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CONVERGENCE CRITERIA OF THE GENERALIZED NEWTON METHOD AND UNIQUENESS OF SOLUTION FOR GENERALIZED EQUATIONS

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Abstract. Under a generalized L-average Lipschitz condition, we establish a convergence criterion around an initial point regarding the generalized Newton method for solving a generalized equation

$$0 \in F(x) + T(x),$$

where F is Fréchet differentiable and T is set-valued and maximal monotone. Moreover, we also get an estimation of uniqueness ball for a solution of the generalized equation. As applications, we obtain Kantorovich type theorem under the classical Lipschitz condition, convergence results under the γ-condition, and Smale's point estimate theory. Our results extend some corresponding results in [22].

1. Introduction

Let $T : H \rightrightarrows H$ be a (set-valued) maximal monotone operator, where H is a Hilbert space. Let $F : H \to H$ be a Fréchet differentiable function. Consider the following generalized equation: Find $x^* \in H$ such that

$$0 \in F(x^*) + T(x^*). \tag{1.1}$$

This problem has been studied in the literature (see [14–17, 22] and references therein). Problems from applied mathematical areas, such as variational inequality problems including linear and nonlinear complementary problems, systems of nonlinear equations, abstract inequality systems, etc., can be cast as problem (1.1). Furthermore, such problems have important applications in the physical and engineering sciences and in many other fields (cf. [3, 4, 13]).

The generalized Newton method is one of the most important methods for generalized problem (1.1), which is given as follows (cf. [22]): Let $x_0 \in H$ be given. Having $x_0, x_1, \ldots, x_n$, we define x_{n+1} such that

$$0 \in F(x_n) + F'(x_n)(x_{n+1} - x_n) + T(x_{n+1}).$$

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In [22], Uko established convergence analysis for the generalized Newton method under the classical Lipschitz condition. In particular, for the special case of variational inequality problems, the convergence of the generalized Newton method has been studied by Eaves [5], Robinson [16], Josephy [11], by Pang and Chan [13] (see also [10]) and some recent works (see [2, 8]).

Recall that the classical Newton method is one of the most important methods known for solving systems of nonlinear equations when they are continuously differentiable. It has been studied and used extensively (see [12, 19–21, 24, 25, 27] and the references therein). One of the most important results on Newton's method is Kantorovich's theorem (cf. [12]). Under the mild condition that the second Fréchet derivative of F is bounded (or more general, the first derivative is Lipschitz continuous) on a proper open metric ball of the initial point x_0, Kantorovich's theorem provides a simple and clear criterion, based on the knowledge of the first derivative around the initial point, ensuring the existence, uniqueness of the solution of the equation and the quadratic convergence of Newton method. Another important result on Newton method is Smale's point estimate theory (i.e., α-theory and γ-theory) in [19], where the rules to judge an initial point x_0 to be an approximate zero were established, depending on the information of the analytic nonlinear operator at this initial point and at a solution x^*, respectively. There are a lot of works on the weakness and/or the extension of the Lipschitz continuity made on the mappings; see for example, [6, 7, 9, 25] and references therein. In particular, Wang [25] introduced the notion of Lipschitz conditions with L-average to unify both Kantorovich's and Smale's criteria.

In sprit of Smale's point estimate theory [19, 20] and Wang's work in [25], the purpose of the present paper is to continue the study of the generalized Newton method for (1.1) under more generalized Lipschitz condition. Under a generalized L-average Lipschitz condition, we give a convergence criterion ensuring the convergence of the generalized Newton method around an initial point x_0 for solving the generalized equation. Moreover, we also get an estimation of uniqueness ball for the solution of (1.1). As applications, we obtain Kantorovich type theorem under the classical Lipschitz condition, convergence results under the γ-condition, and Smale's point estimate theory. Hence, our results extend some corresponding results in [22].

The paper is organized as follows. In Section 2, some notions, notations and preliminaries are provided. In Section 3, the convergence criterion is established under a generalized L-average Lipschitz condition, while in Section 4, we present an estimation of uniqueness ball of the solution of (1.1). In the final section, as applications, we get the Kantorovich type theorem under the classical Lipschitz condition, convergence results under the γ-condition, and Smale's point estimate theory.

2. Notions and preliminaries

Let $x \in H$ and $r > 0$. As usual, we use $\mathbf{B}(x, r)$ and $\overline{\mathbf{B}(x, r)}$ to denote, respectively, the open metric ball and the closed metric ball at x with radius r, that is,

$$\mathbf{B}(x, r) := \{y \in H | \, \|x - y\| < r\} \quad \text{and} \quad \overline{\mathbf{B}(x, r)} := \{y \in H | \, \|x - y\| \leq r\}.$$

Recall that a bounded linear operator $G: H \to H$ is called a positive operator if G is self-conjugate and $\langle Gx, x\rangle \geq 0$ for each $x \in H$ (cf. [18, p. 313]). The following lemma about properties of positive operators is taken from [23].

Lemma 2.1. *Let G be a positive operator. Then the following conclusions hold:*
(i) $\|G^2\| = \|G\|^2$.
(ii) *If G^{-1} exists, then G^{-1} also is a positive operator and*

$$\langle Gx, x\rangle \geq \frac{\|x\|^2}{\|G^{-1}\|} \quad \textit{for each } x \in H. \tag{2.1}$$

Let $T : H \rightrightarrows H$ be a set-valued operator. The domain dom$T$ of T is defined as dom$T := \{x \in H |\ T(x) \neq \emptyset\}$. Below, we recall notions of monotonicity for set-valued operators (see [1, 28] for details).

