}$$

Combining this with (3.1) and (3.2) yields that

$$\begin{aligned} \|F'(x^*)^{-1}F'(x) - I\| &\le \gamma\|x-x^*\| + \int_0^1 L\tau\|x-x^*\|d\tau\|x-x^*\| \\ &\le \gamma\|x-x^*\| + \frac{L}{2}\|x-x^*\|^2 \\ &= 1 + (-1 + \gamma\|x-x^*\| + \frac{L}{2}\|x-x^*\|^2) \\ &= 1 + h'(t). \end{aligned}$$

This implies that $\|F'(x^*)^{-1}F'(x) - I\| < 1$ because $t \in (0, r_0)$ and $h'(t) \in (-1, 0)$ by Lemma 2.1. Thus, by the well known Banach lemma, one has that $F'(x)^{-1}$ and

$$\|F'(x)^{-1}F'(x^*)\| \le -h'(t)^{-1}.$$

□

Let $x^* \in X$. Below, we always assume that

$$F(x^*) = 0.$$

Write

$$x^\tau = x^* + \tau(x - x^*) \quad \text{for each } 0 \le \tau \le 1.$$

Since

$$\begin{aligned} F(x) &= F(x) - F(x^*) + F'(x)(x - x^*) - F'(x)(x - x^*) \\ &= F'(x)(x - x^*) - \int_0^1 \tau F''(x^\tau)(x - x^*)^2 d\tau, \end{aligned}$$

it follows that

$$F'(x)^{-1}F(x) = \left(I - \int_0^1 \tau F'(x)^{-1}F''(x^\tau)d\tau(x - x^*)\right)(x - x^*). \tag{3.5}$$

Lemma 3.2. *Suppose that* (3.1) *and* (3.2) *hold. Then, for each* $x \in \mathbf{B}(x^*, r_0)$*, the following assertions hold:*

$$\|F'(x)^{-1}F(x)\| \le h'(t)^{-1}h(t), \tag{3.6}$$

$$\|F'(x)^{-1}F''(x)\| \le -h'(t)^{-1}h''(t) \tag{3.7}$$

and

$$\|H(x, y)\| \le 1 - U(t), \tag{3.8}$$

where $t = \|x - x^*\|$.

Proof. By (3.5), we get

$$\begin{aligned} &\|F'(x)^{-1}F(x)\| \\ &\le \left(1 + \int_0^1 \tau \|F'(x)^{-1}F'(x^*)\| \|F'(x^*)^{-1}F''(x^\tau)\| d\tau \|(x - x^*)\|\right) \|(x - x^*)\|. \end{aligned} \tag{3.9}$$

Combing this with (3.4) and (3.3) yields that

$$\begin{aligned} \|F'(x)^{-1}F(x)\| &= \left(1 - h'(t)^{-1}\int_0^1 \tau h''(t\tau)d\tau t\right) t \\ &\le [1 - h'(t)^{-1}(h'(t) - \frac{h(t) - h(0)}{t})]t \\ &= h'(t)^{-1}h(t). \end{aligned}$$

Hence, (3.6) is seen to hold. As $\|F'(x)^{-1}F''(x)\| \le \|F'(x)^{-1}F'(x^*)\| \|F'(x^*)^{-1}F''(x)\|$, it follows from (3.4) and (3.3) that

$$\|F'(x)^{-1}F''(x)\| \le -h'(t)^{-1}h''(t)$$

and so (3.7) holds. Note that $y = x - F'(x)^{-1}F(x)$. Then, by definition,

$$H(x, y) = \frac{1}{\lambda}F'(x)^{-1}[F'(x - \lambda F'(x)^{-1}F(x)) - F'(x)]$$

$$\begin{aligned} &= -F'(x)^{-1}\int_0^1 F''(x-\tau\lambda F'(x)^{-1}F(x))F'(x)^{-1}F(x)d\tau \\ &= -F'(x)^{-1}F''(x)F'(x)^{-1}F(x) \\ &+ F'(x)^{-1}\int_0^1 [F''(x)-F''(x-\tau\lambda F'(x)^{-1}F(x))]F'(x)^{-1}F(x)d\tau. \end{aligned}$$

This gives that

$$\begin{aligned} \|H(x,y)\| \le\ & \|-F'(x)^{-1}F''(x)\|\|F'(x)^{-1}F(x)\| \\ & +\|F'(x)^{-1}F'(x^*)\|\int_0^1 \|F'(x^*)^{-1}[F''(x) \\ & -F''(x-\tau\lambda F'(x)^{-1}F(x))]\|d\tau\|F'(x)^{-1}F(x)\|. \end{aligned}$$