Definition 2.2. Let $T : H \rightrightarrows H$ be a set-valued operator. T is said to be
(a) *monotone* if the following condition holds for any $x, y \in \text{dom}T$:

$$\langle u - v, y - x\rangle \geq 0 \quad \text{for each } u \in T(y) \text{ and } v \in T(x); \tag{2.2}$$

(b) *maximal monotone* if it is monotone and the following implication holds for any $x, u \in H$:

(2.3)

$$\langle u - v, x - y\rangle \geq 0 \ \text{ for each } y \in \text{dom}T \text{ and } v \in T(y) \Longrightarrow x \in \text{dom}T \text{ and } u \in T(x).$$

Throughout the whole paper, let R be a positive constant and $L(\cdot)$ be a non-negative nondecreasing integrable function on $[0, R)$ satisfying

$$\int_0^R L(s)\mathrm{d}s \geq 1.$$

A generalized Lipschitz condition with L-average has been introduced in [25]. Below, we extend generalized Lipschitz condition with L-average for operators on Hilbert spaces which is slightly different from that in [25]. Throughout the whole paper, for any bounded linear operator $G : H \to H$, we always adopt the convention that $\widehat{G} := \frac{1}{2}(G + G^*)$ where G^* is the conjugate operator of G. Clearly, $\widehat{G}$ is a self-conjugate operator. Throughout the whole paper, we always assume that $T : H \rightrightarrows H$ is a (set-valued) maximal monotone operator and $F : H \to H$ is a Fréchet differentiable function.

Definition 2.3. Let $r > 0$ and $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists. Then $\|\widehat{F'(\bar{x})}^{-1}\|F'$ is said to satisfy
(a) the center Lipschitz condition with L-average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$ if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x) - F'(\bar{x})\| \leq \int_0^{\|x-\bar{x}\|} L(u)\mathrm{d}u \quad \text{for each } x \in \mathbf{B}(\bar{x}, r).$$

(b) the radius Lipschitz condition with L-average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$ if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x) - F'(x^\tau)\| \leq \int_{\tau\|x-\bar{x}\|}^{\|x-\bar{x}\|} L(u)\mathrm{d}u \quad \text{for each } x \in \mathbf{B}(\bar{x}, r), 0 \leq \tau \leq 1,$$

where $x^\tau = \bar{x} + \tau(x - \bar{x})$.

(c) the center Lipschitz condition in the inscribed sphere with L-average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$ if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x') - F'(x)\| \leq \int_{\|x-\bar{x}\|}^{\|x-\bar{x}\|+\|x'-x\|} L(u)\mathrm{d}u \quad \text{for each } x, x' \in \mathbf{B}(\bar{x}, r),$$

where $\|x - \bar{x}\| + \|x - x'\| < r$.

Let $r_0 > 0$ be such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1. \tag{2.4}$$

The following lemma is taken from [23] and is useful in the next section.

Lemma 2.4. *Let $r < r_0$. Let $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}$ is a positive operator and $\widehat{F'(\bar{x})}^{-1}$ exists. Suppose that $\|\widehat{F'(\bar{x})}^{-1}\|F'$ satisfies the center Lipschitz condition with L-average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$. Then, for each $x \in \mathbf{B}(\bar{x}, r)$, $\widehat{F'(x)}$ is a positive operator and $\widehat{F'(x)}^{-1}$ exists. Moreover,*

$$\|\widehat{F'(x)}^{-1}\| \leq \frac{\|\widehat{F'(\bar{x})}^{-1}\|}{1 - \int_0^{\|x-\bar{x}\|} L(u)\mathrm{d}u}. \tag{2.5}$$

3. Convergence criterion

Let $T : H \rightrightarrows H$ be a (set-valued) maximal monotone operator. Let $F : H \to H$ be a Fréchet differentiable function. Consider the following generalized equation: Find $x^* \in H$ such that

$$0 \in F(x^*) + T(x^*). \tag{3.1}$$

Newton's method for the generalized equation (3.1) is given as follows:

Algorithm **3.1**. Let $x_0 \in H$ be given. Have $x_0, x_1, \ldots, x_n$. Define x_{n+1} such that

$$0 \in F(x_n) + F'(x_n)(x_{n+1} - x_n) + T(x_{n+1}). \tag{3.2}$$

Remark 3.1. Have x_n. If there exists a constant $c > 0$ such that

$$\langle F'(x_n)y, y\rangle \geq c\|y\|^2 \quad \text{for each } y \in H, \tag{3.3}$$

then there exists a unique point x_{n+1} such that (3.2) holds because T is maximal monotone (see [22, Lemma 2.2]). Hence, if for each n, there exists a constant $c > 0$ such that (3.3) holds, then the sequence generated by (3.2) is well defined.

The majorizing function h defined in the following, which was first introduced and studied by Wang (cf. [25]), is a powerful tool in our study. For $\beta > 0$, define the majorizing function h by

$$h(t) = \beta - t + \int_0^t L(u)(t - u)\mathrm{d}u \quad \text{for each } 0 \leq t \leq R. \tag{3.4}$$

Some useful properties are described in the following proposition, see [25]. Let $r_0 > 0$ and $b > 0$ be such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1 \quad \text{and} \quad b = \int_0^{r_0} L(u)u\mathrm{d}u. \tag{3.5}$$

Proposition 3.2. *The function h is monotonic decreasing on $[0, r_0]$ and monotonic increasing on $[r_0, R]$. Moreover, if $\beta \le b$, h has a unique zero respectively in $[0, r_0]$ and $[r_0, R]$, which are denoted by r_1 and r_2. They satisfy*

$$\beta < r_1 < \frac{r_0}{b}\beta < r_0 < r_2 < R \tag{3.6}$$

if $\beta < b$ and $r_1 = r_2$ if $\beta = b$.

Let $\{t_n\}$ denote the sequence generated by Newton's method with the initial value $t_0 = 0$ for h, that is,

$$t_{n+1} = t_n - h'(t_n)^{-1}h(t_n) \quad \text{for each } n = 0, 1, \ldots. \tag{3.7}$$

Let $x_0 \in \Omega$ be such that $\widehat{F'(x_0)}^{-1}$ exists. The main result of this paper is as follows.

Theorem 3.3. *Suppose that $\beta \le b$ and $\|\widehat{F'(x_0)}^{-1}\|F'$ satisfies the center Lipschitz condition in the inscribed sphere with L-average at x_0 on $\mathbf{B}(x_0, r_1)$ and $F'(x_0)$ is a positive operator (not necessary symmetric). Let $\{x_n\}$ be a sequence generated by Newton's method* (3.2) *with initial point x_0 and*

$$\|x_1 - x_0\| \le \beta. \tag{3.8}$$

Then, $\{x_n\}$ is well defined, and converges to a solution x^* of (3.1) *in $\overline{\mathbf{B}(x_0, r_1)}$. Moreover, there hold*

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \quad \text{and} \quad \|x_n - x^*\| \le r_1 - t_n \quad \text{for each } n = 0, 1, \ldots. \tag{3.9}$$

Proof. We will use mathematical induction to prove that $\{x_n\}$ is well defined and

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \tag{3.10}$$

holds for each $n = 0, 1, \ldots$. The case when $n = 0$ is trivial because of assumption (3.8). Suppose that (3.10) holds for $n = 0, 1, \ldots, k-1$. Below, we show that (3.10) holds for $n = k$. Note that

$$\|x_k - x_0\| \le \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \le t_k - t_{k-1} + \cdots + t_1 - t - 0 = t_k - t_0 < r_1.$$

Since $F'(x_0)$ is a positive operator and $\widehat{F'(x_0)}^{-1}$ exists, it follows from Lemma 2.4 that $\widehat{F'(x_k)}$ is a positive operator, $\widehat{F'(x_k)}^{-1}$ exists and

$$\|\widehat{F'(x_k)}^{-1}\| \le \frac{\|\widehat{F'(x_0)}^{-1}\|}{1 - \displaystyle\int_0^{\|x_k - x_0\|} L(u)\mathrm{d}u}. \tag{3.11}$$

Then, one obtains from Lemma 2.1(ii) that

$$\frac{\|x\|^2}{\|\widehat{F'(x_k)}^{-1}\|} \le \langle \widehat{F'(x_k)}x, x\rangle = \langle F'(x_k)x, x\rangle \quad \text{for each } x \in H. \tag{3.12}$$

Consequently, we get from Remark 3.1 that there exists a unique point x_{k+1} such that

$$0 \in F(x_k) + F'(x_k)(x_{k+1} - x_k) + T(x_{k+1}). \tag{3.13}$$

Observe from assumption that

$$0 \in F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1}) + T(x_k). \tag{3.14}$$

Since T is maximal monotone, we get from (3.14) and (3.13) that

$$\langle -F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1}) + F(x_k) + F'(x_k)(x_{k+1} - x_k), x_k - x_{k+1}\rangle \geq 0$$

This gives that

$$\langle F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1}), x_k - x_{k+1}\rangle \geq \langle F'(x_k)(x_k - x_{k+1}), x_k - x_{k+1}\rangle. \tag{3.15}$$

Observe from (3.12) that

$$\frac{\|x_k - x_{k+1}\|^2}{\|\widehat{F'(x_k)}^{-1}\|} \leq \langle \widehat{F'(x_k)}(x_k - x_{k+1}), x_k - x_{k+1}\rangle = \langle F'(x_k)(x_k - x_{k+1}), x_k - x_{k+1}\rangle.$$

Combing this with (3.15) yields that

$$\|x_k - x_{k+1}\| \leq \|\widehat{F'(x_k)}^{-1}\|\|F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1})\|. \tag{3.16}$$

Note that

$$\begin{aligned} & F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1}) \\ &= \int_0^1 F'(x_{k-1} + t(x_k - x_{k-1}))(x_k - x_{k-1})\mathrm{d}t - F'(x_{k-1})(x_k - x_{k-1}) \\ &= \int_0^1 (F'(x_{k-1} + t(x_k - x_{k-1})) - F'(x_{k-1}))(x_k - x_{k-1})\mathrm{d}t \end{aligned} \tag{3.17}$$