Thus, it follows from (3.2) that

$$\begin{aligned} \|H(x,y)\| \le\ & \|-F'(x)^{-1}F''(x)\|\|F'(x)^{-1}F(x)\| \\ & +\|F'(x)^{-1}F'(x^*)\|\int_0^1 L\|\lambda\tau F'(x)^{-1}F(x)\|d\tau\|F'(x)^{-1}F(x)\| \\ \le\ & \|-F'(x)^{-1}F''(x)\|\|F'(x)^{-1}F(x)\| \\ & +\frac{L\lambda}{2}\|F'(x)^{-1}F'(x^*)\|\|F'(x)^{-1}F(x)\|^2. \end{aligned}$$

Combing this with (3.7) and (3.6) implies that

$$\|H(x,y)\| \le -h'(t)^{-2}h(t)[h''(t)+\frac{L\lambda}{2}(h'(t)^{-1}h(t))] = 1-U(t).$$

Therefore, (3.8) holds. □

Lemma 3.3. *Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 1.1. Then*

(3.10)

$$\begin{aligned} Q(x_n,y_n)(y_n-x_n) &= -\frac{1}{2}F'(x_n)^{-1}F''(x_n)(x_n-x^*)^2 \\ &+ \frac{1}{2}F'(x_n)^{-1}F''(x_n)(x_n-x^*-F'(x_n)^{-1}F(x_n))(x_n-x^*) \\ &+ \frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)[I+H(x_n,y_n)]^{-1} \\ &\cdot \big(H(x_n,y_n)(x_n-x^*)+x_n-x^*-F'(x_n)^{-1}F(x_n)\big) \\ &+ \frac{1}{2}F'(x_n)^{-1}\int_0^1 [F''(x_n)-F''(x_n-\lambda(1-\tau)F'(x_n)^{-1}F(x_n))]\mathrm{d}\tau F'(x_n)^{-1}F(x_n) \\ &\cdot [I+H(x_n,y_n)]^{-1}F'(x_n)^{-1}F(x_n). \end{aligned}$$

Proof. By definition of Algorithm 1.1, we get

$$\begin{aligned} Q(x_n,y_n)(y_n-x_n) &= -\frac{1}{2\lambda}F'(x_n)^{-1}[F'(x_n)-F'(x_n-\lambda F'(x_n)^{-1}F(x_n))] \\ &\cdot [I+H(x_n,y_n)]^{-1}F'(x_n)^{-1}F(x_n) \\ &= -\frac{1}{2}F'(x_n)^{-1}\int_0^1 F''(x_n-\lambda(1-\tau)F'(x_n)^{-1}F(x_n))\mathrm{d}\tau F'(x_n)^{-1}F(x_n) \end{aligned}$$

$$\cdot [I + H(x_n, y_n)]^{-1}F'(x_n)^{-1}F(x_n)$$
$$= -\frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)[I + H(x_n, y_n)]^{-1}F'(x_n)^{-1}F(x_n)$$
$$+ \frac{1}{2}F'(x_n)^{-1}\int_0^1 [F''(x_n) - F''(x_n - \lambda(1-\tau)F'(x_n)^{-1}F(x_n))]\mathrm{d}\tau F'(x_n)^{-1}F(x_n)$$
$$\cdot [I + H(x_n, y_n)]^{-1}F'(x_n)^{-1}F(x_n).$$

Hence, it follows that

$$Q(x_n, y_n)(y_n - x_n) = -\frac{1}{2}F'(x_n)^{-1}F''(x_n)(x_n - x^*)^2$$
$$+ \frac{1}{2}F'(x_n)^{-1}F''(x_n)(x_n - x^* - F'(x_n)^{-1}F(x_n))(x_n - x^*)$$
$$+ \frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)[I + H(x_n, y_n)]^{-1}$$
$$\cdot \big(H(x_n, y_n)(x_n - x^*) + x_n - x^* - F'(x_n)^{-1}F(x_n)\big)$$
$$+ \frac{1}{2}F'(x_n)^{-1}\int_0^1 [F''(x_n) - F''(x_n - \lambda(1-\tau)F'(x_n)^{-1}F(x_n))]\mathrm{d}\tau F'(x_n)^{-1}F(x_n)$$
$$\cdot [I + H(x_n, y_n)]^{-1}F'(x_n)^{-1}F(x_n)$$

and so (3.10) is seen to hold. □

Now we are ready to have the main theorem about convergence of sequences generated by Algorithm 1.1.