Since $\|\widehat{F'(x_0)}^{-1}\|F'$ satisfies the center Lipschitz condition in the inscribed sphere with L-average at x_0 on $\mathbf{B}(x_0, r_1)$, we get from (3.11), (3.16) and (3.17) that

$$\begin{aligned} & \|x_k - x_{k+1}\| \\ & \leq \frac{\|\widehat{F'(x_0)}^{-1}\|}{1 - \int_0^{\|x_k - x_0\|} L(u)\mathrm{d}u} \int_0^1 \|F'(x_{k-1} + \tau(x_k - x_{k-1})) - F'(x_{k-1})\|\|x_k - x_{k-1}\|\mathrm{d}\tau \\ & \leq \frac{1}{1 - \int_0^{\|x_k - x_0\|} L(u)\mathrm{d}u} \int_0^1 \int_{\|x_{k-1} - x_0\|}^{\|x_{k-1} - x_0\| + \tau\|x_k - x_{k-1}\|} L(u)\mathrm{d}u\|x_k - x_{k-1}\|\mathrm{d}\tau \\ & \leq \frac{1}{1 - \int_0^{t_k - t_0} L(u)\mathrm{d}u} \int_0^1 \int_{t_{k-1} - t_0}^{t_{k-1} - t_0 + \tau(t_k - t_{k-1})} L(u)\mathrm{d}u(t_k - t_{k-1})\mathrm{d}\tau \\ & = t_{k+1} - t_k. \end{aligned}$$

Hence, (3.10) holds for $n = k$. So $\{x_k\}$ is a Cauchy sequence which has a limit $x^* \in \overline{\mathbf{B}(x_0, r_1)}$. Since F is a C^1 mapping and T is maximal monotone, we get that x^* solves (3.1). □

4. Uniqueness ball of a solution around initial point

In this section, we give an estimation of uniqueness ball of solution around initial point. Let

$$t_{n+1} = t_n + h(t_n) \quad \text{for each } n = 0, 1, \ldots, \tag{4.1}$$

where $t_0 = 0$**.** Note that the function $t \mapsto t + h(t)$ increases monotonically on $[0, r_1]$ and $t_0 = 0 < t_1 = \beta < r_1$. It's easy to verify that the sequence $\{t_n\}$ increases monotonically and $\lim_{n\to\infty} t_n = r_1$.

Theorem 4.1. *Let $\beta \le b$. Let $r_1 \le r < r_2$ if $\beta < b$, and $r = r_1$ if $\beta = b$. Suppose that $\|\widehat{F'(x_0)}^{-1}\|F'$ satisfies the center Lipschitz condition with L-average at x_0 on $\overline{\mathbf{B}(x_0, r)}$ and $F'(x_0)$ is a positive operator (not necessary symmetric). Let $x_1 \in H$ be such that*

$$0 \in F(x_0) + F'(x_0)(x_1 - x_0) + T(x_1)$$

and $\|x_1 - x_0\| \le \beta$. Then, there exists a unique solution x^* of (3.1) *in $\overline{\mathbf{B}(x_0, r)}$.*

Proof. Let $\{t_n\}$ be a sequence given by (4.1). Let $\{x_n\}$ be a sequence generated by the following algorithm with initial point x_0:

$$0 \in F(x_n) + F'(x_0)(x_{n+1} - x_n) + T(x_{n+1}).$$

Since $\widehat{F'(x_0)}^{-1}$ exists and $F'(x_0)$ is a positive operator, $\{x_n\}$ is well defined because of Lemma 2.1(ii) and Remark 3.1. Below, we show that

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \quad \text{for each } n = 0, 1, \ldots. \tag{4.2}$$

Granting this, $\{x_n\}$ is a Cauchy sequence and converges to a solution x^* of (3.1) due to the fact that F is Fréchet differentiable and T is maximal monotone. The case when $n = 0$ is trivial because of assumption that $\|x_1 - x_0\| \le \beta = t_1 - t_0$. To proceed, assume that (4.2) holds for $n = 0, 1, \ldots, k-1$. Observe that

$$0 \in F(x_{k-1}) + F'(x_0)(x_k - x_{k-1}) + T(x_k)$$

and

$$0 \in F(x_k) + F'(x_0)(x_{k+1} - x_k) + T(x_{k+1}).$$

Since T is maximal monotone, it follows that

$$\langle F(x_k) + F'(x_0)(x_{k+1} - x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1}), x_k - x_{k+1}\rangle \ge 0$$

and so

$$\langle F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1}), x_k - x_{k+1}\rangle \ge \langle F'(x_0)(x_k - x_{k+1}), x_k - x_{k+1}\rangle. \tag{4.3}$$

Using Lemma 2.1(ii), we get

$$\begin{aligned} \frac{\|x_k - x_{k+1}\|^2}{\|\widehat{F'(x_0)}^{-1}\|} &\le \langle \widehat{F'(x_0)}(x_k - x_{k+1}), x_k - x_{k+1}\rangle \\ &= \langle F'(x_0)(x_k - x_{k+1}), x_k - x_{k+1}\rangle. \end{aligned}$$

Combing this with (4.3) yields that

$$\|x_k - x_{k+1}\| \le \|\widehat{F'(x_0)}^{-1}\|\|F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1})\|. \tag{4.4}$$

Note that

$$F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1}) = \int_0^1 (F'(x_{k-1} + \tau(x_k - x_{k-1})) - F'(x_0))(x_k - x_{k-1})\mathrm{d}\tau.$$

This, together with (4.4) and the assumption that $\|\widehat{F'(x_0)}^{-1}\|F'$ satisfies the center Lipschitz condition with L-average at x_0 on $\overline{\mathbf{B}(x_0, r)}$, implies that

$$\begin{aligned} \|x_k - x_{k+1}\| &\le \|\widehat{F'(x_0)}^{-1}\| \int_0^1 \|F'(x_{k-1} + \tau(x_k - x_{k-1})) - F'(x_0)\|\|x_k - x_{k-1}\|\mathrm{d}\tau \\ &\le \int_0^1 \int_0^{\|x_{k-1}-x_0\|+\tau\|x_k - x_{k-1}\|} L(u)\mathrm{d}u\|x_k - x_{k-1}\|\mathrm{d}\tau \\ &\le \int_0^1 \int_0^{t_{k-1}+\tau(t_k - t_{k-1})} L(u)\mathrm{d}u(t_k - t_{k-1})\mathrm{d}\tau \\ &= \int_0^{t_k} L(u)(t_k - u)\mathrm{d}u - \int_0^{t_{k-1}} L(u)(t_{k-1} - u)\mathrm{d}u \\ &= t_{k+1} - t_k. \end{aligned}$$

Hence, (4.2) holds for $n = k$.

Let $x_0' \in \overline{\mathbf{B}(x_0, r)}$, and let $t_0' = \|x_0' - x_0\|$. Set

$$t_{n+1}' = t_n' + h(t_n') \quad \text{for each } n = 0, 1, \ldots.$$

Then, it's easy to verify that $\{t_n'\}$ converges to r_1. Consider the following algorithm with initial point x_0':

$$0 \in F(x_n') + F'(x_0)(x_{n+1}' - x_n') + T(x_{n+1}').$$

Since $\widehat{F'(x_0)}^{-1}$ exists and $F'(x_0)$ is a positive operator, it follows from Lemma 2.1(ii) and Remark 3.1 that $\{x_n'\}$ is well defined. Below, we show that

$$\|x_n' - x_n\| \le t_n' - t_n \quad \text{for each } n = 0, 1, \ldots. \tag{4.5}$$

Grating this, we have

$$\lim_{n\to\infty} x_n' = \lim_{n\to\infty} x_n = x^*,$$

which implies that x^* is a unique solution of (3.1) in $\overline{\mathbf{B}(x_0, r)}$. The case when $n = 0$ is trivial because of assumption that $\|x_0' - x_0\| = t_0' - t_0$. To proceed, assume that (4.5) holds for $n = 0, 1, \ldots, k$. Observe that

$$0 \in F(x_k') + F'(x_0)(x_{k+1}' - x_k') + T(x_{k+1}')$$

and

$$0 \in F(x_k) + F'(x_0)(x_{k+1} - x_k) + T(x_{k+1}).$$

Since T is maximal monotone, it follows that

$$\langle F(x_k') + F'(x_0)(x_{k+1}' - x_k') - F(x_k) - F'(x_0)(x_{k+1} - x_k), x_{k+1} - x_{k+1}'\rangle \ge 0$$

and so

$$\langle F(x_k') - F(x_k) - F'(x_0)(x_k' - x_k), x_{k+1} - x_{k+1}'\rangle \ge \langle F'(x_0)(x_{k+1} - x_{k+1}'), x_{k+1} - x_{k+1}'\rangle. \tag{4.6}$$

Using Lemma 2.1(ii), we get

$$\begin{aligned}\frac{\|x'_{k+1}-x_{k+1}\|^2}{\|\widehat{F'(x_0)}^{-1}\|} &\leq \langle \widehat{F'(x_0)}(x_{k+1}-x'_{k+1}), x_{k+1}-x'_{k+1}\rangle \\ &= \langle F'(x_0)(x_{k+1}-x'_{k+1}), x_{k+1}-x'_{k+1}\rangle.\end{aligned}$$

Combing this with (4.6) yields that

$$\|x_{k+1}-x'_{k+1}\| \leq \|\widehat{F'(x_0)}^{-1}\|\|F(x'_k)-F(x_k)-F'(x_0)(x'_k-x_k)\|. \tag{4.7}$$

Note that

$$F(x'_k)-F(x_k)-F'(x_0)(x'_k-x_k) = \int_0^1 (F'(x_k+\tau(x'_k-x_k))-F'(x_0))(x'_k-x_k)\mathrm{d}\tau.$$

This, together with (4.7) and the assumption that $\|\widehat{F'(x_0)}^{-1}\|F'$ satisfies the center Lipschitz condition with L-average at x_0 on $\overline{\mathbf{B}(x_0,r)}$, implies that

$$\begin{aligned}\|x_{k+1}-x'_{k+1}\| &\leq \|\widehat{F'(x_0)}^{-1}\|\int_0^1 \|F'(x_k+\tau(x'_k-x_k))-F'(x_0)\|\|x'_k-x_k\|\mathrm{d}\tau \\ &\leq \int_0^1\int_0^{\|x_k-x_0\|+\tau\|x'_k-x_k\|} L(u)\mathrm{d}u\|x'_k-x_k\|\mathrm{d}\tau \\ &\leq \int_0^1\int_0^{t_k+\tau(t'_k-t_k)} L(u)\mathrm{d}u(t'_k-t_k)\mathrm{d}\tau \\ &= \int_0^{t'_k} L(u)(t'_k-u)\mathrm{d}u - \int_0^{t_k} L(u)(t_k-u)\mathrm{d}u \\ &= t'_{k+1}-t_{k+1}.\end{aligned}$$

Hence, (4.5) holds for $n=k+1$. The proof is completed. □

5. Applications

This section is devoted to the application of our previous results for some special cases such as the classical Lipschitz condition and the γ-condition.