Theorem 3.4. *Suppose that* (3.2) *holds for all* $x, x' \in \overline{\mathbf{B}(x^*, r_\lambda)}$ *and that* (3.1) *holds. Let* $x_0 \in \mathbf{B}(x^*, r_\lambda)$. *Then the sequence* $\{x_n\}$ *generated by Algorithm 1.1 is well defined and converges cubically to* x^*. *Furthermore,*

$$\|x_n - x^*\| \le q^{(3)^n - 1} \cdot \|x_0 - x^*\|, n = 0, 1, \ldots, \tag{3.11}$$

where

$$q = \sqrt{\frac{P_\lambda(t_0)}{t_0}} < 1, \quad t_0 = \|x_0 - x^*\| < r_\lambda. \tag{3.12}$$

Proof. Since $t_0 = \|x_0 - x^*\| < r_\lambda$, it follows from Lemma 2.3(ii) that $q = \sqrt{\frac{P_\lambda(t_0)}{t_0}} < 1$. Below, we will use mathematical induction to show that the sequence $\{x_n\}$ is well defined, $\{x_n\} \subset \mathbf{B}(x^*, r_\lambda)$ and (3.11) holds. Clearly, the case when $n = 0$ is trivial. Now we assume that $x_n \in \mathbf{B}(x^*, r_\lambda)$ and (3.11) holds for n. Below, we show that x_{n+1} is well defined, $x_{n+1} \in \mathbf{B}(x^*, r_\lambda)$ and (3.11) holds for $n+1$. To do this, since $x_n \in \mathbf{B}(x^*, r_\lambda)$, it follows from Lemma 3.1 that $F'(x_n)^{-1}$ exists. Let $t_n = \|x_n - x^*\|$. Then, by Lemma 3.2, we have

$$H(x_n, y_n) \le 1 - U(t_n).$$

This implies that

$$\|I + H(x_n, y_n)\| \ge 1 - \|H(x_n, y_n)\| \ge U(t_n) > U(r_\lambda) > U(r_1) = 0$$

because by Lemma 2.2 that U is monotone increasing on $(0, r_0)$, $U(r_1) = 0$ and $t_n < r_\lambda < r_1 < r_0$. Consequently, we obtain that $(I + H(x_n, y_n))^{-1}$ exists and so x_{n+1} is well defined. Thus, to complete the proof, it's sufficient to show that (3.11) holds for $n + 1$. By the definition of Algorithm 1.1, we have

$$x_{n+1} - x^* = x_n - x^* - F'(x_n)^{-1}F(x_n) + Q(x_n, y_n)(y_n - x_n).$$

Then, it follows from (3.5) and (3.10) that

$$\begin{aligned} x_{n+1} - x^* = &-F'(x_n)^{-1}\int_0^1 \tau[F''(x_n) - F''(x_n^\tau)]\mathrm{d}\tau(x_n - x^*)^2 \\ &+ \frac{1}{2}F'(x_n)^{-1}F''(x_n)\int_0^1 \tau F'(x_n)^{-1}F''(x_n^\tau)\mathrm{d}\tau(x_n - x^*)^3 \\ &+ \frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)[I + H(x_n, y_n)]^{-1} \\ &\cdot\left(H(x_n, y_n)(x_n - x^*) + \int_0^1 \tau F'(x_n)^{-1}F''(x_n^\tau)\mathrm{d}\tau(x_n - x^*)^2\right) \\ &+ \frac{1}{2}F'(x_n)^{-1}\int_0^1 [F''(x_n) \\ &- F''(x_n - \lambda(1-\tau)F'(x_n)^{-1}F(x_n))]\mathrm{d}\tau F'(x_n)^{-1}F(x_n) \\ &\cdot[I + H(x_n, y_n)]^{-1}F'(x_n)^{-1}F(x_n). \end{aligned} \tag{3.13}$$