5.1. **The classical Lipschitz condition.** Let $L>0$ be a constant, and let $r>0$. Let $x_0\in H$ be such that $\widehat{F'(x_0)}^{-1}$ exists. Then $\|\widehat{F'(x_0)}^{-1}\|F'$ is said to satisfy the Lipschitz condition on $\mathbf{B}(x_0,r)$ if

$$\|\widehat{F'(x_0)}^{-1}\|\|F'(x)-F'(x')\| \leq L\|x-x'\| \quad \text{for each } x,x'\in\mathbf{B}(x_0,r),$$

where $\|x-x_0\|+\|x'-x\|<r$.

Since $L(\cdot)\equiv L$, the majorizing function h is reduced to

$$h(t)=\beta-t+\frac{1}{2}Lt^2.$$

Furthermore, it follows from (3.5) that

$$r_0=\frac{1}{L} \quad \text{and} \quad b=\frac{1}{2L}.$$

If $\lambda = L\beta \le \frac{1}{2}$, h has two zeroes

$$r_1 = \frac{1-\sqrt{1-2\lambda}}{L} \quad \text{and} \quad r_2 = \frac{1+\sqrt{1-2\lambda}}{L}.$$

Moreover,

$$\beta \le r_1 \le 2\beta \le \frac{1}{L} \le r_2 \le \frac{2}{L}.$$

Let $\{t_n\}$ denote the sequence generated by Newton's method with the initial value $t_0 = 0$ for h. Then,

$$t_n = \frac{1-q^{2^n-1}}{1-q^{2^n}} r_1,$$

where $q = \frac{1-\sqrt{1-2\lambda}}{1+\sqrt{1-2\lambda}}$.

Hence, the following two corollaries follow directly from Theorems 3.3 and 4.1,respectively.

Corollary 5.1. *Suppose that* $\beta \le \frac{1}{2L}$ *and* $\|\widehat{F'(x_0)}^{-1}\|F'$ *satisfies the Lipschitz condition on* $\mathbf{B}(x_0, r_1)$ *and* $F'(x_0)$ *is a positive operator (not necessary symmetric). Let* $\{x_n\}$ *be a sequence generated by Newton's method* (3.2) *with initial point* x_0 *and*

$$\|x_1 - x_0\| \le \beta.$$

Then, $\{x_n\}$ *is well defined, and converges to a solution* x^* *of* (3.1) *in* $\overline{\mathbf{B}(x_0, r_1)}$. *Moreover, there holds*

$$\|x_n - x^*\| \le \frac{1-q}{1-q^{2^n}} q^{2^n-1} r_1 \le q^{2^n-1} r_1.$$

Corollary 5.2. *Let* $\beta \le \frac{1}{2L}$. *Let* $r_1 \le r < r_2$ *if* $\beta < \frac{1}{2L}$, *and* $r = r_1$ *if* $\beta = \frac{1}{2L}$. *Suppose that* $\|\widehat{F'(x_0)}^{-1}\|F'$ *satisfies the Lipschitz condition at* x_0 *on* $\overline{\mathbf{B}(x_0, r)}$ *and* $F'(x_0)$ *is a positive operator (not necessary symmetric). Let* $x_1 \in H$ *be such that*

$$0 \in F(x_0) + F'(x_0)(x_1 - x_0) + T(x_1)$$

and $\|x_1 - x_0\| \le \beta$. *Then, there exists a unique solution* x^* *of* (3.1) *in* $\overline{\mathbf{B}(x_0, r)}$.

Remark 5.3. Note that Corollary 5.1 has been given in [22, Theroem 2.11], while Corollary 5.2 extends corresponding results in [22, Theroem 2.10] by bigger radius of uniqueness ball.

5.2. **The γ-condition.** Let $r > 0$ and $\gamma > 0$ be such that $\gamma r \le 1$. In this subsection, we always assume that $F : H \to H$ is a C^2 function. The γ-conditions for operators in Banach space were first presented by Wang [27] for the study of Smale's point estimate theory. Below, it's an analogue of γ-condition for operators, which has been given in [23] and is slightly different from the one given in [27].

Definition 5.4. Let $x_0 \in H$ be such that $\widehat{F'(x_0)}^{-1}$ exists. F is said to satisfy the γ-condition at x_0 in $\mathbf{B}(x_0, r)$, if

$$\|\widehat{F'(x_0)}^{-1}\| \cdot \|F''(x)\| \le \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3} \quad \text{for each } x \in \mathbf{B}(x_0, r). \tag{5.1}$$

The following proposition shows that the γ-condition implies the radius Lipschitz condition with L-average, where the function L is defined by

$$L(u) := \frac{2\gamma}{(1-\gamma u)^3}, \quad \forall u \in [0, r). \tag{5.2}$$

Proposition 5.5. *Let $x_0 \in H$ be such that $\widehat{F'(x_0)}^{-1}$ exists. Suppose that F satisfies the γ-condition at x_0 in $\mathbf{B}(x_0, r)$. Then $\|\widehat{F'(x_0)}^{-1}\|F'$ satisfies the center Lipschitz condition in the inscribed sphere with L-average at x_0 on $\mathbf{B}(x_0, r)$, where L is given by* (5.2)*.*

Proof. Let $x, x' \in \mathbf{B}(x_0, r)$ be such that $\|x - x_0\| + \|x' - x\| < r$. Then

$$F'(x') - F'(x) = \int_0^1 F''(x + s(x' - x))(x' - x)\mathrm{d}s.$$

Hence, it follows

$$\begin{aligned}\|\widehat{F'(x_0)}^{-1}\|\|F'(x') - F'(x)\| &\leq \int_0^1 \|\widehat{F'(x_0)}^{-1}\|\|F''(x + s(x' - x))\|\|x' - x\|\mathrm{d}s \\ &\leq \int_0^1 \frac{2\gamma\|x' - x\|}{(1 - \gamma(\|x - x_0\| + s\|x' - x\|))^3}\mathrm{d}s \\ &= \int_{\|x - x_0\|}^{\|x - x_0\| + \|x' - x\|} \frac{2\gamma}{(1 - \gamma u)^3}\mathrm{d}u.\end{aligned}$$

Thus, the conclusion follows. □

For $L(\cdot)$ given by (5.2), the majoring function h is reduced to

$$h(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t} \quad \text{for each } 0 \leq t < \frac{1}{\gamma}. \tag{5.3}$$

Furthermore, it follows from (3.5) that

$$r_0 = \frac{2 - \sqrt{2}}{2\gamma} \quad \text{and} \quad b = \frac{3 - 2\sqrt{2}}{\gamma}.$$

Let $\{t_k\}$ denote the sequence generated by Newton's method with the initial value $t_0 = 0$ for h, that is,

$$t_{k+1} = t_k - h'(t_k)^{-1}h(t_k) \quad \text{for each } k = 0, 1, \ldots. \tag{5.4}$$

Then we have the following proposition which was proved in [24, 26].

Proposition 5.6. *Suppose that $\alpha = \gamma\beta \leq 3 - 2\sqrt{2}$. Then the zeros of h are*

$$r_1 = \frac{1 + \alpha - \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma}, \quad r_2 = \frac{1 + \alpha + \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma} \tag{5.5}$$

and satisfy

$$\beta \leq r_1 \leq (1 + \frac{1}{\sqrt{2}})\beta \leq (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma} \leq r_2 \leq \frac{1}{2\gamma}. \tag{5.6}$$

Moreover,

$$t_k = \frac{1-\mu^{2^k-1}}{1-\mu^{2^k-1}\eta} r_1 \tag{5.7}$$

and

$$t_{k+1} - t_k = \frac{(1-\mu^{2^k})\sqrt{(1+\alpha)^2 - 8\alpha}}{2\alpha(1-\eta\mu^{2^k-1})(1-\eta\mu^{2^{k+1}-1})} \eta\mu^{2^k-1}\beta, \quad k = 0, 1, \ldots, \tag{5.8}$$

where

$$\mu = \frac{1-\alpha-\sqrt{(1+\alpha)^2-8\alpha}}{1-\alpha+\sqrt{(1+\alpha)^2-8\alpha}} \tag{5.9}$$

and

$$\eta = \frac{1+\alpha-\sqrt{(1+\alpha)^2-8\alpha}}{1+\alpha+\sqrt{(1+\alpha)^2-8\alpha}}. \tag{5.10}$$

Lemma 5.7 below was known in [24, 26].

Lemma 5.7. *Suppose that* $\alpha < 3 - 2\sqrt{2}$. *Then*

$$\frac{(1-\mu^{2^k})\sqrt{(1+\alpha)^2 - 8\alpha}}{2\alpha(1-\eta\mu^{2^k-1})(1-\eta\mu^{2^{k+1}-1})} \eta \le 1, \quad k = 0, 1, \ldots. \tag{5.11}$$

Recall that F is a C^2 mapping. In the remainder of this section, let $x_0 \in H$ be such that $\widehat{F'(x_0)}^{-1}$ exists and define $\alpha := \gamma\beta$.

Then, the following corollary follows directly from Propositions 5.5 and 5.6, and Theorem 3.3.

Corollary 5.8. *Let*

$$\alpha = \beta\gamma \le 3 - 2\sqrt{2}.$$

Suppose that F satisfies the γ-condition at x_0 in $\mathbf{B}(x_0, r_1)$ and $F'(x_0)$ is a positive operator (not necessary symmetric). Let $\{x_n\}$ be a sequence generated by Newton's method (3.2) *with initial point x_0 and*

$$\|x_1 - x_0\| \le \beta.$$

Then, $\{x_n\}$ is well defined, and converges to a solution x^* of (3.1) *in $\overline{\mathbf{B}(x_0, r_1)}$. Moreover, there holds*

$$\|x_{k+1} - x_k\| \le \frac{(1-\mu^{2^k})\sqrt{(1+\alpha)^2 - 8\alpha}}{2\alpha(1-\eta\mu^{2^k-1})(1-\eta\mu^{2^{k+1}-1})} \eta\mu^{2^k-1} \|x_1 - x_0\|$$

for all $k = 0, 1, 2, \ldots,$ where μ, η are given by (5.9) *and* (5.10) *respectively.*

By (5.11), we arrive at the following corollary from Theorem 3.3.