Write

$$x_n^\tau = x^* + \tau(x_n - x^*) \quad \text{for each } 0 \le \tau \le 1.$$

Then, it follows from (3.13) that

$$\begin{aligned} \|x_{n+1} - x^*\| \le &\|F'(x_n)^{-1}F'(x^*)\|\int_0^1 \tau\|F'(x^*)^{-1}[F''(x_n) - F''(x_n^\tau)]\|\mathrm{d}\tau\|x_n - x^*\|^2 \\ &+ \frac{1}{2}\|F'(x_n)^{-1}F''(x_n)\|\int_0^1 \tau\|F'(x_n)^{-1}F''(x_n^\tau)\|\mathrm{d}\tau\|x_n - x^*\|^3 \\ &+ \frac{1}{2(1 - \|H(x_n, y_n)\|)}\|F'(x_n)^{-1}F''(x_n)\|\|F'(x_n)^{-1}F(x_n)\| \\ &\cdot\left(\|H(x_n, y_n)\| + \int_0^1 \tau\|F'(x_n)^{-1}F''(x_n^\tau)\|\mathrm{d}\tau\|x_n - x^*\|\right)\|x_n - x^*\| \\ &+ \frac{\|F'(x_n)^{-1}F'(x^*)\|}{2(1 - \|H(x_n, y_n)\|)}\int_0^1 \|F'(x^*)^{-1}[F''(x_n) \\ &- F''(x_n - \lambda(1-\tau)F'(x_n)^{-1}F(x_n))]\|\mathrm{d}\tau\|F'(x_n)^{-1}F(x_n)\|^2. \end{aligned}$$

Combing this with (3.2) yields that

$$\begin{aligned} \|x_{n+1} - x^*\| \le &\|F'(x_n)^{-1}F'(x^*)\|\int_0^1 \tau L\|x_n - x_n^\tau\|\mathrm{d}\tau\|x_n - x^*\|^2 \\ &+ \frac{1}{2}\|F'(x_n)^{-1}F''(x_n)\|\int_0^1 \tau\|F'(x_n)^{-1}F''(x_n^\tau)\|\mathrm{d}\tau\|x_n - x^*\|^3 \\ &+ \frac{1}{2(1 - \|H(x_n, y_n)\|)}\|F'(x_n)^{-1}F''(x_n)\|\|F'(x_n)^{-1}F(x_n)\| \end{aligned}$$

$$\cdot\left(\|H(x_n,y_n)\| + \int_0^1 \tau\|F'(x_n)^{-1}F''(x_n^\tau)\|\mathrm{d}\tau\|x_n - x^*\|\right)\|x_n - x^*\|$$
$$+\frac{\|F'(x_n)^{-1}F'(x^*)\|}{2(1-\|H(x_n,y_n)\|)}\int_0^1 L\|\lambda(1-\tau)F'(x_n)^{-1}F(x_n)\|\mathrm{d}\tau\|F'(x_n)^{-1}F(x_n)\|^2$$

This, together with (3.4), (3.7) and (3.6) (where t is replaced by t_n), gives that

$$\begin{aligned}\|x_{n+1}-x^*\| \le{}& \frac{-Lh'(t_n)^{-1}t_n^3}{6} + \frac{h'(t_n)^{-2}h''(t_n)(\gamma+2h''(t_n))}{12}t_n^3\\ &-\frac{L\lambda h'(t_n)^{-1}(h'(t_n)^{-1}h(t_n))^3}{4(1-\|H(x_n,y_n)\|)}\\ &-\frac{h'(t_n)^{-2}h''(t_n)h(t_n)}{2(1-\|H(x_n,y_n)\|)}t_n\left(\|H(x_n,y_n)\| + \frac{-h'(t_n)^{-1}(\gamma+2h''(t_n))t_n}{6}\right).\end{aligned}$$

Thus, it follows from (3.8) (with t replaced by t_n) that

$$\begin{aligned}&\|x_{n+1}-x^*\| \le -\frac{Lh'(t_n)^{-1}t_n^3}{6} + \frac{(\gamma+2h''(t_n))h'(t_n)^{-2}h''(t_n)t_n^3}{12}\\ &-\frac{L\lambda(h'(t_n)^{-1}h(t_n))^3h'(t_n)^{-1}}{4(1+h'(t_n)^{-2}h(t_n)[h''(t_n)+\frac{L\lambda}{2}(h'(t_n)^{-1}h(t_n))])}\\ &+\frac{h'(t_n)^{-3}h''(t_n)h(t_n)t_n(\frac{\gamma+2h''(t_n)}{6}t_n + \ h'(t_n)^{-1}h(t_n)[h''(t_n)+\frac{L\lambda}{2}(h'(t_n)^{-1}h(t_n))])}{2(1+h'(t_n)^{-2}h(t_n)[h''(t_n)+\frac{L\lambda}{2}(h'(t_n)^{-1}h(t_n))])}\\ &= P_\lambda(t_n).\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\| &\le P_\lambda(t_n) = \tfrac{P_\lambda(t_n)}{t_n^3}t_n^3 \le \tfrac{P_\lambda(t_0)}{t_0^3}(q^{(3)^n-1})^3\|x_0-x^*\|^3\\ &< q^{(3)^{n+1}-1}\|x_0-x^*\|.\end{aligned}$$

Then, (3.11) is seen to hold for $n+1$. The proof is completed. □

4. Application to a nonlinear integral equation of Hammerstein type

In this section, we provide an application of the main result to a special nonlinear Hammerstein integral equation of the second kind (*cf.* [15]). Letting $\mu \in \mathbf{R}$, we consider

$$x(s) = l(s) + \int_a^b G(s,t)[x(t)^3 + \mu x(t)^2]\mathrm{d}t, \quad s \in [a,b], \tag{4.1}$$

where l is a continuous function such that $l(s) > 0$ for all $s \in [a,b]$ and the kernel G is a non-negative continuous function on $[a,b]\times[a,b]$.