Corollary 5.9. *Let*

$$\alpha = \beta\gamma < 3 - 2\sqrt{2}.$$

Suppose that F satisfies γ-condition at x_0 in $\mathbf{B}(x_0, r_1)$ and $F'(x_0)$ is a positive operator (not necessary symmetric). Let $\{x_n\}$ be a sequence generated by Newton's method (3.2) *with initial point x_0 and*

$$\|x_1 - x_0\| \leq \beta.$$

Then, $\{x_n\}$ is well defined, and converges to a solution x^* of (3.1) *in $\overline{\mathbf{B}(x_0, r_1)}$. Moreover,*

$$\|x_{k+1} - x_k\| \leq \mu^{2^k - 1}\|x_1 - x_0\|, \quad k = 0, 1, \dots,$$

where μ is defined by (5.9).

The following corollary follows directly from Propositions 5.5 and 5.6, and Theorem 4.1.

Corollary 5.10. *Let $\beta \leq \frac{3-2\sqrt{2}}{\gamma}$. Let $r_1 \leq r < r_2$ if $\beta < \frac{3-2\sqrt{2}}{\gamma}$, and $r = r_1$ if $\beta = \frac{3-2\sqrt{2}}{\gamma}$. Suppose that F satisfies the γ-condition at x_0 in $\overline{\mathbf{B}(x_0, r)}$ and $F'(x_0)$ is a positive operator (not necessary symmetric). Let $x_1 \in H$ be such that*

$$0 \in F(x_0) + F'(x_0)(x_1 - x_0) + T(x_1)$$

and $\|x_1 - x_0\| \leq \beta$. Then, there exists a unique solution x^* of (3.1) *in $\overline{\mathbf{B}(x_0, r)}$.*

5.3. **Analytic cases.** In the remainder of this section, we assume that F is analytic on $\mathbf{B}(x_0, r)$. Let $x \in \mathbf{B}(x_0, r)$ be such that $\widehat{F'(x)}^{-1}$ exists. Define

$$\gamma(F, x) := \|\widehat{F'(x)}^{-1}\| \sup_{k \geq 2} \left\| \frac{F^k(x)}{k!} \right\|^{\frac{1}{k-1}}. \tag{6.1}$$

Also we adopt the convention that $\gamma(F, x) = \infty$ if $\widehat{F'(x)}$ is not invertible. Note that this definition is justified, and in the case when $\widehat{F'(x)}$ is invertible, by analyticity, $\gamma(F, x)$ is finite. The following lemma shows that if F is analytic , then F satisfies the γ-condition. Its proof is easy and so is omitted here (see also [24, 25]).

Lemma 5.11. *Let $x_0 \in \Omega$ and let $\gamma := \gamma(F, x_0)$. Let $0 < r \leq \frac{2-\sqrt{2}}{2\gamma}$. Then F satisfies γ-condition at x_0 in $\mathbf{B}(x_0, r)$.*

Let $x_0 \in H$ be such that $\widehat{F'(x_0)}^{-1}$ exists, and let $\gamma := \gamma(F, x_0)$. Define $\alpha := \gamma\beta$.

Corollary 5.12. *Let*

$$\alpha = \beta\gamma < 3 - 2\sqrt{2}.$$

Suppose that $F'(x_0)$ is a positive operator (not necessary symmetric). Let $\{x_n\}$ be a sequence generated by Newton's method (3.2) *with initial point x_0 and*

$$\|x_1 - x_0\| \leq \beta.$$

Then, $\{x_n\}$ is well defined, and converges to a solution x^* of (3.1) *in $\overline{\mathbf{B}(x_0, r_1)}$. Moreover, there holds*

$$\|x_{k+1} - x_k\| \leq \mu^{2^k - 1}\|x_1 - x_0\|$$

for all $k = 0, 1, 2, \dots,$ where μ, η are given by (5.9) *and* (5.10) *respectively.*

Proof. By Lemma 5.11, F satisfies γ-condition at x_0 in $\mathbf{B}(x_0, r_1)$. Thus, Corollary 5.9 is applicable and the conclusion follows. □

Corollary 5.13. *Let $\beta \leq \frac{3-2\sqrt{2}}{\gamma}$. Suppose that $F'(x_0)$ is a positive operator (not necessary symmetric). Let $x_1 \in H$ be such that*

$$0 \in F(x_0) + F'(x_0)(x_1 - x_0) + T(x_1)$$

and $\|x_1 - x_0\| \leq \beta$. Then, there exists a unique solution x^* of (3.1) *in* $\overline{\mathbf{B}(x_0, r_1)}$.

Proof. By Lemma 5.11, F satisfies γ-condition at x_0 in $\mathbf{B}(x_0, r_1)$. Thus, Corollary 5.10 is applicable and the conclusion follows. □

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