Note that if G is the Green function defined by

$$G(s,t) = \begin{cases} \frac{(b-s)(t-a)}{b-a}, & t \le s,\\ \frac{(s-a)(b-t)}{b-a}, & s \le t,\end{cases}$$

equation (4.1) is equivalent to the following boundary value problem (*cf.* [16]):

$$\begin{cases} x'' = -x^3 - \mu x^2\\ x(a) = v(a),\ x(b) = v(b).\end{cases}$$

To apply Theorem 3.4, let $X = Y = C[a,b]$, the Banach space of real-valued continuous functions on $[a,b]$ with the uniform norm, and let $\Omega = C[a,b]$. Define $F : \Omega \to C[a,b]$ by

$$[F(x)](s) = x(s) - l(s) - \int_a^b G(s,t)[x(t)^3 + \mu x(t)^2]\mathrm{d}t, \quad s \in [a,b]. \tag{4.2}$$

Then solving equation (4.1) is equivalent to solving equation (1.1) with F being defined by (4.2).

We start by calculating the parameters γ and L in the study. Firstly, we have

$$[F'(x)u](s) = u(s) - \int_a^b G(s,t)[3x(t)^2 + 2\mu x(t)]u(t)\mathrm{d}t, \quad s \in [a,b]$$

and

$$[F''(x)u\,z](s) = -\int_a^b G(s,t)[6x(t) + 2\mu]u(t)z(t)\mathrm{d}t, \quad s \in [a,b].$$

Let $x^* \in \Omega_p$ be fixed. Then

$$\|\mathbf{I} - F'(x^*)\| \le M(3\|x^*\|^2 + 2\mu\|x^*\|),$$

where

$$M = \max_{s\in[a,b]} \int_a^b |G(s,t)|\mathrm{d}t.$$

By the Banach Lemma, if

$$M(3\|x^*\|^2 + 2\mu\|x^*\|) < 1, \tag{4.3}$$

one has

$$\|F'(x^*)^{-1}\| \le \frac{1}{1 - M(3\|x^*\|^2 + 2\mu\|x^*\|)}.$$

Since

$$\|F''(x^*)\| \le M(6\|x^*\| + 2\mu),$$

it follows that

$$\|F'(x^*)^{-1}F''(x^*)\| \le \frac{M(6\|x^*\| + 2\mu)}{1 - M(3\|x^*\|^2 + 2\mu\|x^*\|)}. \tag{4.4}$$

Therefore,

$$\gamma = \frac{M(6\|x^*\| + 2\mu)}{1 - M(3\|x^*\|^2 + 2\mu\|x^*\|)} \tag{4.5}$$

is estimated.

On the other hand, for $x,\, y \in \Omega_p$,

$$[(F''(x) - F''(y))u\,z](s) = -6\int_a^b G(s,t)(x(t) - y(t))u(t)z(t)\mathrm{d}t, \quad s \in [a,b]$$

and consequently,

$$\|F'(x^*)^{-1}(F''(x) - F''(y))\| \le \frac{6M}{1 - M(3\|x^*\|^2 + 2\mu\|x^*\|)}\|x - y\|, \quad x,\, y \in \Omega.$$

This means that

$$L = \frac{6M}{1 - M(3\|x^*\|^2 + 2\mu\|x^*\|)}. \tag{4.6}$$

Thus, we can establish the following result from Theorem 3.4.

Theorem 4.1. *Let x^* be a solution of $F(x) = 0$ with F being defined by* (4.2)*. Suppose that* (4.3) *holds. Let γ and L be given by* (4.5) *and* (4.6)*, respectively. Let $x_0 \in \mathbf{B}(x^*, r_\lambda)$. Then the sequence $\{x_n\}$ generated by Algorithm 1.1 is well defined and converges cubically to x^*. Furthermore,*

$$\|x_n - x^*\| \le q^{(3)^n - 1} \cdot \|x_0 - x^*\|, n = 0, 1, \dots,$$

where

$$q = \sqrt{\frac{P_\lambda(t_0)}{t_0}} < 1, \ \ t_0 = \|x_0 - x^*\| < r_\lambda.$$

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